Aliens in QCD: Towards precise parton evolution

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Theory seminar HUN-REN Wigner RCP

18 October 2024

hep-ph/2409.02870

Introduction

At HL-LHC: Statistical/systematic uncertainties $\sim 1\%$
$\Rightarrow$ Theory needs to keep up!

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	δ(scale)	δ(PDF-TH)
$gg \to$ Higgs	m_H	3.5%	$^{+0.21\%}_{-2.37\%}$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	$^{+3.0\%}_{-4.8\%}$	$\pm 2.5\%$
NCDY	30	-4.8%	$^{+1.53\%}_{-2.54\%}$	$\pm 2.8\%$
	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.2\%$
	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.2\%$
	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

Table: [Baglio et al., 2022]

$$\delta(\text{PDF-TH}) = \frac{1}{2} \frac{\left|\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})\right|}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}$$

PDF scale dependence

Scale evolution of PDFs is set by the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d} f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

with P_{ij} the QCD splitting functions. These are perturbative quantities and can be computed as the anomalous dimensions of the leading-twist operators that define the PDFs

$$\frac{\mathrm{d}[\mathcal{O}_i]}{\mathrm{d}\ln\mu^2} = \gamma^{ij}[\mathcal{O}_j], \quad \gamma^{ij} \equiv a_s\gamma^{ij,(0)} + a_s^2\gamma^{ij,(1)} + \dots$$

$$\gamma^{ij} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x)$$

We can distinguish 2 sets of leading-twist operators based on their representation in the QCD flavour group.

Leading-twist operators

- Flavour non-singlet quark operator

$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\lambda^{\alpha}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

- Flavour singlet quark operator + gluon operator

$$\mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\,F^{a_1}{}_{\mu\mu_1}\,D^{a_1a_2}_{\mu_2}\ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$

$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

with

$$D_\mu = \partial_\mu - ig_s T^a A^a_\mu$$

$$D^{ac}_\mu = \partial_\mu\delta^{ac} + g_s f^{abc}A^b_\mu$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu$$

Renormalization of gauge invariant operators

To extract the anomalous dimensions of interest, we now need to renormalize the operators. For this, one needs to take into account mixing of operators in the same representation. This implies that

- the non-singlet quark operators renormalize multiplicatively

$$\mathcal{O}_{q\,\mathrm{NS}}^{(N)} = Z_N [\mathcal{O}_{q\,\mathrm{NS}}^{(N)}]$$

- the singlet quark and gluon operators mix under renormalization

$$\begin{pmatrix} \mathcal{O}_{q\,\mathrm{S}}^{(N)} \\ \mathcal{O}_{g}^{(N)} \end{pmatrix} = \begin{pmatrix} Z_N^{qq} & Z_N^{qg} \\ Z_N^{gq} & Z_N^{gg} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{q\,\mathrm{S}}^{(N)}] \\ [\mathcal{O}_{g}^{(N)}] \end{pmatrix}$$

Note: Use the $\overline{\mathrm{MS}}$-scheme and $D = 4 - 2\varepsilon$ dimensional regularization.

Renormalization of gauge invariant operators

$$Z_N^{qq} = 1 + \frac{a_s}{\varepsilon}\gamma_N^{qq,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{1}{\varepsilon}\left[\gamma_N^{qq,(0)}(\gamma_N^{qq,(0)} - \beta_0) + \gamma_N^{qg,(0)}\gamma_N^{gq,(0)}\right] + \gamma_N^{qq,(1)}\right\}\ldots$$

$$Z_N^{qg} = \frac{a_s}{\varepsilon}\gamma_N^{qg,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{\gamma_N^{qg,(0)}}{\varepsilon}(\gamma_N^{qq,(0)} + \gamma_N^{gg,(0)} - 2\beta_0) + \gamma_N^{qg,(1)}\right\} + \ldots$$

$$Z_N^{gq} = \frac{a_s}{\varepsilon}\gamma_N^{gq,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{\gamma_N^{gq,(0)}}{\varepsilon}(\gamma_N^{qq,(0)} + \gamma_N^{gg,(0)} - 2\beta_0) + \gamma_N^{gq,(1)}\right\} + \ldots$$

$$Z_N^{gg} = 1 + \frac{a_s}{\varepsilon}\gamma_N^{gg,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{1}{\varepsilon}\left[\gamma_N^{gg,(0)}(\gamma_N^{gg,(0)} - \beta_0) + \gamma_N^{gq,(0)}\gamma_N^{qg,(0)}\right] + \gamma_N^{gg,(1)}\right\}\ldots$$

Renormalization of gauge invariant operators

Unfortunately, the mixing pattern of the operators is even more complicated as alluded to above when computing off-shell matrix elements. In particular, one needs to take into account mixing with non-gauge-invariant (👽) operators.

Aliens through history

- The appearance of alien[1] operators in the renormalization of the physical ones has been known since the early seventies [Gross and Wilczek, 1974]. They obtained the physical anomalous dimensions without accounting for aliens by using lightcone gauge.
- The origin of the issue was provided by Dixon and Taylor [Dixon and Taylor, 1974]. In particular, they showed that the bare Yang-Mills Lagrangian is invariant under a different set of gauge transformations as the renormalized one.
  $\rightarrow$ Construction of the aliens relevant for the computation of the 1-loop anomalous dimensions
- 2 years later, Joglekar and Lee worked out the general theory of the renormalization of gauge invariant operators. Their main results are summarized in 3 theorems [Joglekar and Lee, 1976]

[1] Term coined in '94 by [Collins and Scalise, 1994].

Side-step: Joglekar-Lee theorems

1. The basis of alien operators A_i that mix with the gauge invariant ones can be chosen such that they are BRST exact

$$A_i \sim \delta_{\mathrm{BRST}} B_i .$$

   Here B_i is called the ancestor of A_i.

2. Physical matrix elements of the aliens vanish.
3. The mixing matrix is triangular

$$\begin{pmatrix} [\mathcal{O}_G] \\ [\mathcal{O}_A] \\ [\mathcal{O}_E] \end{pmatrix} = \begin{pmatrix} Z_{GG} & Z_{GA} & Z_{GE} \\ 0 & Z_{AA} & Z_{AE} \\ 0 & 0 & Z_{EE} \end{pmatrix} \begin{pmatrix} \mathcal{O}_G \\ \mathcal{O}_A \\ \mathcal{O}_E \end{pmatrix} .$$

Aliens through history

- The 2-loop anomalous dimensions were computed a few years later using different gauges: [Floratos et al., 1979, Gonzalez-Arroyo and Lopez, 1980, Floratos et al., 1981] used the covariant gauge while [Furmanski and Petronzio, 1980] used the axial gauge
- The computations using covariant gauge agreed with one another but disagreed with the axial gauge one
- The issue was solved a decade later by Hamberg and van Neerven in favour of the axial gauge result [Hamberg and van Neerven, 1992]
- Unfortunately, the way forward was not clear; the generalization of the basis of aliens to higher orders in perturbation theory was unknown.
- Nevertheless, the 3-loop anomalous dimensions were computed using different methods [Vogt et al., 2004]

Aliens through history

- Finally, after 30 years, significant progress was made in the alien issue by 2 independent groups
- G. Falcioni and F. Herzog were able to derive constraints to consistently derive the aliens at fixed orders which were solved for fixed $N \leq 20$ [Falcioni and Herzog, 2022]

  $\rightarrow$ All 4-loop splitting functions now known to $N = 20$ [Falcioni et al., 2023b, Falcioni et al., 2023a, Gehrmann et al., 2024a, Falcioni et al., 2024d, Falcioni et al., 2024b, Falcioni et al., 2024a]
- On the other hand, [Gehrmann et al., 2023] developed a method to derive the counterterm Feynman rules for the aliens

  $\rightarrow$ n_f^2 contributions to the pure-singlet splitting functions at 4 loops [Gehrmann et al., 2024a]

Focus on method by Giulio and Franz in what's next

Construction of the alien operators

The complete gauge-fixed QCD action is written as

$$S = \int \mathrm{d}^D x \, (\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF+G}}) \, .$$

Here $\mathcal{L}_0$ represents the classical part of the QCD Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_a^{\mu\nu} \, F^a_{\mu\nu} + \sum_{f=1}^{n_f} \overline{\psi}^f (i\not{D} - m_f) \psi^f \, ,$$

with

$$\mathcal{L}_{\mathsf{GF+G}} = -\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 - \overline{c}^a \, \partial^\mu D^{ab}_\mu \, c^b$$

and

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \\ D_\mu &= \partial_\mu - i g_s T^a A^a_\mu \\ D^{ac}_\mu &= \partial_\mu \delta^{ac} + g_s f^{abc} A^b_\mu \end{aligned}$$

f^{abc} are the standard QCD structure constants.

Construction of the alien operators

The QCD Lagrangian can be extended to also include the leading-twist spin-N gauge-invariant operators, which we define as

$$\mathcal{O}_{\mathrm{g}}^{(N)}(x) = \frac{1}{2} F_\nu(x)\, D^{N-2} F^\nu(x)\,,$$
$$\mathcal{O}_{\mathrm{q}}^{(N)}(x) = \overline{\psi}(x) \not{\Delta}\, D^{N-1} \psi(x)\,.$$

Here Δ_μ is a lightlike vector and we introduced the notation

$$F^{\mu;a} = \Delta_\nu\, F^{\mu\nu;a}, \qquad A^a = \Delta_\mu A^{\mu;a}, \qquad D = \Delta_\mu\, D^\mu, \qquad \partial = \Delta_\mu \partial^\mu\,.$$

These physical operators now mix under renormalization with aliens, which are (a) proportional to the field EOMs and (b) contain 👻. Schematically the complete Lagrangian is then

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\mathrm{i}}\, \mathcal{O}_{\mathrm{i}} + \mathcal{O}_{\mathsf{EOM}}^{(N)} + \mathcal{O}_c^{(N)}$$

Construction of the alien operators

The most general form of the EOM operator is [Falcioni and Herzog, 2022]

$$\mathcal{O}_{\text{EOM}}^{(N)} = \left(D \cdot F^a + g_s \overline{\psi} T^a \not{\Delta} \psi\right) \mathcal{G}^a(A^a, \partial A^a, \partial^2 A^a, ...)$$

with $\mathcal{G}^a$ a generic local function of the gauge field and its derivatives. Expanding $\mathcal{G}^a$ in a series of contributions with an increasing number of gauge fields then leads to

$$\mathcal{O}_{\text{EOM}}^{(N)} = \mathcal{O}_{\text{EOM}}^{(N),I} + \mathcal{O}_{\text{EOM}}^{(N),II} + \mathcal{O}_{\text{EOM}}^{(N),III} + \mathcal{O}_{\text{EOM}}^{(N),IV} + \ldots$$

Construction of the alien operators

$$
\begin{aligned}
\mathcal{O}_{\text{EOM}}^{(N),I} &= \eta(N)\,\left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\,\left(\partial^{N-2}A^a\right),\\
\mathcal{O}_{\text{EOM}}^{(N),II} &= g_s\left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\sum_{\substack{i+j\\=N-3}} C_{ij}^{abc}(\partial^i A^b)(\partial^j A^c),\\
\mathcal{O}_{\text{EOM}}^{(N),III} &= g_s^2\,\left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k\\=N-4}} C_{ijk}^{abcd}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),\\
\mathcal{O}_{\text{EOM}}^{(N),IV} &= g_s^3\,\left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k+l\\=N-5}} C_{ijkl}^{abcde}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e).
\end{aligned}
$$

Construction of the alien operators

The coefficients $C^{a_1 \dots a_n}_{i_1 \dots i_{n-1}}$ appearing can be written in terms of a set of independent colour tensors, each of them multiplying an associated coupling constant, as follows

$$C^{abc}_{ij} = f^{abc} \kappa_{ij},$$
$$C^{abcd}_{ijk} = (f\,f)^{abcd} \kappa^{(1)}_{ijk} + d_4^{abcd} \kappa^{(2)}_{ijk} + d^{abcd}_{\widehat{4ff}} \kappa^{(3)}_{ijk},$$
$$C^{abcde}_{ijkl} = (f\,f\,f)^{abcde} \kappa^{(1)}_{ijkl} + d_{4f}^{abcde} \kappa^{(2)}_{ijkl}$$

To avoid overcounting: κ-couplings inherit properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$

The standard gauge transformations leave $\mathcal{L}_0$ and $\mathcal{O}_i$ invariant, but not $\mathcal{O}^{(N)}_{\text{EOM}}$

$\Rightarrow$ generalized gauge transformation

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

Construction of the alien operators

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

$$\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b(x),$$

$$\delta^\Delta_\omega A^a_\mu = -\Delta_\mu \Bigg[\eta(N)\, \partial^{N-1}\omega^a + g_s \sum_{\substack{i+j \\ =N-3}} \widetilde{C}^{aa_1a_2}_{ij} \left(\partial^i A^{a_1}\right) \left(\partial^{j+1}\omega^{a_2}\right)$$

$$+ g_s^2 \sum_{\substack{i+j+k \\ =N-4}} \widetilde{C}^{aa_1a_2a_3}_{ijk} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^{k+1}\omega^{a_3}\right)$$

$$+ g_s^3 \sum_{\substack{i+j+k+l \\ =N-5}} \widetilde{C}^{aa_1a_2a_3a_4}_{ijkl} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^k A^{a_3}\right) \left(\partial^{l+1}\omega^{a_4}\right) + \mathcal{O}(g_s^4) \Bigg]$$

Construction of the alien operators

$$\widetilde{C}^{abc}_{ij} = f^{abc}\eta_{ij},$$
$$\widetilde{C}^{abcd}_{ijk} = (f\,f)^{abcd}\eta^{(1)}_{ijk} + d_4^{abcd}\eta^{(2)}_{ijk} + d_{\widehat{4ff}}^{abcd}\eta^{(3)}_{ijk},$$
$$\widetilde{C}^{abcde}_{ijkl} = (f\,f\,f)^{abcde}\eta^{(1)}_{ijkl} + d_{4f}^{abcde}\eta^{(2a)}_{ijkl} + d_{4f}^{aebcd}\eta^{(2b)}_{ijkl}.$$

The generalized gauge symmetry implies that the couplings $\eta^{(k)}_{n_1 \ldots n_j}$ are related to $\kappa^{(k)}_{n_1 \ldots n_j}$

Construction of the alien operators

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i},$$

$$\eta^{(1)}_{ijk} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2[\kappa^{(1)}_{ijk} + \kappa^{(1)}_{kji}],$$

$$\eta^{(2)}_{ijk} = 3\kappa^{(2)}_{ijk},$$

$$\eta^{(3)}_{ijk} = 2[\kappa^{(3)}_{ijk} - \kappa^{(3)}_{kji}],$$

$$\eta^{(1)}_{ijkl} = 2[\kappa^{(1)}_{ij(l+k+1)} + \kappa^{(1)}_{(l+k+1)ji}]\binom{l+k+1}{k} + 2[\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ilkj} + \kappa^{(1)}_{likj} + \kappa^{(1)}_{lkij}],$$

$$\eta^{(2a)}_{ijkl} = 3\kappa^{(2)}_{ij(k+l+1)}\binom{k+l+1}{k} + 2\kappa^{(2)}_{ijkl},$$

$$\eta^{(2b)}_{ijkl} = 2\kappa^{(2)}_{lijk}.$$

Construction of the alien operators

The generalized gauge transformation can now be promoted to a generalized BRST (gBRST) transformation

$$A^a_\mu \to A^a_\mu + \delta_c A^a_\mu + \delta^\Delta_c A^a_\mu$$

The ghost operator is now generated by the action of gBRST on a suitable ancestor operator [Falcioni and Herzog, 2022], giving

$$\mathcal{O}^{(N)}_c = \mathcal{O}^{(N),I}_c + \mathcal{O}^{(N),II}_c + \mathcal{O}^{(N),III}_c + \mathcal{O}^{(N),IV}_c + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}_c^{(N),I} = -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a),$$
$$\mathcal{O}_c^{(N),II} = -g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c),$$
$$\mathcal{O}_c^{(N),III} = -g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u),$$
$$\mathcal{O}_c^{(N),IV} = -g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e).$$

Renormalization

The complete Lagrangian is now

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\mathrm{i}}\,\mathcal{O}_{\mathrm{i}} + \mathcal{O}^{(N)}_{\mathsf{EOM}} + \mathcal{O}^{(N)}_c$$
$$= \mathcal{L}_0(A^a_\mu, g_s) + \mathcal{L}_{\mathsf{GF+G}}(A^a_\mu, c^a, \bar{c}^a, g_s, \xi) + \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}\,\mathcal{O}_{\mathrm{k}},$$

where $\mathcal{C}_{\mathrm{k}}$ labels all the distinct couplings of the operators, $\mathcal{C}_{\mathrm{k}} = \{w_{\mathrm{i}}, \eta(N), \kappa^{(i)}_{n_1 \dots n_j}, \eta^{(k)}_{n_1 \dots n_j}\}$. The UV singularities associated with the QCD Lagrangian are absorbed by introducing the bare fields/parameters

$$A^{a;\mathsf{bare}}_\mu(x) = \sqrt{Z_3} A^a_\mu(x)$$
$$c^{a;\mathsf{bare}}(x) = \sqrt{Z_c} c^a(x)$$
$$\bar{c}^{a;\mathsf{bare}}(x) = \sqrt{Z_c} \bar{c}^a(x)$$
$$g_s^{\mathsf{bare}} = \mu^\epsilon Z_g g_s$$
$$\xi^{\mathsf{bare}} = \sqrt{Z_3} \xi$$

Renormalization

This is not enough to make the OMEs finite. Instead they need an additional renormalization

$$\mathcal{O}_{\rm i}^{\text{ren}}(x) = Z_{\rm ij}\,\mathcal{O}_{\rm j}^{\text{bare}}(x),$$

The renormalized Lagrangian becomes

$$\begin{aligned}\widetilde{\mathcal{L}} &= \mathcal{L}_0(A_\mu^{a;\text{bare}}, g_s^{\text{bare}}) + \mathcal{L}_{\text{GF+G}}(A_\mu^{a;\text{bare}}, c^{a;\text{bare}}, \bar{c}^{a;\text{bare}}, g_s^{\text{bare}}, \xi^{\text{bare}}) \\ &\quad + \sum_{\rm k} \mathcal{C}_{\rm k}^{\text{bare}}\,\mathcal{O}_{\rm k}^{\text{bare}}, \\ \mathcal{C}_{\rm i}^{\text{bare}} &= \sum_{\rm k} \mathcal{C}_{\rm k}\,Z_{\rm k\,i},\end{aligned}$$

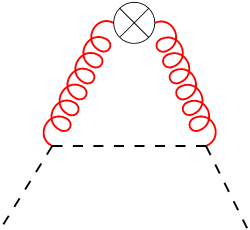
where $\mathcal{C}_{\rm k}$ is the (finite) renormalized coupling of the operator $\mathcal{O}_{\rm k}$. The UV-finite OMEs featuring a single insertion of $\mathcal{O}_{\rm g/q}^{\text{ren}}$ are computed by setting the renormalized couplings $\mathcal{C}_{\rm i} = \delta_{\rm i\,g/q}$, which gives

$$\mathcal{C}_{\rm i}^{\text{bare}} = Z_{\rm g/q\,i}.$$

Renormalization

⇒ The couplings of the bare operators $\eta^{\text{bare}}(N)$, ... are interpreted as the renormalization constants that mix the physical operators into the aliens

→ Extracted from the direct calculation of the singularities of the OMEs, e.g.



$$\eta^{\text{bare}}(N) = Z_{\mathrm{g}\,c} = -\frac{a_s}{\epsilon}\frac{C_A}{N(N-1)} + O(a_s^2)$$

We note that this quantity is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

Analytic reconstructions based on fixed moments

In [Falcioni and Herzog, 2022, Falcioni et al., 2024b], this setup was used for fixed $N \leq 20$. The available moments can be used to try and constrain an Ansatz for $\gamma_{ij}^{(3)}$

- Denominators of the form $\frac{1}{N+\alpha}$ $\quad (\alpha > 0)$
- Harmonic sums $S_{\pm m_1, m_2, \ldots, m_d}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m_1} \, S_{m_2, \ldots, m_d}(i)$

Ansatz large: $2 \cdot 3^6 = 1458$ weight-7 S-sums

$\Rightarrow$ Difficult for generic colour structures, simplifications for terms $\sim n_f^a$ or containing ζ_k

More details can be found in HP2 talk by **G. Falcioni**
`https://agenda.infn.it/event/35067/contributions/233420/`

Can the method of Falcioni and Herzog be generalized to arbitrary N?

Identities between alien couplings

- The κ couplings in the EOM operators are chosen to inherit the properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$
- Because of gBRST, the η couplings are connected to the κ ones.
- An equivalent approach to generate the ghost operators would be to start from anti-gBRST, for which $\omega^a(x)$ in the generalized gauge transformation should be replaced by the anti-ghost field $\overline{c}^a(x)$

$$A^a_\mu \to A^a_\mu + \delta_{\overline{c}} A^a_\mu + \delta^{\Delta}_{\overline{c}} A^a_\mu$$

  $\to$ the functional form of the resulting operators is different from those derived from gBRST
  $\Rightarrow$ non-trivial identities for the η-couplings!
- These identities allow one to restrict the function space of the couplings and hence constrain their generic N-dependence.
- During this talk: Focus on couplings coming with a string of f's

Class II couplings

$$\mathcal{O}_{\mathrm{EOM}}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) f^{abc} \sum_{\substack{i+j \\ =N-3}} \kappa_{ij} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_{c}^{(N),II} = -g_s \, f^{abc} \sum_{\substack{i+j \\ =N-3}} \eta_{ij} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Class II couplings

Note that the anti-gBRST relation is an example of a conjugation relation, in the sense that a second application of the sum leads to

$$\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\eta_{(i-t)(j+t)} = -\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s+j+t}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}$$

and hence

$$\eta_{ij} = \sum_{t=0}^{i}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}.$$

- Already encountered in the computation of the anomalous dimensions of leading-twist operators in non-forward kinematics, see e.g. [Moch and Van Thurenhout, 2021, Van Thurenhout, 2024]
- Great predictive power!
- Valuable information about the function space

Solving conjugation relations

- To take full advantage of the anti-gBRST conjugation relations, one needs to be able to evaluate them analytically
- Use principles of symbolic summation: Telescoping and Gosper's algorithm!

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned}\sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a).\end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that

$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$

and

$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$

2. Solve the so-called Gosper equation,

$$f(x) = g(x)y(x+1) - h(x)y(x),$$

for the polynomial $y(x)$.

3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by

$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Classical telescoping and Gosper's algorithm

Example: $S(N,m) = \sum_{k=0}^{N} (-1)^k \binom{m}{k}$

$$\frac{a_{k+1}}{a_k} = \frac{k-m}{k+1} \Rightarrow q(x) = \frac{x-m}{x+1}$$

1. Choose $f(x) = 1$, $g(x) = x - m$ and $h(x) = x$

$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$

2. $$1 = (x-m)y(x+1) - x\,y(x)$$

   Solve with Ansatz: $y(x) = a + b\,x \Rightarrow y(x) = -1/m$

3. $$t(x) = \frac{h(x)}{f(x)} y(x) = -\frac{x}{m} \Rightarrow b(N) = -\frac{N}{m}(-1)^N \binom{m}{N}$$

$$\Rightarrow S(N,m) = b(N+1) - b(0) = (-1)^N \binom{m-1}{N}$$

Creative telescoping

Generalization to definite summation problems: creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{N} f(N,k) \equiv S(N)$$

with $f(N,k)$ hypergeometric in both N and k. The way to go about this is by attempting to find d functions $c_0(N),\ldots,c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N)f(N,k) + \ldots + c_d(N)f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,N+1) - g(N,a) = c_0(N)\sum_{k=a}^{N} f(N,k) + \ldots + c_d(N)\sum_{k=a}^{N} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N+d).$$

The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N+1)S_1(N) - (2N+3)S_1(N+1) + (N+2)S_1(N+2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovšek et al., 1996].

Solving conjugation relations

→ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

- `Sigma` generates and solves recurrence for given summation problem
- Solution consists of solution set for homogeneous recurrence + particular solution
- For final closed expression of summation: Determine linear combination of solutions that has same initial values as the given sum

→ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Class II couplings

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i}(-1)^{s+j}\binom{s+j}{j}\eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Combining anti-symmetry with gBRST we have

$$\eta_{ij} + \eta_{ji} = \eta(N)\left[\binom{i+j+1}{i} + \binom{i+j+1}{j}\right]$$

which gives an idea about the function space of η_{ij}.

Class II couplings

Using the RHS of the previous equation as an Ansatz for η_{ij} gives

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = c_1\, \eta(N) \left[(-1)^j + \binom{i+j+1}{i} \right]$$

for even values of N. Hence, we find a consistent solution if $c_1 = 0$ while c_2 remains unconstrained. Assuming that κ_{ij} lives in the same function space as η_{ij}, the full set of relations fixes both couplings uniquely

$$\eta_{ij} = \eta(N) \binom{N-2}{j},$$
$$\kappa_{ij} = \frac{\eta(N)}{2} \left[\binom{N-2}{j} - \binom{N-2}{i} \right]$$

Check: Compare with some fixed-N computations

$\rightarrow$ Correct for $N = 4$ 😸

$\rightarrow$ Incorrect for $N > 4$ 😾

Class II couplings

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = c_1\, \eta(N) \left[(-1)^j + \binom{i+j+1}{i} \right]$$

The RHS however suggests the inclusion of a new structure: $(-1)^j$. With

$$\eta_{ij} = \eta(N) \left[c_1 (-1)^j + c_2 \binom{i+j+1}{i} + c_3 \binom{i+j+1}{j} \right]$$

we find

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = (c_1 + c_2) \eta(N) \left[\binom{i+j+1}{i} + (-1)^j \right]$$

and hence $c_1 = -c_2$.

Class II couplings

Assuming that κ_{ij} lives in the same function space as η_{ij}, the full set of relations fixes both couplings up to 1 free parameter

$$\eta_{ij} = \eta(N)\left\{(1+2c)\left[\binom{i+j+1}{i} - (-1)^j\right] - 2c\binom{i+j+1}{j}\right\}$$
$$\kappa_{ij} = \eta(N)\left\{c\left[\binom{i+j+1}{i} - \binom{i+j+1}{j}\right] - \frac{1}{2}(1+2c)(-1)^j\right\}$$

The unknown c can be determined by the computation of 1 fixed-N matrix element computation. E.g. for $N=6$ we have $\kappa_{30} = 1/24$ which sets $c = -3/8$

$$\eta_{ij} = -\frac{\eta(N)}{4}\left[(-1)^j - 3\binom{N-2}{i+1} - \binom{N-2}{i}\right]$$
$$\kappa_{ij} = -\frac{\eta(N)}{8}\left[(-1)^j + 3\binom{i+j+1}{i} - 3\binom{i+j+1}{i+1}\right]$$

The solution above exactly agrees with the known solution

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992].

Word of caution: Kernel functions

If an Ansatz is generated using (anti-)gBRST relations, one is in principle free to add non-zero functions that live in the kernel of these relations. For example, if one adds a term of the form

$$-\frac{f(N)}{4}\left((-1)^j + \binom{N-2}{i+1} - \binom{N-2}{i}\right)$$

the corresponding expression for η_{ij} still obeys the constraints. In particular, substituting in the constraint coming from anti-symmetry and gBRST one finds

$$\left[(-1)^i + (-1)^j\right] f(N) = 0.$$

The left-hand side of this expression always vanishes for all physical (even) values of N, independent of the functional form of $f(N)$. In general, the exclusion of this type of function can only be confirmed by comparison with fixed-N computations.

Class III couplings

$$\mathcal{O}_{\text{EOM}}^{(N),III} = g_s^2 \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) (f\,f)^{abcd} \sum_{\substack{i+j+k\\=N-4}} \kappa_{ijk}^{(1)} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}_{c}^{(N),III} = -g_s^2\,(f\,f)^{abcd} \sum_{\substack{i+j+k\\=N-4}} \eta_{ijk}^{(1)} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^{k+1} c^d)$$

$$\kappa_{ijk}^{(1)} + \kappa_{ikj}^{(1)} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\kappa_{ijk}^{(1)} + \kappa_{jki}^{(1)} + \kappa_{kij}^{(1)} = 0, \qquad \text{[Jacobi identity]}$$

$$\eta_{ijk}^{(1)} = 2\kappa_{i(j+k+1)} \binom{j+k+1}{j} + 2[\kappa_{ijk}^{(1)} + \kappa_{kji}^{(1)}], \qquad \text{[gBRST]}$$

$$\eta_{ijk}^{(1)} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!\,n!\,k!} (-1)^{m+n+k} \eta_{(j-n)(i-m)(k+m+n)}^{(1)}. \qquad \text{[anti-gBRST]}$$

Class III couplings

The combination of the Jacobi identity with gBRST leads to

$$\eta^{(1)}_{ijk} + \eta^{(1)}_{kij} + \eta^{(1)}_{jki} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2\kappa_{k(i+j+1)}\binom{i+j+1}{i} + 2\kappa_{j(i+k+1)}\binom{i+k+1}{k}.$$

→ relates the class III coupling $\eta^{(1)}_{ijk}$ to the class II coupling κ_{ij}, at one order lower in perturbation theory!

⇒ use it to determine the function space of the all-N expression of $\eta^{(1)}_{ijk}$

→ leads to 18-dimensional function space

$$\Bigg\{(-1)^{i+j}\binom{i+j+1}{i}, \binom{N-2}{k+1}\binom{i+j+1}{i}, \binom{N-2}{k}\binom{i+j+1}{i}, (-1)^{j+k}\binom{j+k+1}{j},$$
$$\binom{N-2}{i+1}\binom{j+k+1}{j}, \binom{N-2}{i}\binom{j+k+1}{j}, (-1)^{i+k}\binom{i+k+1}{k}, \binom{N-2}{j+1}\binom{i+k+1}{k},$$
$$\binom{N-2}{j}\binom{i+k+1}{k} + \text{ independent permutations of } i, j \text{ and } k\Bigg\}.$$

Class III couplings

We assume $\kappa^{(1)}_{ijk}$ to live in the same function space. Hence in total we have 36 free parameters. Using the relations described above we are able to fix 34 of these. The final 2 free parameters are then fixed using $\kappa^{(1)}_{110} = 0$ and $\kappa^{(1)}_{121} = 13/336$, which follow from the explicit operator renormalization for $N = 6$ and $N = 8$ respectively. Our final result for $\kappa^{(1)}_{ijk}$ then becomes [new!]

$$\kappa^{(1)}_{ijk} = \frac{\eta(N)}{48}\Bigg\{2(-1)^{i+j}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 3(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\Bigg[2(-1)^{i+k+1} + 5\binom{N-1}{j+1}\Bigg] + \binom{j+k+1}{k}\Bigg[3(-1)^{j+k} - 10\binom{N-2}{i} + 4\binom{N-2}{i+1}\Bigg] + \binom{i+j+1}{j}\Bigg[(-1)^{i+j+1} + 5\binom{N-2}{k} - 9\binom{N-2}{k+1}\Bigg]\Bigg\}.$$

Class III couplings

We have checked that the above expression agrees with explicitly computed values, following from the renormalization of the operators, up to $N = 20$. Substituting this expression into the gBRST relation allows one to also reconstruct the full N-dependence of $\eta^{(1)}_{ijk}$ [new!]

$$\eta^{(1)}_{ijk} = -\frac{\eta(N)}{24}\Bigg\{5(-1)^{i+j+1}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 2(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\left[(-1)^{i+k} + 4\binom{N-2}{j+1}\right] + \binom{j+k+1}{k}\left[5(-1)^{j+k+1} - 3\binom{N-2}{i} + \binom{N-2}{i+1}\right] + \binom{i+j+1}{j}\left[4(-1)^{i+j} - 15\binom{N-2}{k} - 5\binom{N-2}{k+1}\right]\Bigg\}.$$

Class IV couplings

$$\mathcal{O}^{(N),IV}_{\text{EOM}} = g_s^3 \, \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l \\ =N-5}} \kappa^{(1)}_{ijkl} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e),$$

$$\mathcal{O}^{(N),IV}_{c} = -g_s^3 \, (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l \\ =N-5}} \eta^{(1)}_{ijkl} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1} c^e)$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ijlk} = 0, \qquad \text{[anti-symmetry]}$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{iklj} + \kappa^{(1)}_{iljk} = 0, \qquad \text{[Jacobi]}$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{jilk} + \kappa^{(1)}_{lkji} + \kappa^{(1)}_{klij} = 0, \qquad \text{[double Jacobi]}$$

$$\eta^{(1)}_{ijkl} = 2[\kappa^{(1)}_{ij(l+k+1)} + \kappa^{(1)}_{(l+k+1)ji}] \binom{l+k+1}{k} + 2[\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ilkj} + \kappa^{(1)}_{likj} + \kappa^{(1)}_{lkij}], \qquad \text{[gBRST]}$$

$$\eta^{(1)}_{ijkl} = -\sum_{s_1=0}^{i} \sum_{s_2=0}^{j} \sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\, s_2!\, s_3!\, l!} (-1)^{s_1+s_2+s_3+l} \eta^{(1)}_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)} \qquad \text{[anti-gBRST]}$$

Class IV couplings

Combining the double Jacobi identity with the gBRST one allows one to write $\eta^{(1)}_{ijkl}$ in terms of $\kappa^{(1)}_{ijk}$ appearing already in the class III operators at one order lower in perturbation theory!

$$\eta^{(1)}_{ijkl} + \eta^{(1)}_{jilk} + \eta^{(1)}_{lkji} + \eta^{(1)}_{klij} = 2[\kappa^{(1)}_{ij(k+l+1)} + \kappa^{(1)}_{(k+l+1)ji}]\binom{k+l+1}{k} + 2[\kappa^{(1)}_{ji(k+l+1)} + \kappa^{(1)}_{(k+l+1)ij}]\binom{k+l+1}{l}$$
$$+ 2[\kappa^{(1)}_{lk(i+j+1)} + \kappa^{(1)}_{(i+j+1)kl}]\binom{i+j+1}{j} + 2[\kappa^{(1)}_{kl(i+j+1)} + \kappa^{(1)}_{(i+j+1)lk}]\binom{i+j+1}{i}.$$

Again this tells us something about the function space for $\eta^{(1)}_{ijkl}$. Taking into account all the independent permutations of the indices i, k, j and l this space is now 264-dimensional. Assuming that the functional form of $\kappa^{(1)}_{ijkl}$ is similar to the one of $\eta^{(1)}_{ijkl}$ then implies that in total we now have 528 parameters to fix. However, after implementing all of the above relations, only 8 remain in the end!

→ Explicit expressions in [Falcioni et al., 2024c]

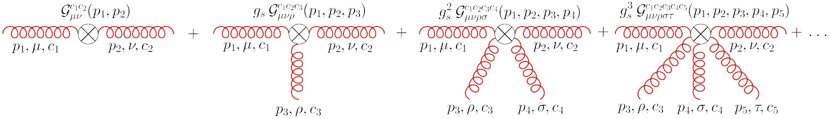
Application: Alien Feynman rules

With the couplings known, one can derive the Feynman rules of the alien operators

- The Feynman rules for the gauge-invariant quark and gluon operators, up to the four-loop level, can be found e.g. in [Falcioni and Herzog, 2022, Gehrmann et al., 2023, Floratos et al., 1977, Floratos et al., 1979, Mertig and van Neerven, 1996, Kumano and Miyama, 1997, Hayashigaki et al., 1997, Bierenbaum et al., 2009, Klein, 2009, Blümlein, 2001, Velizhanin, 2012, Velizhanin, 2020, Moch et al., 2017, Moch et al., 2022, Falcioni et al., 2023b, Falcioni et al., 2023a, Falcioni et al., 2024d, Moch et al., 2024, Gehrmann et al., 2024b, Kniehl and Velizhanin, 2023] and references therein. The generalization to arbitrary orders in perturbation theory can be found in [Somogyi and Van Thurenhout, 2024] [2]
- The alien rules were computed up to two loops in [Hamberg and van Neerven, 1992],[Matiounine et al., 1998],[Blümlein et al., 2022], and an extension to the three-loop level was recently presented in [Gehrmann et al., 2023]

[2] Note that the latter also presents the corresponding rules for the operators with total derivatives, relevant for non-zero momentum flow through the operator vertex.

Application: Alien Feynman rules



$$
\begin{aligned}
\mathcal{G}^{c_1c_2c_3c_4c_5}_{\mu\nu\rho\sigma\tau}(p_1,p_2,p_3,p_4,p_5) = {}& \frac{1+(-1)^N}{2}\, i^{N-1} f^{c_1c_2x} f^{xc_3y} f^{yc_4c_5} \Bigg\{ \\
& - g_{\mu\rho}\Delta_\nu\Delta_\sigma\Delta_\tau \sum_{i+j=N-3} \kappa_{ij}(\Delta\cdot p_4)^i(\Delta\cdot p_5)^j + \Delta_\rho\Delta_\sigma\Delta_\tau[(p_1+2p_2)_\mu\Delta_\nu \\
& - (\Delta\cdot p_2)g_{\mu\nu}] \sum_{i+j+k=N-4} \kappa^{(1)}_{ijk}(\Delta\cdot p_3)^i(\Delta\cdot p_4)^j(\Delta\cdot p_5)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(1)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Bigg\} \\
& + \frac{1+(-1)^N}{2}\, i^{N-1} d_{4f}^{c_1c_2c_3c_4c_5} \Bigg\{ \Delta_\mu\Delta_\nu\Delta_\rho[(p_4+2p_5)_\sigma\Delta_\tau \\
& - (\Delta\cdot p_5)g_{\sigma\tau}] \sum_{i+j+k=N-4} \kappa^{(2)}_{ijk}(\Delta\cdot p_1)^i(\Delta\cdot p_2)^j(\Delta\cdot p_3)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(2)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Bigg\} \\
& + \text{permutations}
\end{aligned}
$$

Application: Alien Feynman rules

- Ghost vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0- and 1-gluon vertices and $(f\,f)$, d_4 parts of the 2-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 2-gluon vertex new!
  (c) 3-gluon vertex new!
- Alien gluon vertices:
  (a) Agreement with [Blümlein et al., 2022, Gehrmann et al., 2023] for 2- and 3-gluon vertices; agreement with [Gehrmann et al., 2023] for $(f\,f)$, d_4 parts of the 4-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 4-gluon vertex new!
  (c) 5-gluon vertex new! [Recently also obtained in [Gehrmann et al., 2024c], comparison in progress]
- Alien quark vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0-, 1- and 2-gluon vertices
  (b) 3- and 4-gluon vertices new!

Summary and outlook

- Accuracy @ hadron colliders: Need higher-order corrections to PDF evolution (i.e. splitting functions = operator anomalous dimensions)
- Off-shell renormalization of OMEs ⇒ 
- One way to reconstruct the functional form of the alien operators is based on the use of generalized gauge symmetry, which is then promoted to a generalized (anti)-BRST symmetry
- One then finds classes of EOM and ghost operators, the couplings of which obey interesting consistency relations
- We used these relations to reconstruct the full N-dependence of the 1-loop alien couplings necessary to perform the operator renormalization to 4 loops
- This should be useful in the reconstruction of the full N-dependence of the 4-loop splitting functions!
- Next steps: Generalization to higher orders

End [3]

Thank you for your attention!

[3] Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Colour structures

f^{abc} are the QCD structure constants. The other colour structures are in turn defined as

$$
\begin{aligned}
(f\,f)^{abcd} &= f^{abe} f^{cde}, \\
(f\,f\,f)^{abcde} &= f^{abm} f^{mcn} f^{nde}, \\
d_4^{abcd} &= \frac{1}{4!}[\mathrm{Tr}(T_A^a T_A^b T_A^c T_A^d) + \text{symmetric permutations}], \\
d_{4ff}^{abcd} &= d_4^{abmn} f^{mce} f^{edn}, \\
d_{\widehat{4ff}}^{abcd} &= d_{4ff}^{abcd} - \frac{1}{3} C_A d_4^{abcd}, \\
d_{4f}^{abcde} &= d_4^{abcm} f^{mde}.
\end{aligned}
$$

Jacobi identity: $(f\,f)^{abcd} + (f\,f)^{acdb} + (f\,f)^{adbc} = 0$
Double Jacobi identity:
$(f\,f\,f)^{abcde} + (f\,f\,f)^{acbed} + (f\,f\,f)^{adebc} + (f\,f\,f)^{aedcb} = 0$

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