Three-loop evolution kernel for transversity operator

Leonid Shumilov

based on [2407.12696]
in collaboration with Alexander Manashov and Sven-Olaf Moch

8 November 2024

Perturbative QCD

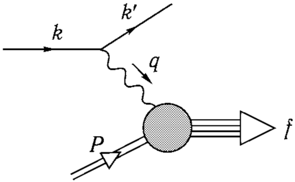

Figure: Deep Inelastic Scattering. (Pic. from the Peskin-Schröder book)

- $Q^2 = -q^2 \to \infty$
- $x = \dfrac{Q^2}{2P \cdot q}$ — fixed

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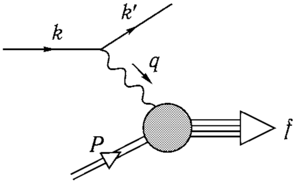

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Hadronic tensor [Björken'1966], [Feynman 1969]

$$W^{\mu\nu} = F_1(x,Q^2)\Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big) + \frac{1}{x}F_2(x,Q^2)\Big(p^\mu - \frac{p\cdot q}{q^2}q^\mu\Big)\Big(p^\nu - \frac{p\cdot q}{q^2}q^\nu\Big)$$

- $F_1(x,Q^2), F_2(x,Q^2)$ — scalar **structure functions**.

Factorization [Collins, Soper, Sterman' 1985]

$$W^{\mu\nu} = \sum_a \int_x^1 \frac{d\xi}{\xi} f_{a/P}(\xi,\mu) H_a^{\mu\nu}(q,\xi,\mu,\alpha_s(\mu)) + O\left(\frac{1}{Q^2}\right).$$

- $f_{a/P}(\xi,\mu)$ — non-perturbative **PDFs** (Parton distribution functions).
- $H_a^{\mu\nu}(q,\xi,\mu,\alpha_s(\mu))$ — perturbative hard scattering coefficient.

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DGLAP equation [Gribov, Lipatov'1972], [Altarelli, Parisi'1977], [Dokshitzer'1977]

$$\frac{df(x,\mu)}{d\ln\mu} = \int_x^1 \frac{dy}{y} P(y) f\Big(\frac{x}{y},\mu\Big).$$

$P(y) = \alpha_s P^{(0)}(y) + \alpha_s^2 P^{(1)}(y) + \alpha_s^3 P^{(2)} + \ldots$ — perturbative **splitting functions**.

- Full three-loop result. [Moch, Vermaseren, Vogt' 2004]
- Partial four-loop result. [Falconi, Herzog, Moch, Vermaseren, Vogt'2023-2024]

Close to the light-cone

- Choosing two light-like directions n^μ, $\bar{n}^\mu$ ($n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 1$):

$$a^\mu = a_- n^\mu + a_+ \bar{n}^\mu + a_\perp^\mu .$$

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Non-local operators [Balitsky, Braun'1988]

$$f_q(x,\mu) = \int \frac{dz}{2\pi} e^{-i\pi z P_+ x} \langle P| \underbrace{\bar{q}\left(\frac{z}{2}n\right) \not{n} q \left(-\frac{z}{2}n\right)}_{\mathcal{O}_q} |P\rangle .$$

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Scale evolution of the operator:

$$\frac{df(x,\mu)}{d\ln\mu} \quad \Leftrightarrow \quad \frac{d\mathcal{O}_q}{d\ln\mu} .$$

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- Distribution amplitudes: $\langle P|\mathcal{O}_q|0\rangle$ — **ERBL** evolution equation for the meson wave functions.
- Deeply virtual Compton scattering: $\langle P|\mathcal{O}_q|P'\rangle$ — evolution equation for Generalised parton distributions.

Problem formulation

Euclidean QCD in $d = 4 - 2\epsilon$ dimensions

$$S = \int d^d x \left\{ \bar{q} \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{1}{2\xi} (\partial A)^2 + \partial_\mu \bar{c}^a (D^\mu c)^a \right\}.$$

Light-ray operator

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n] \not{\Gamma} q(z_2 n).$$

Notations:

- $\bar{q}(x), q(x)$ are quark fields of the different flavour (**non-singlet case**).

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- $[z_1 n, z_2 n]$ is the Wilson line

$$[z_1 n, z_2 n] = \mathrm{Pexp} \left\{ i g z_{12} \int_0^1 d\alpha n^\mu A_\mu (z_{12}^\alpha) \right\}.$$

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- $z_{12}^\alpha = z_1 \bar{\alpha} + z_2 \alpha$, where $\bar{\alpha} = 1 - \alpha$, and $z_{12} = z_1 - z_2$.

Renormalization

Renormalized light ray operator

$$\Big[\mathcal{O}(z_1, z_2)\Big]_{\overline{MS}} = Z(a)\mathcal{O}(z_1, z_2),$$

where Z is an integral operator acting on the variables z_1, z_2.

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RG – equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathbb{H}(a)\Big)\Big[\mathcal{O}(z_1, z_2)\Big] = 0.$$

- Operator $\mathbb{H}(a)$ is the **evolution kernel**

$$\mathbb{H}(a)f(z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\alpha, \beta) f(z_{12}^{\alpha}, z_{21}^{\beta}).$$

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- Connection to the renormalization operator

$$\mathbb{H}(a) = -\mu\frac{dZ(a)}{d\mu}Z^{-1}(a) + 2\gamma_q(a).$$

Local operators

Expansion in terms of local operators

$$\mathcal{O}(z_1, z_2) = \sum_{N,k} \Psi_{N,k}(z_1, z_2)\mathcal{O}_{N,k}(0).$$

Twist-2 quark operators [Gross, Wilczek'1973]

$$\mathcal{O}_{N,k}(x) = (\partial_+)^k \underbrace{\left(\bar{q}(x)\not{v}\,(D_+)^{N-k}\,q(x)\right)}_{\text{twist-2 non-singlet quark operator}}.$$

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Operators of twist-2 $\mathcal{O}_N = \mathcal{O}_{N,k=0}$ mix under renormalization with total derivatives

$$\left[\mathcal{O}_N\right] = Z_N\mathcal{O}_N + \underbrace{\sum_{k=1}^{N} Z_{N,k}\,(\partial_+)^k\,\mathcal{O}_{N-k}}_{\text{non-forward}}.$$

Anomalous dimension matrix

RG – equation for local operators

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}\right)\left[\mathcal{O}_{N,k}\right]=-\sum_{k'=0}^{N}\gamma_{k,k'}\left[\mathcal{O}_{N,k'}\right].$$

- $\gamma_{k,k'}$ is the **anomalous dimension matrix**

$$\hat{\gamma}=\begin{pmatrix}\gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \cdots & \gamma_{0,N}\\ 0 & \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,N}\\ 0 & 0 & \gamma_{2,2} & \cdots & \gamma_{2,N}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \gamma_{N,N}\end{pmatrix}.$$

- $\gamma_{k,k}$ corresponds to the $\gamma(N)$ – **forward** anomalous dimensions for twist-2 operators.

Eigenvalues

The set of eigenfunctions of the evolution kernel $\psi_N(z_1, z_2) = z_{12}^{N-1}$:

$$\mathbb{H}(a)\psi_N(z_1, z_2) = \gamma(N)\psi_N(z_1, z_2).$$

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$$\gamma(N) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, h(\alpha, \beta)(1 - \alpha - \beta)^{N-1}.$$

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Another connection

$$\gamma(N) = -\int_0^1 dx\, x^{N-1} P(x),$$

where $P(x)$ is the splitting function (DGLAP kernel).

Results

- $\mathcal{O}^V(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\gamma_+ q(z_2 n)$ — three-loop evolution kernel [Braun, Manashov, Moch, Strohmaier'2017].
- $\mathcal{O}^A(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\gamma_+ \gamma_5 q(z_2 n)$ — three-loop evolution kernel [Braun, Manashov, Moch, Strohmaier'2021].

Transversity case

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\sigma_{\perp +} q(z_2 n)$$

- One-loop evolution kernel. [Belitsky, Müller'1998]
- Two-loop evolution kernel. [Belitsky, Müller'1999], [Belitsky, Müller, Freund' 1999]
- Two-loop ERBL kernels. [Mikhailov, Vladimirov'2009]
- All loops anomalous dimension matrix in the large n_f limit. [Van Thurenhout'2022]

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- Three-loop evolution kernel for the transversity operator. [Manashov, Moch, **LS**'2024]

Conformal group

Consider d-dimensional Poincare group with two additional transformation

$$x^\mu \mapsto \lambda x^\mu, \qquad x^\mu = \frac{x^\mu}{x^2}.$$

Generators

Action of conformal group is defined on the primary fields $\Phi(x)$:

$$\begin{aligned}
i[\mathbf{P}^\mu, \Phi(x)] &= \partial^\mu \Phi(x);\\
i[\mathbf{M}^{\mu\nu}, \Phi(x)] &= (x^\mu \partial^\nu - x^\nu \partial^\mu - \Sigma^{\mu\nu})\Phi(x);\\
i[\mathbf{D}, \Phi(x)] &= (x \cdot \partial + \Delta_\Phi)\Phi(x);\\
i[\mathbf{K}^\mu, \Phi(x)] &= (2x^\mu(x \cdot \partial) - x^2 \partial^\mu + 2\Delta_\Phi x^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x),
\end{aligned}$$

- Δ_Φ is a scaling dimension
$$\Phi(\lambda x) = \lambda^{-\Delta_\Phi}\Phi(x).$$
- $\Sigma^{\mu\nu}$ generator of the spin s.

Collinear subgroup

Fields are "living" on the light directions $\Phi(zn^\mu) \equiv \Phi(z)$.
[Braun, Korchemsky, Müller' 2003]

- $\mathbf{L}_+ = -in^\mu \mathbf{P}_\mu$.
- $\mathbf{L}_- = \frac{i}{2}\bar{n}^\mu \mathbf{K}_\mu$.
- $\mathbf{L}_0 = \frac{i}{2}\left(\mathbf{D} + n^\mu \bar{n}^\nu \mathbf{M}_{\mu\nu}\right)$.

sl(2) commutation relations

$$[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp, \quad [\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0.$$

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$SL(2,\mathbb{R})$ group of transformations:

$$\Phi(z) \mapsto (cz+d)^{-2j}\,\Phi\left(\frac{az+b}{cz+d}\right),$$

- $a, b, c, d \in \mathbb{R}$.
- $ad - bc = 1$.
- $j = (\Delta_\Phi + s)/2$ — conformal spin.

Symmetries of the kernel

Canonical $sl(2)$ generators acting on the light-ray operator $\mathcal{O}(z_1, z_2)$:

- $S_-^{(0)} = -\partial_{z_1} - \partial_{z_2}$,
- $S_0^{(0)} = z_1\partial_{z_1} + z_2\partial_{z_2} + 2$,
- $S_+^{(0)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2)$.

with commutation relation

$$\left[S_0^{(0)}, S_\pm^{(0)}\right] = \pm S_\pm^{(0)}, \qquad \left[S_+^{(0)}, S_-^{(0)}\right] = 2S_0^{(0)}.$$

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One-loop symmetry [Makeenko'1980]

$$\left[\mathbb{H}^{(1)}, S_{\pm,0}\right] = 0,$$

Perturbative expansion

$$\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + a^3\mathbb{H}^{(3)} + \dots .$$

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Higher loop orders ($l > 1$):

$$\left[S_-^{(0)}, \mathbb{H}^{(l)}\right] = 0, \qquad \left[S_{+,0}^{(0)}, \mathbb{H}^{(l)}\right] \neq 0.$$

Quantum corrections

In higher orders generators receive corrections

$$[S_0, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_0.$$

Corrected generators

$$S_-(a) = S_-^{(0)},$$

$$S_0(a) = S_0^{(0)} + \bar{\beta}(a) + \frac{1}{2}\mathbb{H}(a),$$

$$S_+(a) = S_+^{(0)} + \underbrace{(z_1 + z_2)\left(\bar{\beta}(a) + \frac{1}{2}\mathbb{H}(a)\right) + (z_1 - z_2)\Delta_+(a)}_{\Delta S_+}.$$

- $\bar{\beta}(a) = -\beta(a)/2a - \epsilon = \beta_0 a + \beta_1 a^2 + \ldots$.
- $\Delta_+(a)$ is a **conformal anomaly** and can be calculated only perturbatively. [Müller'1998/]

$$\Delta_+(a) = a\Delta_+^{(1)} + a^2\Delta_+^{(2)} + a^3\Delta_+^{(3)} \ldots.$$

Conformal constraints

Commutation relation can be seen as a constraint

$$[S_{\pm,0}, \mathbb{H}(a)] = 0.$$

Conformal constraint for the kernel

$$\left[S_+^{(0)}, \mathbb{H}^{(1)}\right] = 0,$$
$$\left[S_+^{(0)}, \mathbb{H}^{(2)}\right] = \left[\mathbb{H}^{(1)}, \Delta S_+^{(1)}\right],$$
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Three-loop order

Two-loop order

Intermediate results

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Further questions:

- How to calculate $\Delta_+(a)$?
- How to solve the equation?

Critical point

- In $d = 4 - 2\epsilon$ we can reach the **critical point** $\beta(a^*) = 0 \;\Rightarrow\; a^* = a^*(\epsilon)$.
- Perturbative expansions

$$\Delta_+(a^*) = a^* \Delta_+^{(1)} + (a^*)^2 \Delta_+^{(2)} + \ldots,$$
$$\mathbb{H}(a^*) = a^* \mathbb{H}^{(1)} + (a^*)^2 \mathbb{H}^{(2)} + \ldots.$$

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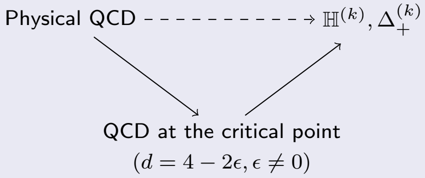
Idea of calculation



Conformal theory

Consider Green's function

$$G(x; z, w) = \Big\langle \Big[\mathcal{O}(z_1, z_2)\Big]\Big[\bar{\mathcal{O}}(x; w)\Big]\Big\rangle$$

- $\bar{n}^2 = 0$ is an independent light-like direction.
- $(x \cdot \bar{n}) = 0$.
- $\bar{\mathcal{O}}(x; w) = \bar{q}(x + w_1\bar{n})\big[x + w_1\bar{n}, x + w_2\bar{n}\big]\not{\Gamma} q(x + w_2\bar{n})$.

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Conformal Ward Identity

$$S_+(\Delta_+)G(x;z,w) = Z\left(S_+^{(0)} - \epsilon(z_1+z_2)\right)Z^{-1}G(x;z,w) + \int d^dy\ (\bar{n}\cdot y)\left\langle \mathcal{N}(y)\left[\mathcal{O}(z_1,z_2)\right]\left[\bar{\mathcal{O}}(x,w)\right]\right\rangle$$

Operator $\mathcal{N}(y)$ can be expanded

$$\mathcal{N}(y) = -\frac{\beta(a)}{a}\left[\mathcal{L}^{\text{YM + gf}}\right] + \text{EOM} + \text{BRST}$$

- EOM operators give simple contribution.
- BRST operators do not contribute at all.

Feynman diagrams

Consider modified QCD action

$$S_{QCD} \mapsto S_\omega = S_{QCD} + \delta^\omega S = S_{QCD} - 2\omega \int d^d y (\bar{n}\cdot y) \left(\frac{1}{4}F^2 + \frac{1}{2\xi}(\partial A)^2\right).$$

- ϵ-expansion of the renormalization operator

$$Z(a) = \mathbb{1} + \frac{1}{\epsilon} Z_1(a) + \frac{1}{\epsilon^2} Z_2(a) + \dots$$

- Deformation of the operator in S_ω

$$Z(a) \mapsto Z_\omega(a) = Z(a) + 2\omega\,(n\cdot\bar{n})\,\tilde{Z}(a).$$

Connection

$$\tilde{Z}_1(a) = z_{12}\Delta_+ + \frac{1}{2}\Big(\mathbb{H}(a) - 2\gamma_q(a)\Big)(z_1 + z_2).$$

- Gluon line:

$$= \ \bar{n}\cdot(x-y)\,\overbracket{A_\mu(x)A_\nu(y)} \ = \ 2ig_{\mu\nu}\int\frac{d^dk}{(2\pi)^d}e^{-ik\cdot(x-y)}\frac{(\bar{n}\cdot k)}{k^4}\,.$$

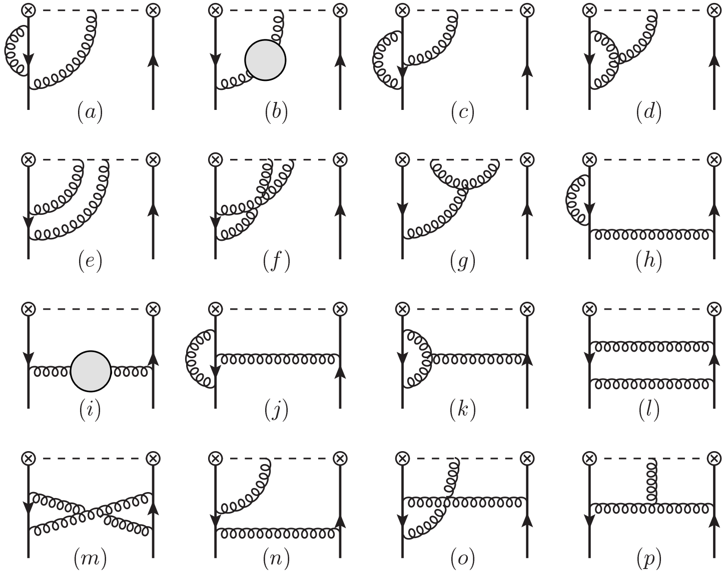
- Quark line:

$$= \ \bar{n}\cdot(x-y)\,\overbracket{q(x)\bar{q}(y)} \ = \ -\int\frac{d^dk}{(2\pi)^d}e^{-ik\cdot(x-y)}\frac{\not{k}\not{\bar{n}}\not{k}}{k^4}\,.$$

- Wilson line:

$$= \ (\bar{n}\cdot n)z_{12}\,[z_1 n, z_2 n]\,.$$

Two-loop diagrams for the evolution kernel



Two-loop conformal anomaly

Two-loop conformal anomaly

$$[\Delta_+^{(2)} f](z_1, z_2) = \int_0^1 du \int_0^1 dt\, \varkappa(t) \Big[f(z_{12}^{ut}, z_2) - f(z_1, z_{21}^{ut})\Big]$$
$$+ \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Big[\omega(\alpha, \beta) + \overline{\omega}(\alpha, \beta)\mathbb{P}_{12}\Big]\Big[f(z_{12}^{\alpha}, z_{21}^{\beta}) - f(z_{12}^{\beta}, z_{21}^{\alpha})\Big].$$

- $\varkappa(t) = C_F^2\, \varkappa_P(t) + \frac{C_F}{N_C}\, \varkappa_{FA}(t) + C_F \beta_0 \varkappa_{bF}(t)$,

$$\begin{aligned}
\varkappa_{bF}(t) =& -2\frac{\bar t}{t}\Big(\ln \bar t + \frac{5}{3}\Big), \\
\varkappa_{FA}(t) =& \frac{2\bar t}{t}\Big\{(2+t)\Big[\operatorname{Li}_2(\bar t) - \operatorname{Li}_2(t)\Big] - (2-t)\Big(\frac{t}{\bar t}\ln t + \ln \bar t\Big) \\
& - \frac{\pi^2}{6}t - \frac{4}{3} - \frac{t}{2}\left(1 - \frac{t}{\bar t}\right)\Big\}, \\
\varkappa_P(t) =& 4\bar t\Big[\operatorname{Li}_2(\bar t) - \operatorname{Li}_2(1)\Big] + 4\left(\frac{t^2}{\bar t} - \frac{2\bar t}{t}\right)\Big[\operatorname{Li}_2(t) - \operatorname{Li}_2(1)\Big] - 2t\ln t \ln \bar t \\
& - \frac{\bar t}{t}(2-t)\ln^2 \bar t + \frac{t^2}{\bar t}\ln^2 t - 2\left(1 + \frac{1}{t}\right)\ln \bar t - 2\left(1 + \frac{1}{\bar t}\right)\ln t \\
& - \frac{16}{3}\frac{\bar t}{t} - 1 - 5t\,.
\end{aligned}$$

Two-loop conformal anomaly

- $\overline{\omega}(\alpha,\beta) = \frac{C_F}{N_C}\overline{\omega}_{NP}(\alpha,\beta)$,

$$\overline{\omega}_{NP}(\alpha,\beta) = -2\left\{\frac{\alpha}{\bar{\alpha}}\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar{\beta}}\right) - \mathrm{Li}_2(\alpha)\right] - \alpha\bar{\tau}\ln\bar{\tau} - \frac{1}{\bar{\alpha}}\ln\bar{\alpha}\ln\bar{\beta} - \frac{\beta}{\bar{\beta}}\ln\bar{\alpha} - \frac{1}{2}\beta\right\}.$$

- $\omega(\alpha,\beta) = C_F^2\,\omega_P(\alpha,\beta) + \frac{C_F}{N_C}\omega_{NP}(\alpha,\beta)$,

$$\begin{aligned}\omega_P(\alpha,\beta) &= \frac{4}{\alpha}\Big[\,\mathrm{Li}_2(\bar{\alpha}) - \zeta_2 + \frac{1}{4}\bar{\alpha}\ln^2\bar{\alpha} + \frac{1}{2}(\beta-2)\ln\bar{\alpha}\Big] \\ &\quad + \frac{4}{\bar{\alpha}}\Big[\,\mathrm{Li}_2(\alpha) - \zeta_2 + \frac{1}{4}\alpha\ln^2\alpha + \frac{1}{2}(\bar{\beta}-2)\ln\alpha\Big],\end{aligned}$$

$$\begin{aligned}\omega_{NP}(\alpha,\beta) &= 2\Big\{\frac{\bar{\alpha}}{\alpha}\left[\mathrm{Li}_2\left(\frac{\beta}{\bar{\alpha}}\right) - \mathrm{Li}_2(\beta) - \mathrm{Li}_2(\alpha) + \mathrm{Li}_2(\bar{\alpha}) - \zeta_2\right] - \ln\alpha - \frac{1}{\alpha}\ln\bar{\alpha} \\ &\quad + \alpha\left(\frac{\bar{\tau}}{\tau}\ln\bar{\tau} + \frac{1}{2}\right)\Big\}.\end{aligned}$$

Solving the conformal constraints

- Equation for the three-loop kernel

$$\left[S_+^{(0)}, \mathbb{H}^{(3)}\right] = \left[\mathbb{H}^{(1)}, \Delta S_+^{(2)}\right] + \left[\mathbb{H}^{(2)}, \Delta S_+^{(1)}\right].$$

Solution

$$\begin{aligned}
\mathbb{H}^{(3)} =& \mathbf{H}_{\mathsf{inv}}^{(3)} + \mathrm{T}_1^{(1)}\left(\beta_1 + \frac{1}{2}\mathbf{H}_{\mathsf{inv}}^{(2)}\right) + \frac{1}{2}\mathrm{T}_2^{(1)}\left(\beta_0 + \frac{1}{2}\mathbf{H}_{\mathsf{inv}}^{(1)}\right)^2 + \left(\mathrm{T}_1^{(2)} + \frac{1}{2}\left(\mathrm{T}_1^{(1)}\right)^2\right)\left(\beta_0 + \frac{1}{2}\mathbf{H}_{\mathsf{inv}}^{(1)}\right) \\
&+ \left[\mathbf{H}_{\mathsf{inv}}^{(2)}, \mathrm{X}^{(1)}\right] + \frac{1}{2}\mathrm{T}_1^{(1)}\left[\mathbf{H}_{\mathsf{inv}}^{(2)}, \mathrm{X}^{(1)}\right] + \left[\mathrm{T}_1^{(1)}, \mathrm{X}^{(2)}\right]\left(\beta_0 + \frac{1}{2}\mathbf{H}_{\mathsf{inv}}^{(1)}\right) \\
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- $\mathrm{X}(a)$ — transformation that "kills" conformal anomaly.

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- $\mathrm{X}(a)$ — transformation that "kills" conformal anomaly.
- $\mathrm{T}_n(a)$ — transformation that maps on the canonically invariant part.
- $\mathbf{H}_{\text{inv}}(a)$ — canonically invariant part of the kernel.

Similarity transformation I

Consider transformation V_1

$$\mathbf{H}(a) = \mathrm{V}_1 \mathbb{H}(a) \mathrm{V}_1^{-1},$$
$$\mathbf{S}_\alpha(a) = \mathrm{V}_1 S_\alpha(a) \mathrm{V}_1^{-1}.$$

Transformed generators

$$\mathbf{S}_-(a) = S_-^{(0)},$$
$$\mathbf{S}_0(a) = S_0^{(0)} + \bar{\beta}(a) + \frac{1}{2}\mathbf{H}(a),$$
$$\mathbf{S}_+(a) = S_+^{(0)} + (z_1 + z_2)\left(\bar{\beta}(a) + \frac{1}{2}\mathbf{H}(a)\right).$$

- It is easier to look at the operator $V_1(a) = \exp\left(\mathrm{X}(a)\right)$

Operator $\mathrm{X}(a)$

- Equation on the operator $\mathrm{X}(a)$:

$$\left[S_+^{(0)}, \mathrm{X}^{(1)}\right] = z_{12}\Delta^{(1)},$$

$$\left[S_+^{(0)}, \mathrm{X}^{(2)}\right] = z_{12}\Delta^{(2)} + \left[\mathrm{X}^{(1)}, z_1 + z_2\right]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \frac{1}{2}\left[\mathrm{X}^{(1)}, z_{12}\Delta^{(1)}\right].$$

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- Form of the operator $\mathrm{X}(a)$:

$$\mathrm{X}f(z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \chi(\alpha,\beta) f(z_{12}^\alpha, z_{21}^\beta) + \int_0^1 d\alpha \frac{\bar\alpha}{\alpha}\chi^\delta(\alpha)\Big(2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big).$$

Differential equation

$$\left[S_+^{(0)}, \mathrm{X}\right] f(z_1, z_2) = z_{12}\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \left(\alpha\bar\alpha\partial_\alpha - \beta\bar\beta\partial_\beta\right)\chi(\alpha,\beta) f(z_{12}^\alpha, z_{21}^\beta) - z_{12}\int_0^1 d\alpha\, \bar\alpha^2\partial_\alpha\chi^\delta(\alpha)\left(f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\right).$$

Similarity transformation II

Consider transformation V_2:

$$\mathbf{H}_{\mathsf{inv}}(a) = \mathrm{V}_2 \mathbf{H} {\mathrm{V}_2}^{-1},$$
$$S_\alpha^{(0)} = \mathrm{V}_2 \mathbf{S}_\alpha {\mathrm{V}_2}^{-1}.$$

Perturbative form of transformation [Ji, Manashov, Moch' 2023]

$$\mathbf{H}(a) = \mathbf{H}_{\mathsf{inv}}(a) + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{T}_n(a) \left(\bar{\beta}(a) + \frac{1}{2}\mathbf{H}(a) \right)^n .$$

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Operators $\mathrm{T_n}(a)$ has the closed form:

$$\mathrm{T}_n(a) f(z_1, z_2) = -\Gamma_{\text{cusp}}(a) \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \ln^n \bar{\alpha} \left(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha) \right)$$
$$+ \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, \ln^n (1 - \alpha - \beta) \Big(h(\tau) + \overline{h}(\tau) \mathbb{P}_{12} \Big) f(z_{12}^\alpha, z_{21}^\beta) .$$

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- $\Gamma_{\mathsf{cusp}}(a)$ — Cusp anomalous dimension. [Polyakov' 1980]

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- $\Gamma_{\mathsf{cusp}}(a)$ — Cusp anomalous dimension. [Polyakov' 1980]
- $\mathbb{P}_{12}$ — operator of permutation $\big(\mathbb{P}_{12} f(z_1, z_2) = f(z_2, z_1)\big)$.

Similarity transformation II

Consider transformation V_2:

$$\mathbf{H}_{\textsf{inv}}(a) = \mathrm{V}_2 \mathbf{H} {\mathrm{V}_2}^{-1},$$
$$S_\alpha^{(0)} = \mathrm{V}_2 \mathbf{S}_\alpha {\mathrm{V}_2}^{-1}.$$

Perturbative form of transformation [Ji, Manashov, Moch' 2023]

$$\mathbf{H}(a) = \mathbf{H}_{\textsf{inv}}(a) + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{T}_n(a) \left(\bar{\beta}(a) + \frac{1}{2}\mathbf{H}(a) \right)^n .$$

Operators $\mathrm{T_n}(a)$ has the closed form:

$$\mathrm{T}_n(a) f(z_1, z_2) = -\Gamma_{\textsf{cusp}}(a) \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \ln^n \bar{\alpha} \left(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha) \right)$$
$$+ \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, \ln^n (1 - \alpha - \beta) \Big(h(\tau) + \overline{h}(\tau) \mathbb{P}_{12} \Big) f(z_{12}^\alpha, z_{21}^\beta) \, .$$

- $\Gamma_{\textsf{cusp}}(a)$ — Cusp anomalous dimension. [Polyakov' 1980]
- $\mathbb{P}_{12}$ — operator of permutation $\big(\mathbb{P}_{12} f(z_1, z_2) = f(z_2, z_1)\big)$.
- $h(\tau)$ — integral kernel of $\mathbf{H}_{\textsf{inv}}(a)$, depending on conformal ration $\tau = \alpha\beta/\bar{\alpha}\bar{\beta}$.

Form of the canonically invariant kernel

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$$\widehat{\mathcal{H}}f(z_1, z_2) = \int_0^1 \frac{d\alpha}{\alpha} \left(2f(z_1, z_2) - \bar{\alpha}\big(f(z_{12}^{\alpha}, z_2) + f(z_1, z_{21}^{\alpha})\big)\right).$$

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Reciprocity

- Eigenvalues of the canonically invariant kernel

$$\mathbf{H}_{\text{inv}}(a)\psi_N(z_1, z_2) = \gamma_{\text{inv}}(N)\psi_N(z_1, z_2).$$

Generalized Gribov-Lipatov reciprocity [Gribov, Lipatov'1972], [Basso, Korchemsky'2007]

$$\gamma(N) = \gamma_{\text{inv}}\Big(N + \frac{1}{2}\gamma(N) + \bar{\beta}(a)\Big).$$

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- $\gamma_{\mathsf{inv}}(N)$ depends only on the **reciprocity respecting** harmonic sums [Beccaria, Forini'2009]

$$\begin{aligned}
\Omega_3(N) &= S_3(N) - \zeta_3, \\
\Omega_{-2}(N) &= (-1)^N \left(S_{-2}(N) + \frac{\zeta_2}{2}\right), \\
\Omega_{1,3}(N) &= S_{1,3}(N) - \frac{1}{2}S_4(N) + \frac{3}{10}\zeta_2^2 - \zeta_3 S_1(N), \\
&\ldots.
\end{aligned}$$

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- Integral kernels [Ji, Manashov, Moch' 2023]

$$\Omega_{1,3}(N) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \frac{\bar{\tau}}{4\tau} \underbrace{\left(\mathrm{H}_2(\tau) + \mathrm{H}_{11}(\tau)\right)}_{\text{Harmonic polylogarithms}}.$$

Results for the invariant part

$$
\begin{aligned}
h^{(3)}(\tau) = & -C_F n_f^2 \frac{16}{9} + C_F^2 n_f \left(\frac{352}{9} - \frac{8}{3}\mathrm{H}_0 + \frac{16}{3}\frac{\bar{\tau}}{\tau}\left(\mathrm{H}_2 - \mathrm{H}_{10}\right)\right) \\
& + \frac{C_F n_f}{N_c}\left(8 - \frac{8}{3}\mathrm{H}_1 - \frac{4}{3}\mathrm{H}_0 + \frac{\bar{\tau}}{\tau}\left(8\mathrm{H}_2 - \frac{8}{3}\mathrm{H}_{10} + \frac{16}{3}\mathrm{H}_{11} + \frac{160}{9}\mathrm{H}_1\right)\right) \\
& + C_F^3\left(-\frac{1936}{9} + \frac{88}{3}\mathrm{H}_0 + 32\frac{\bar{\tau}}{\tau}\left(\mathrm{H}_3 + \mathrm{H}_{12} - \mathrm{H}_{110} - \mathrm{H}_{20} - \frac{1}{3}\mathrm{H}_2 + \frac{1}{3}\mathrm{H}_{10} + \frac{1}{2}\mathrm{H}_1\right)\right) \\
& + \frac{C_F^2}{N_c}\left(- \frac{152}{3} - 96\zeta_3 - \left(\frac{8}{3} - 48\zeta_2\right)\mathrm{H}_0 + \frac{76}{3}\mathrm{H}_1 - 32\mathrm{H}_{10} + 4\mathrm{H}_2 - 48\mathrm{H}_{20} - 16\mathrm{H}_{11} \right. \\
& - 24\mathrm{H}_{21} + \frac{\tau}{\bar{\tau}}\left(- 24\zeta_2 - 48\zeta_3 + 64\mathrm{H}_0\right) + \frac{\tau+1}{\bar{\tau}}\left(- (32 - 16\zeta_2)\mathrm{H}_0 \right. \\
& \left. + 12\mathrm{H}_2 - 16\mathrm{H}_{20} - 8\mathrm{H}_{21}\right) + \frac{\bar{\tau}}{\tau}\left(- \left(\frac{2000}{9} + 16\zeta_2\right)\mathrm{H}_1 + \frac{32}{3}\mathrm{H}_{10} - \frac{208}{3}\mathrm{H}_2 \right. \\
& \left.\left. - 64\mathrm{H}_{20} - \frac{32}{3}\mathrm{H}_{11} - 32\mathrm{H}_{110} + 64\mathrm{H}_3 + 80\mathrm{H}_{12} + 64\mathrm{H}_{21} + 96\mathrm{H}_{111}\right)\right) \\
& + \frac{C_F}{N_c^2}\left(\frac{544}{9} + 16\zeta_2 - 96\zeta_3 - \left(\frac{68}{3} - 36\zeta_2\right)\mathrm{H}_0 + \frac{68}{3}\mathrm{H}_1 - 24\mathrm{H}_{10} + 4\mathrm{H}_2 - 36\mathrm{H}_{20} \right. \\
& + \frac{\tau}{\bar{\tau}}\left(- 8\zeta_2 - 48\zeta_3 + 48\mathrm{H}_0\right) + \frac{\tau+1}{\bar{\tau}}\left(\left(- 24 + 12\zeta_2\right)\mathrm{H}_0 + 4\mathrm{H}_2 - 12\mathrm{H}_{20}\right) \\
& + \frac{\bar{\tau}}{\tau}\left(- \left(\frac{1072}{9} + 16\zeta_2\right)\mathrm{H}_1 + \frac{44}{3}\mathrm{H}_{10} - 44\mathrm{H}_2 - 32\mathrm{H}_{20} - \frac{16}{3}\mathrm{H}_{11} - 16\mathrm{H}_{110} \right. \\
& \left.\left. + 32\mathrm{H}_3 + 32\mathrm{H}_{12} + 48\mathrm{H}_{21} + 32\mathrm{H}_{111}\right)\right).
\end{aligned}
$$

Results for the invariant part

$$
\begin{aligned}
\overline{h}^{(3)}(\tau) = &-\frac{C_F n_f}{N_c}\left(\frac{104}{9} + \frac{8}{3}\mathrm{H}_0 + \frac{8}{9}\Big(23 - 20\tau\Big)\mathrm{H}_1 + \frac{16}{3}\bar{\tau}\Big(\mathrm{H}_{11} + \mathrm{H}_{10}\Big)\right) \\
&+ \frac{C_F^2}{N_c}\Bigg(\frac{1480}{9} - 40\zeta_2 - 48\zeta_3 + \left(\frac{28}{3} + 24\zeta\right)\mathrm{H}_0 + \frac{76}{3}\mathrm{H}_1 + 16\mathrm{H}_{10} - 4\mathrm{H}_2 - 24\mathrm{H}_{20} \\
&- 16\mathrm{H}_{11} + 24\mathrm{H}_{21} + \frac{\tau}{\bar{\tau}}\Big(-24\zeta_2 + 48\zeta_3 - 32\mathrm{H}_0\Big) + \frac{\tau+1}{\bar{\tau}}\Big(\Big(16 - 8\zeta_2\Big)\mathrm{H}_0 + 12\mathrm{H}_2 \\
&+ 8\mathrm{H}_{20} - 8\mathrm{H}_{21}\Big) + \bar{\tau}\Big(-24 + 48\zeta_2 + 48\zeta_3 - 16\zeta_2\mathrm{H}_0 + \left(\frac{2144}{9} + 16\zeta_2\right)\mathrm{H}_1 + \frac{104}{3}\mathrm{H}_{10} \\
&- 24\mathrm{H}_2 + 16\mathrm{H}_{20} + \frac{32}{3}\mathrm{H}_{11} - 16\mathrm{H}_{110} - 32\mathrm{H}_{12} - 32\mathrm{H}_{21} - 96\mathrm{H}_{111}\Big)\Bigg) \\
&+ \frac{C_F}{N_c^2}\Bigg(\frac{1028}{9} - 24\zeta_2 - 48\zeta_3 + \left(\frac{44}{3} + 36\zeta_2\right)\mathrm{H}_0 + \frac{68}{3}\mathrm{H}_1 + 24\mathrm{H}_{10} - 4\mathrm{H}_2 - 36\mathrm{H}_{20} \\
&+ \frac{\tau}{\bar{\tau}}\Big(-8\zeta_2 + 48\zeta_3 - 48\mathrm{H}_0\Big) + \frac{\tau+1}{\bar{\tau}}\Big(\Big(24 - 12\zeta_2\Big)\mathrm{H}_0 + 4\mathrm{H}_2 + 12\mathrm{H}_{20}\Big) \\
&+ \bar{\tau}\Big(-24 + 24\zeta_2 + 48\zeta_3 - 32\zeta_2\mathrm{H}_0 + \left(\frac{1072}{3} + 16\zeta_2\right)\mathrm{H}_1 + \frac{88}{3}\mathrm{H}_{10} \\
&- 24\mathrm{H}_2 + 32\mathrm{H}_{20} + \frac{16}{3}\mathrm{H}_{11} - 32\mathrm{H}_{110} + 16\mathrm{H}_{12} + 16\mathrm{H}_{21} - 32\mathrm{H}_{111}\Big)\Bigg).
\end{aligned}
$$

Obtaining the result

Schematically

$$\mathbb{H}(a) \xrightarrow[\mathrm{V}_1]{} \mathbf{H}(a) \xrightarrow[\mathrm{V}_2]{} \mathbf{H}_{\mathsf{inv}}(a) \xrightarrow[\text{eigenvalue}]{} \gamma_{\mathsf{inv}}(N) \xrightarrow[\text{reciprocity}]{} \gamma(N)$$

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- Three-loop result for the transversity forward anomalous dimensions.
  [Velizhanin' 2012], [Blümlein, Marquard, Schneider, Schönwald' 2021]

To the local

Gegenbauer basis

$$\mathcal{O}_{nk}(0) = (\partial_{z_1} + \partial_{z_2})^k C_n^{(3/2)} \left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) [\mathcal{O}](z_1, z_2) \Big|_{z_1 = z_2 = 0},$$

- RG-equation in the Gegenbauer basis

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} \right) O_{nk} = - \sum_{n'=0}^{n} \gamma_{nn'} O_{n'k}.$$

- Separate contributions

$$\gamma_{\text{off}}^{(3)} = \gamma_1^{(3)} + n_f \gamma_{n_f}^{(3)} + n_f^2 \gamma_{n_f^2}^{(3)}.$$

Results for the matrix

- $N_c = 3$ and $0 \leq n, n' \leq 5$

$$\gamma_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{44992}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1316680}{2187} & 0 & 0 & 0 & 0 \\ \frac{1977808}{10125} & 0 & \frac{54669748}{91125} & 0 & 0 & 0 \\ 0 & \frac{68848018}{273375} & 0 & \frac{443231668}{759375} & 0 & 0 \end{pmatrix},$$

$$\gamma_{n_f}^{(3)} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{21008}{243} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{200060}{2187} & 0 & 0 & 0 & 0 \\ \frac{998842}{30375} & 0 & \frac{898436}{10125} & 0 & 0 & 0 \\ 0 & \frac{745418}{18225} & 0 & \frac{4266496}{50625} & 0 & 0 \end{pmatrix},$$

$$\gamma_{n_f^2}^{(3)} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{160}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{520}{243} & 0 & 0 & 0 & 0 \\ \frac{1012}{2025} & 0 & \frac{4088}{2025} & 0 & 0 & 0 \\ 0 & \frac{3268}{3645} & 0 & \frac{416}{225} & 0 & 0 \end{pmatrix}.$$

Result for the matrix

- The diagonal elements have the form:

$$\gamma_{00}^{(3)} = \frac{105110}{81} - \frac{1856}{27}\zeta_3 - \left(\frac{10480}{81} + \frac{320}{9}\zeta_3\right) n_f - \frac{8}{9}n_f^2,$$
$$\gamma_{11}^{(3)} = \frac{19162}{9} - \left(\frac{5608}{27} + \frac{320}{3}\zeta_3\right) n_f - \frac{184}{81}n_f^2,$$
$$\gamma_{22}^{(3)} = \frac{17770162}{6561} + \frac{1280}{81}\zeta_3 - \left(\frac{552308}{2187} + \frac{4160}{27}\zeta_3\right) n_f - \frac{2408}{729}n_f^2,$$
$$\gamma_{33}^{(3)} = \frac{206734549}{65610} + \frac{560}{27}\zeta_3 - \left(\frac{3126367}{10935} + \frac{5120}{27}\zeta_3\right) n_f - \frac{14722}{3645}n_f^2,$$
$$\gamma_{44}^{(3)} = \frac{144207743479}{41006250} + \frac{9424}{405}\zeta_3 - \left(\frac{428108447}{1366875} + \frac{5888}{27}\zeta_3\right) n_f - \frac{418594}{91125}n_f^2,$$
$$\gamma_{55}^{(3)} = \frac{183119500163}{47840625} + \frac{3328}{135}\zeta_3 - \left(\frac{1073824028}{3189375} + \frac{2176}{9}\zeta_3\right) n_f - \frac{3209758}{637875}n_f^2.$$