

# On the running and the UV limit of Wilsonian renormalization group flows

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# Outline

- Mathematics of Euclidean Feynman functional integral.
- Mathematics of Wilsonian regularization (Euclidean/Lorentzian signature).
- Mathematics of Wilsonian renormalization (Euclidean/Lorentzian signature).
- Wilsonian renormalizability of an Eucl. signature model.

# Recap on distribution theory

Will consider only scalar and bosonic fields for simplicity.

Will consider only flat (affine) spacetime manifold for simplicity.

●  $\mathcal{E}$  : space of all **smooth fields** over spacetime. collection of “open” sets

They form a vector space with a topology:

$\varphi_i \in \mathcal{E} (i \in \mathbb{N}) \rightarrow 0$  iff all derivatives locally uniformly converge to zero.

●  $\mathcal{S}$  : space of rapidly decreasing smooth fields (**Schwartz fields**) over spacetime.

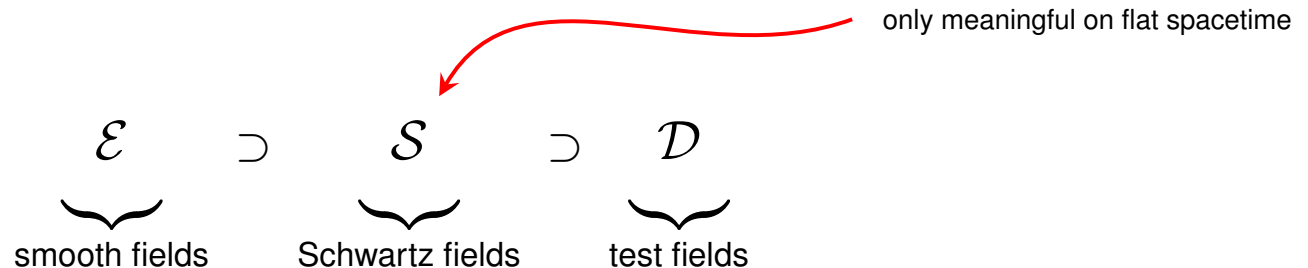
They form a vector space with a topology:

$\varphi_i \in \mathcal{S} (i \in \mathbb{N}) \rightarrow 0$  iff all derivatives  $\times$  all polynomials uniformly converge to zero.

●  $\mathcal{D}$  : space of compactly supported smooth fields (**test fields**) over spacetime.

They form a vector space with a topology:

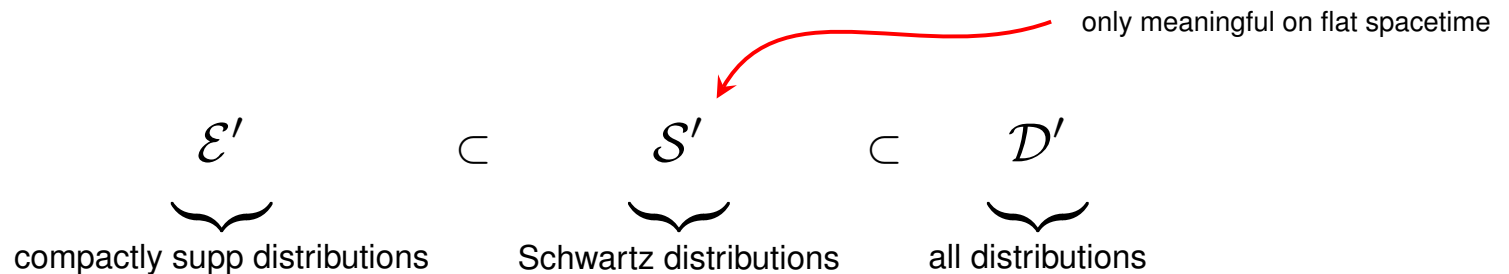
$\varphi_i \in \mathcal{D} (i \in \mathbb{N}) \rightarrow 0$  iff they stay within a compact set and  $\rightarrow 0$  in  $\mathcal{E}$  sense.



Distributions are continuous duals of  $\mathcal{E}$ ,  $\mathcal{S}$ ,  $\mathcal{D}$ .

- $\mathcal{E}'$  : continuous  $\mathcal{E} \rightarrow \mathbb{R}$  linear functionals.  
They are the **compactly supported distributions**.
- $\mathcal{S}'$  : continuous  $\mathcal{S} \rightarrow \mathbb{R}$  linear functionals.  
They are the tempered or **Schwartz distributions**.
- $\mathcal{D}'$  : continuous  $\mathcal{D} \rightarrow \mathbb{R}$  linear functionals.  
They are the space of **all distributions**.

They carry a corresponding natural topology (notion of “open” sets).



[Of course, functions are also distributions, e.g.  $\mathcal{D} \subset \mathcal{E}'$  and  $\mathcal{E} \subset \mathcal{D}'$  etc.]

# Recap on measure / integration / probability theory

● Let  $X$  be a set (its elements called **elementary events**).

● Let  $\Sigma$  be a collection of subsets of  $X$  such that:

●  $X$  is in  $\Sigma$ ,

● for all  $A$  in  $\Sigma$ , its complement is in  $\Sigma$ .

● for all max countably infinite system  $A_i \in \Sigma$  ( $i \in \mathbb{N}$ ), the union  $\bigcup_{i \in \mathbb{N}} A_i$  is in  $\Sigma$ .

Then,  $\Sigma$  is called a sigma-algebra (collection of **composite events** or **measurable sets**).

When  $X$  carries open sets (topology), the sigma-alg generated by them is used (Borel).

$(X, \Sigma)$  is called **measurable space**.

● Let  $\mu : \Sigma \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$  be a weight-assigning function to sets, such that:

●  $\mu(\emptyset) = 0$ ,

● for all max countably inf. disjoint system  $A_i \in \Sigma$  ( $i \in \mathbb{N}$ ):  $\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i)$ ,

●  $\exists$  some max countably infinite system  $A_i \in \Sigma$  ( $i \in \mathbb{N}$ ) with  $\mu(A_i) < \infty$ :  $X = \bigcup_{i \in \mathbb{N}} A_i$ .

Then,  $\mu$  is called **measure**.

$(X, \Sigma, \mu)$  is called **measure space**. [E.g. probability measure space iff  $\mu(X) = \text{finite}$ .]

- A function  $f : X \rightarrow \mathbb{C}$  is called **measurable** iff in good terms with measure theory:  
 for all  $B \in \text{Borel}(\mathbb{C})$ , one has  $f^{-1}(B) \in \Sigma$  of  $X$ .  
 Theorem:  $f$  is measurable iff approximable pointwise by “histograms” with bins from  $\Sigma$ .
- The **integral**  $\int_{x \in X} f(x) d\mu(x)$  is defined via the histogram “area” approximations.  
 Theorem: this is well-defined.
- Let  $(X, \Sigma, \mu)$  be a measure space and  $(Y, \Delta)$  a measurable space.  
 Let  $C : X \rightarrow Y$  be a measurable mapping.  
 Then, one can define the **pushforward** (or marginal) measure  $C_* \mu$  on  $Y$ .  
 [For all  $B \in \Delta$  one defines  $(C_* \mu)(B) := \mu(C^{-1}(B))$ .]
- Pushforward (marginal) measure means simply transformation of integration variable.  
 If forgetful transformation, the “forgotten” d.o.f. are “integrated out”.
- If  $\mu$  is a probability measure e.g. on  $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$ , then  
 $Z(y) := \int_{x \in X} e^{i(y|x)} d\mu(x)$  is its **Fourier transform** (partition function in QFT).

# Mathematics of Euclidean Feynman functional integral

● Take an Euclidean action  $S = T + V$ , with kinetic + potential term splitting.  
 Say,  $T(\varphi) = \int \varphi (-\Delta + m^2)\varphi$ , and  $V(\varphi) = g \int \varphi^4$ .

● Then  $T$ , i.e.  $(-\Delta + m^2)$  has a propagator  $K(\cdot, \cdot)$  which is positive definite:

●  $(-\Delta + m^2)_x K(x, y) = \delta_y(x),$

● for all  $j \in \mathcal{S}$  rapidly decreasing sources:  $(K|j \otimes j) \geq 0.$

● Due to above, the function  $\mathcal{S} \rightarrow \mathbb{C}, j \mapsto e^{-(K|j \otimes j)}$  has “quite nice” properties.

● **Bochner-Minlos theorem:** because of

● “quite nice” properties of  $j \mapsto e^{-(K|j \otimes j)},$

● “quite nice” properties of the space  $\mathcal{S},$

$\exists!$  measure  $\gamma_T$  on  $\mathcal{S}'$ , whose Fourier transform is  $j \mapsto e^{-(K|j \otimes j)}.$

It is the Feynman measure for free theory:  $\int_{\phi \in \mathcal{S}'} (\dots) d\gamma_T(\phi) = \int_{\phi \in \mathcal{S}'} (\dots) e^{-T(\phi)} \text{“d}\phi\text{”}.$

● Tempting definition for Feynman measure of interacting theory:

$$\int_{\phi \in \mathcal{S}'} (\dots) e^{-V(\phi)} d\gamma_T(\phi) \quad \left[ = \int_{\phi \in \mathcal{S}'} (\dots) \underbrace{e^{-(T(\phi)+V(\phi))}}_{=e^{-S(\phi)}} \text{“d}\phi\text{”} \right]$$

# Mathematics of Wilsonian regularization

- Problem, the interacting Feynman measure  $\mu := e^{-V} \gamma_T$  is undefined:

$$\int_{\phi \in \mathcal{S}'} (\dots) \underbrace{d\mu(\phi)}_{\text{wannabe Feynman measure}} := \int_{\phi \in \mathcal{S}'} (\dots) \underbrace{e^{-V(\phi)}}_{\text{lives on function sense fields}} \underbrace{d\gamma_T(\phi)}_{\text{lives on distribution sense fields}}$$

Because  $V$  is spacetime integral of pointwise product of fields, e.g.  $V(\varphi) = \int \varphi^4$ .  
How to bring  $e^{-V}$  and  $\gamma_T$  to common grounds?

- Physicist solution: in desperation, **Wilsonian regularization**.  
Take a continuous linear mapping  $C: (\text{distributional fields}) \rightarrow (\text{function sense fields})$ .  
Take the pushforward Gaussian measure  $C_* \gamma_T$ , which lives on  $\text{Ran}(C)$ .  
Those are functions, so safe to integrate  $e^{-V}$  there:

$$\int_{\varphi \in \text{Ran}(C)} (\dots) e^{-V(\varphi)} d(C_* \gamma_T)(\varphi) \quad \left[ = \int_{\varphi \in \text{Ran}(C)} (\dots) e^{-(T_C(\varphi) + V(\varphi))} \text{“d}\varphi\text{”} \right]$$

a space of UV regularized fields

[Schwartz kernel theorem:  $C$  is convolution by a test function, if translationally invariant.  
I.e., it is a momentum space damping, or coarse-graining of fields.]



# Mathematics of Wilsonian renormalization

- What do we do with the  $C$ -dependence? What is the physics / mathematics behind?
- Take a family  $V_C$  ( $C \in \{\text{coarse-grainings}\}$ ) of interaction terms  $\leftrightarrow \mu_C := e^{-V_C} C_* \gamma_T$ . We say that it is a **Wilsonian renormalization group (RG) flow** iff:

$\exists$  some continuous functional  $z : \{\text{coarse-grainings}\} \rightarrow \mathbb{R}$ , such that

$\forall$  coarse-grainings  $C, C', C''$  with  $C'' = C' C$ :

$$z(C'')_* \mu_{C''} = z(C)_* C'_* \mu_C$$

[ $z$  is called the **running wave function renormalization factor**.]

- If  $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \dots)$  are the moments of  $\mu_C$ , then

$\exists$  some continuous functional  $z : \{\text{coarse-grainings}\} \rightarrow \mathbb{R}$ , such that

$\forall$  coarse-grainings  $C, C', C''$  with  $C'' = C' C$ :

$$z(C'')^n \mathcal{G}_{C''}^{(n)} = z(C)^n \otimes^n C' \mathcal{G}_C^{(n)} \text{ for all } n = 0, 1, 2, \dots$$

[Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

[We can always set  $z(C) = 1$ , by rescaling fields:  $\tilde{\mu}_C := z(C)_* \mu_C$  or  $\tilde{\mathcal{G}}_C^{(n)} := z(C)^n \mathcal{G}_C^{(n)}$ .]

Clean definition:

A family of smooth correlators  $\mathcal{G}_C$  ( $C \in$  coarse-grainings) is **Wilsonian RG flow** iff

$\forall$  coarse-grainings  $C, C', C''$  with  $C'' = C' C$  one has that

$\mathcal{G}_{C''}^{(n)} = \otimes^n C' \mathcal{G}_C^{(n)}$  holds ( $n = 0, 1, 2, \dots$ ). ← rigorous RGE in any signature

Space of Wilsonian RG flows is nonempty:

For any distributional correlator  $G$ , the family

$$\mathcal{G}_C^{(n)} := \otimes^n C G^{(n)} \quad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lász, Z.Tarcsay *Class.Quant.Grav.***41**(2024)125009]:

1. On manifolds it is “quite nice” topological vector space, similar to distributions.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of (\*).

↓  
UV limit.

## Sketch of proofs.

1. On manifolds it is “quite nice” topological vector space, similar to distributions.

- Coarse-grainings have a natural ordering of being less UV than an other:

$$C'' \preceq C \text{ iff } C'' = C \text{ or } \exists C' : C'' = C' C.$$

- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of  $\mathcal{T}(\mathcal{E})$ .

- Check known properties of  $\mathcal{T}(\mathcal{E})$ , some of them are preserved by projective limit.

2. On flat spacetime for bosonic fields, all Wilsonian RG flows are  $\mathcal{G}_C^{(n)} = \otimes^n C G^{(n)}$ .

- On flat spacetime, convolution ops by test functions  $C_\eta := \eta \star (\cdot)$  exist and commute.

- Due to RGE, commutativity of convolution ops, and polarization formula for  $n$ -forms, for bosonic fields  $\mathcal{G}_{C_\eta}^{(n)}$  is  $n$ -order homogeneous polynomial in  $\eta$ .

That is,  $\exists!$   $\mathcal{G}_{\eta_1, \dots, \eta_n}^{(n)}$  symmetric  $n$ -linear map in  $\eta_1, \dots, \eta_n$ , such that  $\mathcal{G}_{C_\eta}^{(n)} = \mathcal{G}_{\eta, \dots, \eta}^{(n)}$ .

- Due to RGE, commutativity of convolution ops, and an improved Banach-Steinhaus thm,

$\mathcal{G}_{\eta_1^t, \dots, \eta_n^t}^{(n)} \Big|_0$  extends to an  $n$ -variate distribution, it will do the job as  $(G^{(n)} | \eta_1 \otimes \dots \otimes \eta_n)$ .

[An improved Banach-Steinhaus theorem (the key lemma – A.László, Z.Tarcsay):

If a sequence of  $n$ -variate distributions pointwise converge on  $\otimes^n \mathcal{D}$ , then also on full  $\mathcal{D}_n$ .]

So, it turns out that Wilsonian RG flow of correlators  $\leftrightarrow$  distributional correlators.  
(under mild conditions)

Executive summary:

- In QFT, the fundamental objects of interest are distributional field correlators.
- Physical ones selected by an “equation of motion”, the master Dyson-Schwinger equation.  
Through their smoothed (Wilsonian regularized) instances [*CQG***39**(2022)185004].

Academic question:

- What about Wilsonian RG flow of measures? (In Euclidean signature QFT.)  
Manuscript in preparation.

# Wilsonian renormalization in Euclidean signature

Let us come back to Euclidean Feynman measures on flat spacetime, for bosonic fields.  
[We work on  $\mathcal{S}$  and  $\mathcal{S}'$ , because we can.]

Take a family  $V_C$  ( $C \in \{\text{coarse-grainings}\}$ ) of interaction terms  $\leftrightarrow \mu_C := e^{-V_C} C_* \gamma_T$ .  
Let it be a **Wilsonian RG flow**:

$\forall$  coarse-grainings  $C, C', C''$  with  $C'' = C' C$ :

$$\mu_{C''} = C'_* \mu_C$$

Space of Wilsonian RG flow of Feynman measures is nonempty:

For any Feynman measure  $\mu$  on  $\mathcal{S}'$ , the family

$$\mu_C := C_* \mu \quad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lász3, Z.Tarcsay *manuscript in prep.*]:

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form (\*).  $\leftarrow$  UV limit
2. There exists some measurable potential  $V : \mathcal{S}' \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , such that  $\mu = e^{-V} \gamma_T$ .
3. For all coarse-graining  $C$ , one has  $V_C(C \phi) = V(\phi)$  for  $\gamma_T$ -a.e.  $\phi \in \mathcal{S}'$ .
4. If  $V_C : C[\mathcal{S}'] \rightarrow \mathbb{R} \cup \{\pm\infty\}$  bounded from below, then  $V$  is  $\gamma_T$ -ess.bounded from below.

## Sketch of proofs.

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form  $\mu_C = C_* \mu$ .
  - Will speak about when we submit the manuscript.
2. There exists some measurable potential  $V : \mathcal{S}' \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , such that  $\mu = e^{-V} \gamma_T$ .
  - Will speak about when we submit the manuscript.
3. For all coarse-graining  $C$ , one has  $V_C(C\phi) = V(\phi)$  for  $\gamma_T$ -a.e.  $\phi \in \mathcal{S}'$ .
  - Will speak about when we submit the manuscript.
4. If  $V_C : C[\mathcal{S}'] \rightarrow \mathbb{R} \cup \{\pm\infty\}$  bounded from below, then  $V$  is  $\gamma_T$ -ess.bounded from below.
  - Will speak about when we submit the manuscript.

Relation to usual RG theory:

Fix some  $\eta \in \mathcal{S}$  such that  $\int \eta = 1$  and  $F(\eta) > 0$ .

Introduce scaled  $\eta$ , that is  $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda x)$  (for all  $x \in \mathbb{R}^N$  and scaling  $1 \leq \Lambda < \infty$ ).

One has  $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$  as  $\Lambda \rightarrow \infty$ .

By our theorem, for all  $\Lambda$ , one has  $V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda} \phi) = V(\phi)$  for  $\gamma_T$ -a.e.  $\phi \in \mathcal{S}'$ .

↓

Informally: ODE for  $V_{C_{\eta_\Lambda}}$ , namely  $\frac{d}{d\Lambda} V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda} \phi) = 0$  for  $1 \leq \Lambda < \infty$ .

QFT people try to solve such flow equation, given initial data  $V_{C_\Lambda}|_{\Lambda=1}$ .

But why bother? By our theorem, all RG flows of such kind has some  $V$  at the UV end.

Look directly for  $V$ ?

# What really the game is about?

Original problem:

- We had  $\mathcal{V} : \{\text{function sense fields}\} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , say  $\mathcal{V}(\varphi) = g \int \varphi^4$ .
- We would need to integrate it against  $\gamma_T$ , but that lives on  $\mathcal{S}'$  fields.
- $\gamma_T$  known to be supported “sparsely”, i.e. not on function fields, but really on  $\mathcal{S}'$ .
- So, we really need to extend  $\mathcal{V}$  at least  $\gamma_T$ -a.e. to make sense of  $\mu := e^{-V} \gamma_T$ .

Concern of physicists: this may be impossible.

- We are afraid that  $V$  on  $\mathcal{S}'$  might not exist.
- Instead, let us push  $\gamma_T$  to smooth fields by  $C$ , do there  $\mu_C := e^{-V_C} C_* \gamma_T$ .
- Then, get rid of  $C$ -dependence of  $\mu_C$  by concept of Wilsonian RG flow.  
Maybe even  $\mu_C \rightarrow \mu$  could exist as  $C \rightarrow \delta$  if we are lucky...

Our result: we are back to the start.

- The UV limit Feynman measure  $\mu$  then indeed exists.
- But we just proved that then there **must** exist some extension  $V$  of  $\mathcal{V}$  to  $\mathcal{S}'$ ,  $\gamma_T$ -a.e.
- So, we'd better look for that ominous extension  $V$ .
- For bounded for below  $\mathcal{V}$ , bounded from below measurable  $V$  needed.

If we find one,  $\mu := e^{-V} \gamma_T$  is then finite measure automatically.

Only pathology: overlap integral of  $e^{-V}$  and  $\gamma_T$  expected small, maybe zero.

We only need to make sure that  $\int_{\phi \in \mathcal{S}'} e^{-V(\phi)} d\gamma_T(\phi) > 0!$



A natural extension[A.László, Z.Tarcsay *manuscript in prep.*]:

If  $\mathcal{V}$  is bounded from below, there is an optimal extension, the “greedy” extension.

$$V(\cdot) := (\gamma_T) \inf_{\{\eta_n \rightarrow \delta\}} \liminf_{\eta_n \rightarrow \delta} \mathcal{V}(\eta_n \star \cdot)$$

This is the lower bound of extensions, i.e. overlap of  $e^{-V}$  and  $\gamma_T$  largest.

But is  $V$  measurable at all? Not evident.

Theorem[A.László, Z.Tarcsay *manuscript in prep.*]:

1. The “greedy extension” is measurable.
2. The interacting Feynman measure  $\mu := e^{-V} \gamma_T$  by greedy extension is nonzero iff a particular inequality on the regularized integrals is satisfied.

This is actually a calculable condition for concrete models!

# Summary

- Wilsonian RG flow of correlators can be defined in any signature and on manifolds.
- Under mild conditions, they originate from a distributional correlator (UV limit).  
[ $\sim$  existence theorem for multiplicative renormalization.]
- Likely to be generically true (on manifolds, in any signature).
- In Euclidean signature, similar for Feynman measures.  
+ a new condition for renormalizability.

# Backup slides

# Followed guidelines

Do not use (unless emphasized):

- Structures specific to an affine spacetime manifold.
- Known fixed spacetime metric / causal structure.
- Known splitting of Lagrangian to free + interaction term.

Consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics.  
(No Schwartz functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.
- Can only use generic, differential geometrically natural objects.

# Outline

Will attempt to set up eom for the key ingredient for the quantum probability space of QFT.

## I. On Wilsonian regularized Feynman functional integral formulation:

- Can be substituted by regularized master Dyson-Schwinger equation for correlators.
- For conformally invariant or flat spacetime Lagrangians, showed an existence condition for regularized MDS solutions, provides convergent iterative solver method.

[*Class.Quant.Grav.***39**(2022)185004]

## II. On Wilsonian renormalization group flows of correlators:

- They form a topological vector space which is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
- On flat spacetime for bosonic fields: in bijection with distributional correlators.

[**arXiv:2303.03740** with *Zsigmond Tarcsay*]

## Part I:

# On Wilsonian regularized Feynman functional integral formulation

# The classical field theory scene

$\mathcal{M}$  a smooth orientable oriented manifold (wannabe spacetime, but no metric, yet).

$V(\mathcal{M})$  a vector bundle over it (its smooth sections are matter fields + metric if dynamical).

Field configurations:

$$\underbrace{(v, \nabla)}_{=: \psi} \in \underbrace{\Gamma\left(V(\mathcal{M}) \times_{\mathcal{M}} \text{CovDeriv}(V(\mathcal{M}))\right)}_{=: \mathcal{E}}$$

Real topological affine space with the  $\mathcal{E}$  smooth function topology.

Field variations:

$$\underbrace{(\delta v, \delta \mathcal{C})}_{=: \delta \psi} \in \underbrace{\Gamma\left(V(\mathcal{M}) \times_{\mathcal{M}} T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M})\right)}_{=: \mathcal{E}}$$

Real topological vector space with the  $\mathcal{E}$  smooth function topology.

Test field variations:  $\delta\psi_T \in \mathcal{D}$ , compactly supported ones from  $\mathcal{E}$  with  $\mathcal{D}$  test funct. top.

# Informal Feynman functional integral in Lorentz signature

Fix a reference field  $\psi_0 \in \mathcal{E}$  for bringing the problem from  $\mathcal{E}$  to  $\mathcal{E}$ , and take  $J_1, \dots, J_n \in \mathcal{E}'$ . Then,  $\psi \mapsto (J_1 | \psi - \psi_0) \cdot \dots \cdot (J_n | \psi - \psi_0)$  defines a  $\mathcal{E} \rightarrow \mathbb{R}$  polynomial observable.

Feynman type quantum vacuum expectation value of this is postulated as:

$$\int_{\psi \in \mathcal{E}} (J_1 | \psi - \psi_0) \cdot \dots \cdot (J_n | \psi - \psi_0) e^{\frac{i}{\hbar} S(\psi)} d\lambda(\psi) \quad / \quad \int_{\psi \in \mathcal{E}} e^{\frac{i}{\hbar} S(\psi)} d\lambda(\psi)$$

Partition function often invoked to book-keep these (formal Fourier transform of  $e^{\frac{i}{\hbar} S} \lambda$ ):

$$Z_{\psi_0} : \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int_{\psi \in \mathcal{E}} e^{i(J | \psi - \psi_0)} e^{\frac{i}{\hbar} S(\psi)} d\lambda(\psi),$$

and from this one can define

$$G_{\psi_0}^{(n)} := \left( (-i)^n \frac{1}{Z_{\psi_0}(J)} \partial_J^{(n)} Z_{\psi_0}(J) \right) \Big|_{J=0}$$

$n$ -field correlator, and their collection  $G_{\psi_0} := (G_{\psi_0}^{(0)}, G_{\psi_0}^{(1)}, \dots, G_{\psi_0}^{(n)}, \dots) \in \bigoplus_{n \in \mathbb{N}_0} \otimes^n \mathcal{E}$ .

Above quantum expectation value expressible via distribution pairing:  $(J_1 \otimes \dots \otimes J_n | G_{\psi_0}^{(n)})$ .



Well known problems:

- No “Lebesgue” measure  $\lambda$  in infinite dimensions.
- Neither  $e^{\frac{i}{\hbar}S} \lambda$  is meaningful. (Can be repaired to some extent in Euclidean signature.)
- Neither the Fourier transform of this undefined measure is meaningful.

Rules in informal QFT:

- as if  $\lambda$  existed as *translation invariant* (Lebesgue) measure,
- as if  $e^{\frac{i}{\hbar}S} \lambda$  existed as *finite measure*, with *finite moments* and *analytic Fourier transform*.

Textbook “theorem”: because of above rules, one has

$Z : \mathcal{E}' \rightarrow \mathbb{C}$  is Fourier transform of  $e^{\frac{i}{\hbar}S} \lambda$  “ $\iff$ ” it satisfies master-Dyson-Schwinger eq

$$\left( \mathbf{E}((-i)\partial_J + \psi_0) + \hbar J \right) Z(J) = 0 \quad (\forall J \in \mathcal{E}')$$

where  $E(\psi) := DS(\psi)$  is the Euler-Lagrange functional at  $\psi \in \mathcal{E}$ .

Does this informal PDE have a meaning? [Yes, on the correlators  $G = (G^{(0)}, G^{(1)}, \dots)$ .]

# Rigorous definition of Euler-Lagrange functional

- Let a **Lagrange form** be given, which is

$$L : V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \oplus T^*(\mathcal{M}) \wedge T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}) \longrightarrow \bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})$$

pointwise bundle homomorphism.

- **Lagrangian expression:**

$$\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} \text{CovDeriv}(V(\mathcal{M}))) \longrightarrow \Gamma\left(\bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})\right), \quad (v, \nabla) \longmapsto L(v, \nabla v, F(\nabla))$$

where  $F(\nabla)$  is the curvature tensor.

- **Action functional:**

$$S : \underbrace{\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} \text{CovDeriv}(V(\mathcal{M})))}_{=: \mathcal{E}} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), \quad \underbrace{(v, \nabla)}_{=: \psi} \longmapsto \left( \mathcal{K} \mapsto S_{\mathcal{K}}(v, \nabla) \right)$$

where  $S_{\mathcal{K}}(v, \nabla) := \int_{\mathcal{K}} L(v, \nabla v, F(\nabla))$  for all  $\mathcal{K} \subset \mathcal{M}$  compact.

Action functional  $S : \mathcal{E} \rightarrow \text{Meas}(\mathcal{M}, \mathbb{R})$  Fréchet differentiable, its Fréchet derivative

$$DS : \mathcal{E} \times \mathcal{E} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \left( \mathcal{K} \mapsto (DS_{\mathcal{K}}(\psi) \mid \delta\psi) \right)$$

is the usual Euler-Lagrange integral on  $\mathcal{K}$  + usual boundary integral on  $\partial\mathcal{K}$ .  
Jointly continuous in its variables, linear in second variable.

**Euler-Lagrange functional:**

We restrict  $DS$  from  $\mathcal{E} \times \mathcal{E}$  to  $\mathcal{E} \times \mathcal{D}$ , to make the EL integral over full  $\mathcal{M}$  finite.

$$E : \mathcal{E} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi, \delta\psi_T) \longmapsto (E(\psi) \mid \delta\psi_T) := (DS_{\mathcal{M}}(\psi) \mid \delta\psi_T)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full  $\mathcal{M}$ , real valued.  
Jointly sequentially continuous, linear in second variable. (Also,  $E : \mathcal{E} \rightarrow \mathcal{D}'$  continuous.)

**Classical field equation** is

$$\psi \in \mathcal{E} ? \quad \forall \delta\psi_T \in \mathcal{D} : (E(\psi) \mid \delta\psi_T) = 0.$$

**Observables** are the  $O : \mathcal{E} \rightarrow \mathbb{R}$  continuous maps.

# Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on  $Z$  to  $n$ -field correlators  $G = (G^{(0)}, G^{(1)}, \dots)$ . These sit in the tensor algebra  $\mathcal{T}(\mathcal{E}) := \bigoplus_{n \in \mathbb{N}_0} \hat{\otimes}_{\pi}^n \mathcal{E}$  of field variations.

More precisely, they sit in a graded-symmetrized subspace, e.g.  $\vee(\mathcal{E})$  or  $\wedge(\mathcal{E})$  of  $\mathcal{T}(\mathcal{E})$ . Naturally topologized: with Tychonoff topology, similar to  $\mathcal{E}$ , i.e. nuclear Fréchet.

- Algebraic tensor algebra  $\mathcal{T}_a(\mathcal{E}') := \bigoplus_{n \in \mathbb{N}_0} \hat{\otimes}_{\pi}^n \mathcal{E}'$  of sources.

Naturally topologized: loc.conv. direct sum topology, similar to  $\mathcal{E}'$ , i.e. dual nuclear Fréchet.

- Schwartz kernel thm gives some simplification:  $\hat{\otimes}_{\pi}^n \mathcal{E} \equiv \mathcal{E}_n$  and  $\hat{\otimes}_{\pi}^n \mathcal{E}' \equiv \mathcal{E}'_n$  ( $n$ -variate).

- One has  $(\mathcal{T}(\mathcal{E}))' \equiv \mathcal{T}_a(\mathcal{E}')$  and  $(\mathcal{T}(\mathcal{E}))'' \equiv \mathcal{T}(\mathcal{E})$  etc, “nice” properties.

Moreover, tensor algebra of field variations is topological unital bialgebra.

Unity  $\mathbb{1} := (1, 0, 0, \dots)$ .

Left-multiplication  $L_x$  by a fix element  $x$  meaningful and continuous linear.

Left-insertion  $\iota_p$  (tracing out) by  $p \in (\mathcal{T}(\mathcal{E}))' \equiv \mathcal{T}_a(\mathcal{E}')$  also meaningful, continuous linear.

Usual graded-commutation:  $(\iota_p L_{\delta\psi} \pm L_{\delta\psi} \iota_p) G = (p|\delta\psi) G \quad (\forall p \in \mathcal{E}', \delta\psi \in \mathcal{E}, G)$ .

Take a classical observable  $O : \mathcal{E} \rightarrow \mathbb{R}$ ,  $\psi \mapsto O(\psi)$ , let  $O_{\psi_0} := O \circ (\mathbf{I}_{\mathcal{E}} + \psi_0)$ .

That is,  $O_{\psi_0}(\psi - \psi_0) \stackrel{!}{=} O(\psi) \quad (\forall \psi \in \mathcal{E})$ , with some fixed reference field  $\psi_0 \in \mathcal{E}$ .

We say that  $O$  is **multipolynomial** iff for some  $\psi_0 \in \mathcal{E}$  there exists  $\mathbf{O}_{\psi_0} \in \mathcal{T}_a(\mathcal{E}')$ , such that

$$\forall \psi \in \mathcal{E} : \underbrace{O_{\psi_0}(\psi - \psi_0)}_{= O(\psi)} = \left( \mathbf{O}_{\psi_0} \mid \left( 1, \overset{1}{\otimes}(\psi - \psi_0), \overset{2}{\otimes}(\psi - \psi_0), \dots \right) \right).$$

Similarly  $E : \mathcal{E} \rightarrow \mathcal{D}'$ ,  $\psi \mapsto E(\psi)$ , let  $E_{\psi_0} := E \circ (\mathbf{I}_{\mathcal{E}} + \psi_0)$  the same re-expressed on  $\mathcal{E}$ .

That is,  $E_{\psi_0}(\psi - \psi_0) \stackrel{!}{=} E(\psi) \quad (\forall \psi \in \mathcal{E})$ , with some fixed reference field  $\psi_0 \in \mathcal{E}$ .

We say that  $E$  is **multipolynomial** iff  $\exists \mathbf{E}_{\psi_0} \in \mathcal{T}_a(\mathcal{E}') \hat{\otimes}_{\pi} \mathcal{D}'$ , such that

$$\forall \psi \in \mathcal{E}, \delta\psi_T \in \mathcal{D} : \underbrace{\left( E_{\psi_0}(\psi - \psi_0) \mid \delta\psi_T \right)}_{= (E(\psi) \mid \delta\psi_T)} = \left( \mathbf{E}_{\psi_0} \mid (1, \overset{1}{\otimes}(\psi - \psi_0), \overset{2}{\otimes}(\psi - \psi_0), \dots) \otimes \delta\psi_T \right).$$

For fixed  $\delta\psi_T \in \mathcal{D}$  one has  $(\mathbf{E}_{\psi_0} \mid \delta\psi_T) \in \mathcal{T}_a(\mathcal{E}')$ , i.e. one can left-insert with it:

$\mathcal{L}_{(\mathbf{E}_{\psi_0} \mid \delta\psi_T)}$  meaningfully acts on  $\mathcal{T}(\mathcal{E})$ .

The master Dyson-Schwinger (MDS) equation is:

we search for  $(\psi_0, G_{\psi_0})$  such that:

$$\underbrace{G_{\psi_0}^{(0)}}_{=: b G_{\psi_0}} = 1,$$

$$\forall \delta\psi_T \in \mathcal{D} : \quad \underbrace{\left( \mathcal{L}_{(\mathbf{E}_{\psi_0} | \delta\psi_T)} - i \hbar L_{\delta\psi_T} \right)}_{=: \mathbf{M}_{\psi_0, \delta\psi_T}} G_{\psi_0} = 0.$$

This substitutes Feynman functional integral formulation, signature independently.  
Also, no fixed background causal structure etc needed.

[Feynman type quantum vacuum expectation value of  $O$  is then  $(\mathbf{O}_{\psi_0} | G_{\psi_0}).$ ]

Example:  $\phi^4$  model.

Euler-Lagrange functional is

$$E : \mathcal{E} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi, \delta\psi_T) \longmapsto \int_{y \in \mathcal{M}} \delta\psi_T(y) \square_y \psi(y) v(y) + \int_{y \in \mathcal{M}} \delta\psi_T(y) \psi^3(y) v(y).$$



MDS operator at  $\psi_0 = 0$  reads

$$\begin{aligned} (\mathbf{M}_{\psi_0, \delta\psi_T} G)^{(n)}(x_1, \dots, x_n) = & \int_{y \in \mathcal{M}} \delta\psi_T(y) \square_y G^{(n+1)}(y, x_1, \dots, x_n) v(y) + \int_{y \in \mathcal{M}} \delta\psi_T(y) G^{(n+3)}(y, y, y, x_1, \dots, x_n) v(y) \\ & - i \hbar \underbrace{n \frac{1}{n!} \sum_{\pi \in \Pi_n} \delta\psi_T(x_{\pi(1)}) G^{(n-1)}(x_{\pi(2)}, \dots, x_{\pi(n)})}_{= (L_{\delta\psi_T} G)^{(n)}(x_1, \dots, x_n)} \end{aligned}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.



Theorem: no solutions with high differentiability (e.g. as smooth functions).

Theorem: for free Minkowski KG case, distributional solution only,  
namely  $G_{\psi_0} = \exp(K_{\psi_0})$ , where

$$\begin{aligned} K_{\psi_0}^{(0)} &= 0, \\ K_{\psi_0}^{(1)} &= 0, \\ K_{\psi_0}^{(2)} &= i \hbar K_{\psi_0}^{(2)} \quad \leftarrow \text{(symmetric propagator)} \\ K_{\psi_0}^{(n)} &= 0 \quad (n \geq 2) \end{aligned}$$

So we expect distributional solutions only, at best.

How can one extend to distributions interaction term like  $G^{(n+3)}(\mathbf{y}, \mathbf{y}, \mathbf{y}, x_1, \dots, x_n)$  ?

With sufficiency condition of Hörmander? (Theorem: not workable.)

Via approximation with functions, i.e. sequential closure? (Theorem: not workable.)

Workaround in QFT: [Wilsonian regularization](#) using coarse-graining (UV damping).

# Wilsonian regularized master Dyson-Schwinger equation

- When  $\mathcal{E}$  (resp  $\mathcal{D}$ ) are smooth sections of some vector bundle, denote by  $\mathcal{E}^\times$  (resp  $\mathcal{D}^\times$ ) the smooth sections of its densitized dual vector bundle. Then, **distributional sections** are  $\mathcal{D}^{\times'}$  (resp  $\mathcal{E}^{\times'}$ ).
- A continuous linear map  $C : \mathcal{E}^{\times'} \rightarrow \mathcal{E}$  is called **smoothing operator**. Schwartz kernel theorem:  $C \longleftrightarrow$  its Schwartz kernel  $\kappa$  which is section over  $\mathcal{M} \times \mathcal{M}$ .
- $C_\kappa$  is **properly supported** iff  $\forall \mathcal{K} \subset \mathcal{M}$  compact:  $\kappa|_{\mathcal{M} \times \mathcal{K}}$  and  $\kappa|_{\mathcal{K} \times \mathcal{M}}$  has compact supp. It extends to  $\mathcal{E}^{\times'}$ ,  $\mathcal{E}$ ,  $\mathcal{D}$ ,  $\mathcal{D}^{\times'}$  and preserves compact support (the transpose similarly).
- A properly supported smoothing operator is **coarse-graining** iff injective as  $\mathcal{E}^{\times'} \rightarrow \mathcal{E}$  and its transpose similarly.  
E.g. ordinary convolution by a nonzero test function over affine (Minkowski) spacetime.

*Coarse-graining ops are natural generalization of convolution by test functions to manifolds.*

Originally: Feynman integral “ $\iff$ ” MDS equation.

Wilsonian regularized Feynman integral:

integrate only on the image space  $C_\kappa[\mathcal{D}^{\times'}] \subset \mathcal{E}$  of some coarse-graining operator  $C_\kappa$ .

Wilsonian regularized Feynman integral “ $\iff$ ” Wilsonian regularized MDS equation:

we search for  $(\psi_0, \gamma(\kappa), \mathcal{G}_{\psi_0, \kappa})$  such that:

$$\underbrace{\mathcal{G}_{\psi_0, \kappa}^{(0)}}_{=: b \mathcal{G}_{\psi_0, \kappa}} = 1,$$

$$\forall \delta\psi_T \in \mathcal{D} : \underbrace{\left( \mathcal{L}_{\gamma(\kappa)}(\mathbf{E}_{\psi_0} | \delta\psi_T) - i \hbar L_{C_\kappa} \delta\psi_T \right)}_{=: \mathbf{M}_{\psi_0, \kappa, \delta\psi_T}} \mathcal{G}_{\psi_0, \kappa} = 0.$$

Brings back problem from distributions to smooth functions, but depends on regulator  $\kappa$ .

Smooth function solution to free KG regularized MDS eq:  $\mathcal{G}_{\psi_0, \kappa} = \exp(\mathcal{K}_{\psi_0, \kappa})$  where

$$\begin{aligned}\mathcal{K}_{\psi_0, \kappa}^{(0)} &= 0, \\ \mathcal{K}_{\psi_0, \kappa}^{(1)} &= 0, \\ \mathcal{K}_{\psi_0, \kappa}^{(2)} &= i \hbar K_{\psi_0, \kappa}^{(2)} \quad \leftarrow \text{(smoothed symmetric propagator)} \\ \mathcal{K}_{\psi_0, \kappa}^{(n)} &= 0 \quad (n \geq 2)\end{aligned}$$

No problem to evaluate interaction term like  $\mathcal{G}^{(n+3)}(y, y, y, x_1, \dots, x_n)$  on functions.

[We proved a convergent iterative solution method at fix  $\kappa$ , see the paper or ask.]

But what we do with  $\kappa$  dependence? (Rigorous Wilsonian renormalization?)

## Part II:

# On Wilsonian RG flows of correlators

# Informal Wilsonian RG flows of Feynman measures

Fix a reference field  $\psi_0 \in \mathcal{E}$  to bring the problem from  $\mathcal{E}$  to  $\mathcal{E}$ .

Fix a coarse-graining  $C_\kappa$  defining a UV regularization strength.

Assume that one has an action  $S_{\psi_0, C_\kappa} : \underbrace{C_\kappa[\mathcal{D}^{\times'}]}_{\subset \mathcal{E}} \rightarrow \mathbb{R}$  for a coarse-graining  $C_\kappa$ .

Informally, one assumes a Lebesgue measure  $\lambda_{C_\kappa}$  on each subspace  $C_\kappa[\mathcal{D}^{\times'}]$  of  $\mathcal{E}$ .  
(In Euclidean signature this inexactness can be remedied by Gaussian measure.)

This defines the Wilsonian regularized Feynman measure  $e^{\frac{i}{\hbar} S_{\psi_0, C_\kappa}} \lambda_{C_\kappa}$ .

A family of actions  $S_{\psi_0, C_\kappa}$  ( $C_\kappa \in$  coarse-grainings) is **Wilsonian RG flow** iff:

$\forall$  coarse-grainings  $C_\kappa, C_\mu, C_\nu$  with  $C_\nu = C_\mu C_\kappa$  one has that

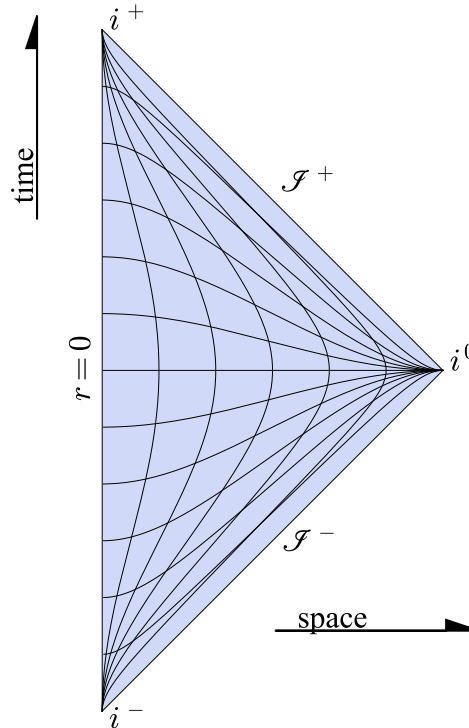
$e^{\frac{i}{\hbar} S_{\psi_0, C_\nu}} \lambda_{C_\nu}$  is the pushforward of  $e^{\frac{i}{\hbar} S_{\psi_0, C_\kappa}} \lambda_{C_\kappa}$  by  $C_\mu$ .

← RGE

Rigorous definition will be this, but expressed on the formal moments ( $n$ -field correlators).

# Existence condition for regularized MDS solutions

If Euler-Lagrange functional  $E : \mathcal{E} \rightarrow \mathcal{D}'$  conformally invariant:  
re-expressible on Penrose conformal compactification.



That is always a compact manifold, with cone condition boundary.

$E : \mathcal{E} \rightarrow \mathcal{D}'$  reformulable over this base manifold.

In such situation,  $\mathcal{E} = \mathcal{D}$  and have nice properties:  
 countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \supset F_1 \supset \dots \supset F_m \supset \dots \supset \mathcal{E}$$

(Intersection of shrinking Hilbert spaces  $F_m$  with Hilbert-Schmidt embedding.)

Theorem [Dubin,Hennings:*P.RIMS***25**(1989)971]:

without penalty, one can equip  $\mathcal{T}(\mathcal{E})$  with a better topology, inheriting CHNF topology.

$$H_0 \supset H_1 \supset \dots \supset H_m \supset \dots \supset \mathcal{T}_h(\mathcal{E})$$

Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0, \kappa} : H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0, \kappa, \delta\psi_T} \mathcal{G}$$

Theorem: one can legitimately trace out  $\delta\psi_T$  variable to form

$$\hat{\mathbf{M}}_{\psi_0, \kappa}^2 : H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i \in \mathbb{N}_0} \mathbf{M}_{\psi_0, \kappa, \delta\psi_T i}^\dagger \mathbf{M}_{\psi_0, \kappa, \delta\psi_T i} \mathcal{G}$$



By construction:  $\mathcal{G}$  is  $\kappa$ -regularized MDS solution  $\iff b\mathcal{G} = 1$  and  $\hat{\mathbf{M}}_{\psi_0, \kappa}^2 \mathcal{G} = 0$ .

Theorem [A.L.]:

(i) the iteration

$$\mathcal{G}_0 := \mathbb{1} \text{ and } \mathcal{G}_{l+1} := \mathcal{G}_l - \frac{1}{T} \hat{\mathbf{M}}_{\psi_0, \kappa}^2 \mathcal{G}_l \quad (l = 0, 1, 2, \dots)$$

is always convergent if  $T > \text{trace norm of } \hat{\mathbf{M}}_{\psi_0, \kappa}^2$ .

(ii) the  $\kappa$ -regularized MDS solution space is nonempty iff

$$\lim_{l \rightarrow \infty} b\mathcal{G}_l \neq 0.$$

(iii) and in this case

$$\lim_{l \rightarrow \infty} \mathcal{G}_l$$

is an MDS solution, up to normalization factor.

Use for lattice-like numerical method in Lorentz signature?

(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

# Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space  $X$  are elementary events.
  - Collection of Borel sets  $\Sigma$  of  $X$  are composite events.
  - A state is a probability measure  $W$  on  $\Sigma$ , i.e.  $(X, \Sigma, W)$  is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space  $\mathcal{H}$  are elementary events,  $X$ .
  - Collection of all closed subspaces  $\Sigma$  of  $\mathcal{H}$  are composite events.
  - A state is a probability measure  $W$  on  $\Sigma$ , i.e.  $(X, \Sigma, W)$  is quantum probability space.

# Fréchet derivative in top.vector spaces

Let  $F$  and  $G$  real top.affine space, Hausdorff.

Subordinate vector spaces:  $\mathbb{F}$  and  $\mathbb{G}$ .

A map  $S : F \rightarrow G$  is **Fréchet-Hadamard differentiable** at  $\psi \in F$  iff:

there exists  $DS(\psi) : \mathbb{F} \rightarrow \mathbb{G}$  continuous linear, such that for all sequence  $n \mapsto h_n$  in  $\mathbb{F}$ , and nonzero sequence  $n \mapsto t_n$  in  $\mathbb{R}$  which converges to zero,

$$(\mathbb{G}) \lim_{n \rightarrow \infty} \left( \frac{S(\psi + t_n h_n) - S(\psi)}{t_n} - DS(\psi) h_n \right) = 0$$

holds.

# Fréchet derivative of action functional

Fréchet derivative of  $S : \mathcal{E} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R})$  is

$$DS : \mathcal{E} \times \mathcal{E} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), (\psi, \delta\psi) \longmapsto \left( \mathcal{K} \mapsto (DS_{\mathcal{K}}(\psi) | \delta\psi) \right)$$

For  $\underbrace{(v, \nabla)}_{=:\psi} \in \mathcal{E}$  given,

$$\underbrace{(\delta v, \delta C)}_{=:\delta\psi} \mapsto (DS_{\mathcal{K}}(v, \nabla) | (\delta v, \delta C)) =$$

$$\int_{\mathcal{K}} \left( D_1 L(v, \nabla v, P(\nabla)) \delta v + D_2^a L(v, \nabla v, P(\nabla)) (\nabla_a \delta v + \delta C_a v) + 2 D_3^{[ab]} L(v, \nabla v, P(\nabla)) \tilde{\nabla}_{[a} \delta C_{b]} \right)$$

$$= \int_{\mathcal{K}} \left( D_1 L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta v - (\tilde{\nabla}_a D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta v \right) +$$

$$\left( D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta C_a v - 2 (\tilde{\nabla}_a D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta C_b \right)$$

$$+ m \int_{\partial \mathcal{K}} \left( D_2^a L(v, \nabla v, P(\nabla))_{[ac_1 \dots c_{m-1}]} \delta v + 2 D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[ac_1 \dots c_{m-1}]} \delta C_b \right)$$

$$(m := \dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

# Distributions on manifolds

$W(\mathcal{M})$  vector bundle,  $W^\times(\mathcal{M}) := W^*(\mathcal{M}) \otimes \bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})$  its **densitized dual**.  
 $W^{\times \times}(\mathcal{M}) \equiv W(\mathcal{M})$ .

Correspondingly:  $\mathcal{E}^\times$  and  $\mathcal{D}^\times$  are densitized duals of  $\mathcal{E}$  and  $\mathcal{D}$ .

$\mathcal{E} \times \mathcal{D}^\times \rightarrow \mathbb{R}$ ,  $(\delta\psi, p_T) \mapsto \int_{\mathcal{M}} \delta\psi p_T$  and  $\mathcal{D} \times \mathcal{E}^\times \rightarrow \mathbb{R}$ ,  $(\delta\psi_T, p) \mapsto \int_{\mathcal{M}} \delta\psi_T p$  jointly sequentially continuous.

Therefore, continuous dense linear injections  $\mathcal{E} \rightarrow \mathcal{E}^{\times'}$  and  $\mathcal{D} \rightarrow \mathcal{D}^{\times'}$ .  
 (hence the name, **distributional sections**)

Let  $A : \mathcal{E} \rightarrow \mathcal{E}$  continuous linear.

It has **formal transpose** iff there exists  $A^t : \mathcal{D}^\times \rightarrow \mathcal{D}^\times$  continuous linear, such that

$$\forall \delta\psi \in \mathcal{E} \text{ and } p_T \in \mathcal{D}^\times : \int_{\mathcal{M}} (A \delta\psi) p_T = \int_{\mathcal{M}} \delta\psi (A^t p_T).$$

Topological transpose of formal transpose  $(A^t)' : (\mathcal{D}^\times)' \rightarrow (\mathcal{D}^\times)'$  is the **distributional extension** of  $A$ . Not always exists.

# Fundamental solution on manifolds

Let  $E : \mathcal{E} \times \mathcal{D} \rightarrow \mathbb{R}$  be Euler-Lagrange functional, and  $J \in \mathcal{D}'$ .

$\mathbb{K}_{(J)} \in \mathcal{E}$  is **solution with source  $J$** , iff  $\forall \delta\psi_T \in \mathcal{D} : (E(\mathbb{K}_{(J)}) | \delta\psi_T) = (J | \delta\psi_T)$ .

Specially: one can restrict to  $J \in \mathcal{D}^\times \subset \mathcal{E}^\times \subset \mathcal{D}'$ .

A continuous map  $\mathbb{K} : \mathcal{D}^\times \rightarrow \mathcal{E}$  is **fundamental solution**, iff for all  $J \in \mathcal{D}^\times$  the field  $\mathbb{K}(J) \in \mathcal{E}$  is solution with source  $J$ .

May not exist, and if does, may not be unique.

If  $\mathbb{K}_{\psi_0} : \mathcal{D}^\times \rightarrow \mathcal{E}$  vectorized fundamental solution is linear (e.g. for linear  $E_{\psi_0} : \mathcal{E} \rightarrow \mathcal{D}'$ ):  
 $\mathbb{K}_{\psi_0} \in \mathcal{Lin}(\mathcal{D}^\times, \mathcal{E}) \subset (\mathcal{D}^\times)' \otimes (\mathcal{D}^\times)'$  is distribution.

# Particular solutions to the free MDS equation

Distributional solutions to free MDS equation:  $G_{\psi_0} = \exp(K_{\psi_0})$  where

$$\begin{aligned}K_{\psi_0}^{(0)} &= 0, \\K_{\psi_0}^{(1)} &= 0, \\K_{\psi_0}^{(2)} &= i \hbar K_{\psi_0}^{(2)} \\K_{\psi_0}^{(n)} &= 0 \quad (n \geq 2)\end{aligned}$$

Smooth function solutions to free regularized MDS equation:  $G_{\psi_0} = \exp(K_{\psi_0, \kappa})$  where

$$\begin{aligned}K_{\psi_0, \kappa}^{(0)} &= 0, \\K_{\psi_0, \kappa}^{(1)} &= 0, \\K_{\psi_0, \kappa}^{(2)} &= i \hbar (C_\kappa \otimes C_\kappa) K_{\psi_0}^{(2)} \\K_{\psi_0, \kappa}^{(n)} &= 0 \quad (n \geq 2)\end{aligned}$$

[Here  $C_\kappa(\cdot) := \eta \star (\cdot)$  is convolution by a test function  $\eta$ .]

# Renormalization from functional analysis p.o.v.

Let  $\mathbb{F}$  and  $\mathbb{G}$  real or complex top.vector space, Hausdorff loc.conv complete.

Let  $M : \mathbb{F} \rightarrow \mathbb{G}$  densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable  $\Leftrightarrow$  where extendable with limits, it is unique.

Multivalued set:

$\text{Mul}(M) := \{y \in \mathbb{G} \mid \exists (x_n)_{n \in \mathbb{N}} \text{ in } \text{Dom}(M) \text{ such that } \lim_{n \rightarrow \infty} x_n = 0 \text{ and } \lim_{n \rightarrow \infty} Mx_n = y\}$ .

$\text{Mul}(M)$  always closed subspace.

Closable  $\Leftrightarrow \text{Mul}(M) = \{0\}$ .

Maximally non-closable  $\Leftrightarrow \text{Mul}(M) = \overline{\text{Ran}(M)}$ . Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!



MDS operator:

$$\mathbf{M} : \mathcal{D} \otimes \mathcal{T}(\mathcal{E}) \rightarrow \mathcal{T}(\mathcal{E}), \quad G \mapsto \mathbf{M}G$$

linear, everywhere defined continuous. So,

$$\mathbf{M} : \mathcal{T}(\mathcal{D}^{\times'}) \rightarrow \mathcal{D}' \otimes \mathcal{T}(\mathcal{D}^{\times'}), \quad G \mapsto \mathbf{M}G$$

linear, densely defined.

Similarly:  $\mathbf{M}_\kappa$  regularized MDS operator ( $\kappa$ : a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times'}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists \mathcal{G}_\kappa \rightarrow G \text{ approximator sequence, such that :}$$
$$\lim_{\kappa \rightarrow \delta} \mathbf{M} \mathcal{G}_\kappa = 0.$$

All  $G$  would be selected, because  $\text{Mul}()$  set of interaction term is full space.

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times'}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists \mathcal{G}_\kappa \rightarrow G \text{ approximator sequence, such that :}$$
$$\lim_{\kappa \rightarrow \delta} \mathbf{M}_\kappa \mathcal{G}_\kappa = 0.$$

All  $G$  would be selected, because  $\text{Mul}()$  set of interaction term is full space.

Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times'}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists \mathcal{G}_\kappa \rightarrow G \text{ approximator sequence, such that :}$$
$$\forall \kappa : \mathbf{M}_\kappa \mathcal{G}_\kappa = 0.$$

That is, as implicit function of  $\kappa$ , not as operator closure kernel.

Running coupling:

If in  $\mathbf{M}_\kappa$  EL terms are combined with  $\kappa$ -dependent weights  $\gamma(\kappa)$ .

(Not just with real factors.)

E.g.:

$$(\gamma, G) \in \mathcal{T}(\mathcal{D}^{\times'}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists \mathcal{G}_\kappa \rightarrow G \text{ approximator sequence, such that :}$$
$$\forall \kappa : \mathbf{M}_{\gamma(\kappa), \kappa} \mathcal{G}_\kappa = 0.$$

Feynman integral “ $\iff$ ” MDS equation.

Wilsonian regularized Feynman integral:

integrate not on  $\mathcal{E}$ , only on the image space  $C_\kappa[\mathcal{E}]$  of a smoothing operator  $C_\kappa : \mathcal{E} \rightarrow \mathcal{E}$ .

[Smoothing operator:  $\sim$  convolution, can be generalized to manifolds. Does UV damping.]

Automatically knows RGE relations.

Wilsonian regularized Feynman integral “ $\iff$ ” regularized MDS equation + RGE:

$$(\psi_0, \kappa \mapsto \gamma(\kappa), \kappa \mapsto \mathcal{G}_{\psi_0, \kappa}) = ? \text{ such that : } \underbrace{\mathcal{G}_{\psi_0, \kappa}^{(0)}}_{=: b \mathcal{G}_{\psi_0, \kappa}} = 1,$$

$$\forall \kappa : \forall \delta\psi_T \in \mathcal{D} : \underbrace{\left( \mathcal{L}_{\gamma(\kappa)}(\mathbf{E}_{\psi_0} | \delta\psi_T) - i\hbar L_{C_\kappa} \delta\psi_T \right)}_{=: \mathbf{M}_{\psi_0, \kappa, \delta\psi_T}} \mathcal{G}_{\psi_0, \kappa} = 0,$$

$$\text{RGE} \longrightarrow \forall \mu, \kappa : \mathcal{G}_{\psi_0, (C_\mu \kappa)}^{(n)} = (\otimes^n C_\mu) \mathcal{G}_{\psi_0, \kappa}^{(n)}.$$

Running coupling is meaningful. Conjecture: RG flow of  $\mathcal{G}_{\psi_0, \kappa} \leftrightarrow$  distributional  $G_{\psi_0}$ .

(Conjecture proved for flat spacetime for bosonic fields.)

# Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\hat{\otimes}_{\pi}^n \mathcal{E} \quad \equiv \quad \mathcal{E}_n \quad \equiv \quad (\hat{\otimes}_{\pi}^n \mathcal{E}')' \quad \equiv \quad \mathcal{L}in(\mathcal{E}', \hat{\otimes}_{\pi}^{n-1} \mathcal{E})$$

$$(\hat{\otimes}_{\pi}^n \mathcal{E})' \quad \equiv \quad \mathcal{E}'_n \quad \equiv \quad \hat{\otimes}_{\pi}^n \mathcal{E}' \quad \equiv \quad \mathcal{L}in(\mathcal{E}, \hat{\otimes}_{\pi}^{n-1} \mathcal{E}')$$

$$\hat{\otimes}_{\pi}^n \mathcal{D} \quad \leftarrow \quad \mathcal{D}_n \quad \equiv \quad (\hat{\otimes}_{\pi}^n \mathcal{D}')'$$

cont.bij.

$$(\hat{\otimes}_{\pi}^n \mathcal{D})' \quad \rightarrow \quad \mathcal{D}'_n \quad \equiv \quad \hat{\otimes}_{\pi}^n \mathcal{D}' \quad \equiv \quad \mathcal{L}in(\mathcal{D}, \hat{\otimes}_{\pi}^{n-1} \mathcal{D}')$$

$\mathcal{E} \times \mathcal{E} \rightarrow F$  separately continuous maps are jointly continuous.

$\mathcal{E}' \times \mathcal{E}' \rightarrow F$  separately continuous bilinear maps are jointly continuous.

For mixed, no guarantee.

For  $\mathcal{D}$  or  $\mathcal{D}'$  spaces, joint continuity from separate continuity of bilinear forms not automatic.

For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed  $\mathcal{E}, \mathcal{E}', \mathcal{D}, \mathcal{D}'$  multilinear forms (separate sequential continuity  $\Leftrightarrow$  joint sequential continuity).