Theoretical High-Energy Physics and Artificial Intelligence

AIME24 November 21, 2024

Matthew Schwartz Harvard University



The NSF AI Institute for Artificial Intelligence and Fundamental Interactions (IAIFI)

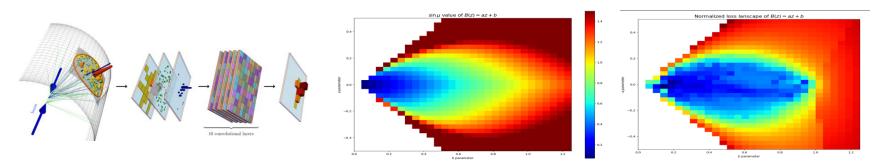
Boston, MA, USA



Outline

The Present

• Symbolic ML for high energy theory



The Future

• Can machines do theoretical physics?



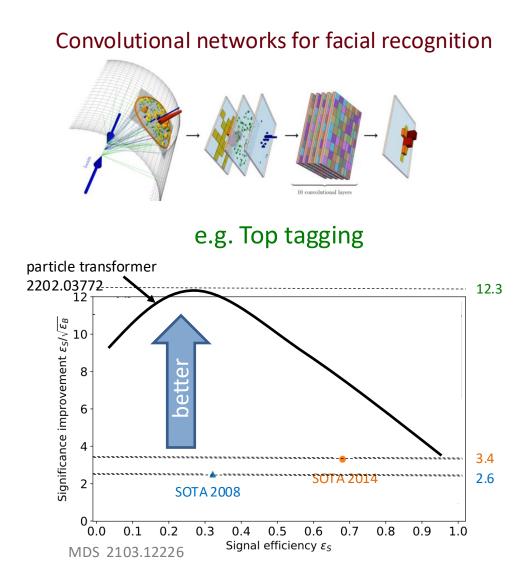
1. The Past



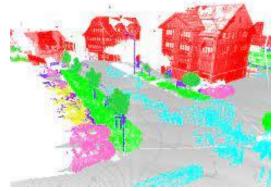
Björn Karlsson, MidJourney January, 2024

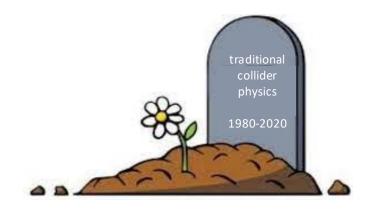
Machine learning approach

Take some tool highly engineered for another puropose and apply it to physics problems



Point clouds/deep sets for self-driving cars





- ML requires less "thinking"
- Better performance
- Provdies less physical insight

2. The present



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What do high-energy theorists actually do?

1. Most papers on hep-th and hep-ph are largely symbolic

- Calculate something
- Find/establish some relationship
- Solve some toy model
- Extract behavior of some theory in some limit

Can ML do these things?

- It's starting to...
- ML is good at helping solve well-defined problems

2. What makes a question interesting?

- It connects to nature
- You can make progress on it
- Someone else thought it was interesting
- It is related to something someone else thought was interesting
- Much harder problem

Can ML answer this?

Not yet. But soon.

Most uses of ML in physics so far are data science

- Machine learning is great at characterizing numerical data
- Has led to revolutionary progress on a great variety of physical questions
- Much much more still to be done

How do we transition from **data science** to **symbolic problems?**

- Large Langauge Models show that ML good for symbolic problems
 - Potential is there
- A lot of what physicists do is study examples, look for patterns, and generalize
- A place to start is with *discrete* symbolic data
 - Find some theoretical physics problems where ML can help
 - Get a feel for what it can (and currently cannot) do

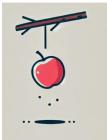
1. Simplifying spinor-helicity amplitudes

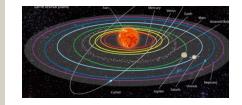
Many advances in theoretical physics come from simplification/unification



Newton: motion of apples and planets governed by the same simple law

$$F=Grac{m_1m_2}{r^2}$$

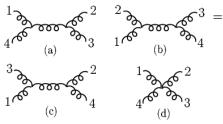




Quantum field theory

calculate some Feynman diagrams for

4 gluon scattering



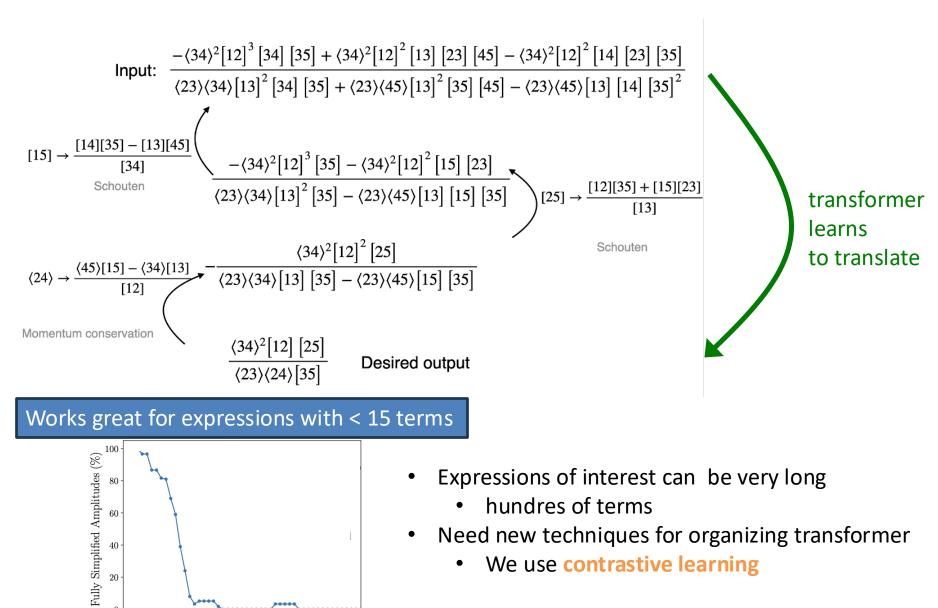
simplify!

get a messy answer

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- simpler form suggests deeper structure
- $-rac{\langle 12
 angle [34]^2}{\langle 34
 angle [14][23]}$ is there a better way to do the calculation?
 - In this case, yes! (BCFW recursion)

Generate training data by scrambling

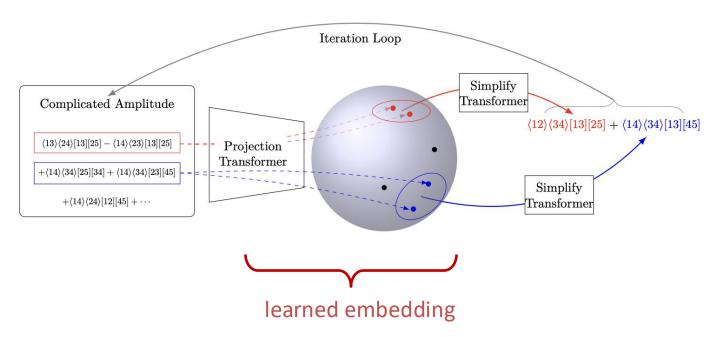


Number of terms in \mathcal{M}

Contrastive learning

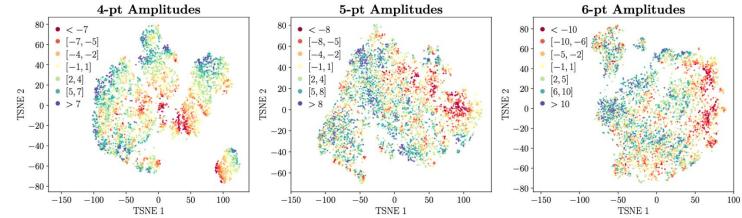
Learn an embedding so that terms that are similar are close

- similar = appear in some identity
- close = metric on embedding space
- 1. Pick subset of terms expected to simplify
- 2. Apply transformer
- 3. Repeat

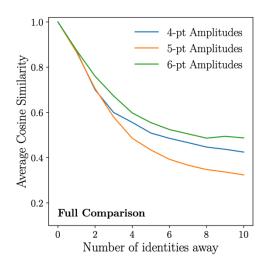


Contrastive learning

• Cross-check: t-SNE on latent space



- color is mass dimension
 - Learns dimensional analysis
 - Learns other features as well

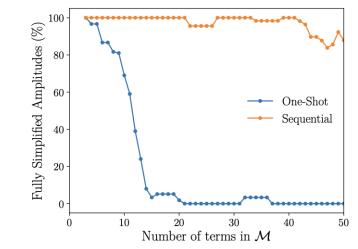


distance inversely correlated with complexity

Does it work?

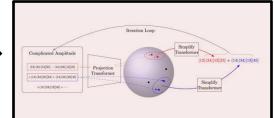
Yes!

It can simplify long expressions now



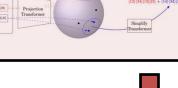
Example application: 5 gluon amplitude

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	$ \begin{array}{c} (12)^{2}(34)(33)(14)(15) \\ (12)^{2}(34)(13)(34)(45) \\ (12)^{2}(34)(13)(34)(45) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15)(15) \\ (12)^{2}(15)(15)(15)(15)(15)(15)(15)(15)(15)(15)$	
+	$\frac{(12)^{-}(34)}{(15)}\frac{(34)}{(45)}\frac{(45)}{(12)}\frac{(12)^{-}(15)}{(13)}\frac{(12)^{-}(15)}{(13)}\frac{(15)}{(13)}\frac{(12)^{-}(15)}{(13)}\frac{(12)^{-}(23)}{(13)}\frac{(23)}{(13)}\frac{(23)}{(13$	
	(10/23)(24)(30)[14][10][23] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)(40)[14][40] (10)(40)(40)(40)(40)(40)(40)(40)(40)(40)(4	
+	$\frac{\langle 12 \rangle^2 \langle 23 \rangle [15] [24] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [45]} + \frac{\langle 12 \rangle^2 \langle 23 \rangle [14] [23] [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [15] [23] [24]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [45]} - \langle 12 \rangle^2 [45]$	
	$ \begin{array}{c} (12)^{2}(23)(15)[24] + (12)^{2}(24)(15)(12)(12)(12)(12)(12)(12)(12)(12)(12)(12$	
+	$\frac{\langle 12 / (22) / (12) / (24)}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14]} + \langle 12 / (24) / (24) / (24) / (24) / (24) / (24) / (24) / (25) / (26) $	
	$\frac{\langle 12 \rangle^2 [24] [45]}{\langle 12 \rangle^2 [34] [45]} = \frac{\langle 12 \rangle^2 (34) [13] [24] [34] [45]}{\langle 12 \rangle^2 [34] [45]} = \frac{\langle 12 \rangle (13) [13] [34] [35]}{\langle 12 \rangle (13) [13] [15] [34]^2} = \langle 12 \rangle (13) [13] [13] [13] [13] [13] [13] [13] [13]$	
+	$\frac{(12)(12)(12)(13)}{(13)(15)(35)(12)(15)} + \frac{(12)(12)(12)(12)(12)(12)(12)(12)}{(13)(15)(24)(35)(12)(12)(12)(12)(12)(12)(12)(12)(12)(12$	
+	$\frac{(12)\langle 13\rangle\langle 23\rangle [13] [34]}{(15)\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14]} + \frac{(12)\langle 13\rangle [34]}{(15)\langle 34\rangle\langle 35\rangle [12]} + \frac{\langle 12\rangle\langle 13\rangle [13] [34]^2}{(15)\langle 24\rangle\langle 35\rangle [12] [14] [23]} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle [13] [45]}{(15)\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [15]}$	_
	$\begin{array}{c} (12)(14) \left[14\right] \left[45\right] \qquad (12)(14) \left[13\right] \left[34\right] \left[45\right] \qquad (12)(15) \left[13\right] \left[15\right] \left[34\right] \qquad (12)(23) \left[13\right] \left[24\right] \left[34\right] \left[35\right] \end{array} \right]$	
+	$\frac{(12)\langle 14\rangle [14] [45]}{(15)\langle 34\rangle \langle 35\rangle [12] [15]} + \frac{(12)\langle 14\rangle [13] [34] [45]}{(15)\langle 24\rangle \langle 35\rangle [12] [15] [23]} - \frac{\langle 12\rangle \langle 15\rangle [13] [15] [34]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [34] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [34] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [34] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [34] [35]} + \langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35] [35] [36] [36] [36] [36] [36] [36] [36] [36$	
	$\frac{\langle 12 \rangle \langle 23 \rangle [34] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [45]} + \frac{\langle 12 \rangle \langle 23 \rangle [15] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [45]} - \frac{\langle 12 \rangle \langle 23 \rangle [13] [24] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} + \frac{\langle 12 \rangle \langle 23 \rangle [14] [23] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]}$	
_	$\frac{1}{(24)(35)(45)[12][45]} + \frac{1}{(24)(35)(45)[12][14][45]} - \frac{1}{(24)(35)(45)[12]^2[45]} + \frac{1}{(24)(35)(45)[12][45]} + \frac$	(
	$\frac{\langle 12 \rangle \langle 23 \rangle [35]}{\langle 24 \rangle \langle 35 \rangle [12]} + \frac{\langle 12 \rangle \langle 23 \rangle [15] [34]}{\langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14]} + \frac{\langle 12 \rangle [34]}{\langle 35 \rangle \langle 45 \rangle [12]} + \frac{\langle 12 \rangle \langle 34 \rangle [13] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} - \frac{\langle 12 \rangle \langle 23 \rangle^2 [13] [23] [45]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]}$	
_	$\frac{1}{(24)(34)(35)[12]} + \frac{1}{(24)(34)(35)[12]} + \frac{1}{(35)(45)[12]} + \frac{1}{(35)(45)[12]} + \frac{1}{(24)(35)(45)[12]} + \frac{1}{(24)(35)(45)[12]} + \frac{1}{(15)(24)(34)(35)[12]} + \frac{1}{(15)(24)(34)$	T+
+	$\frac{(12)(23)^2 [13] [24] [35]}{(15)(24)(34)(35) [12] [14] [15]} - \frac{(12)(23) [23] [45]}{(15)(34)(35) [12] [15]} + \frac{(12)(23) [24] [35]}{(15)(34)(35) [12] [15]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [24] [36]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [24] [36]}{(15)(24)(35) [12] [14] [15] [23] [24] [34] [35]} + \frac{(12)(23) [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23] [24] [34] [35] [24] [34] [35] [35] [36] [36] [36] [36] [36] [36] [36] [36$	-
Ŧ	$\frac{1}{(15)(24)(35)(12)[14][15]} - \frac{1}{(15)(34)(35)[12][15]} + \frac{1}{(15)(34)(35)[12][15]} + \frac{1}{(15)(24)(35)[12][15]} + \frac{1}{(15)(24)(35)[12][14][15][23]} + \frac{1}{(15)(24)(35)[12][14][15][23]} + \frac{1}{(15)(24)(35)[12][15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)($	
+	$\frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [13] [45]^2}{\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14] [15]} + \frac{\langle 12\rangle [34] [45]}{\langle 15\rangle\langle 35\rangle [12] [15]} + \frac{\langle 12\rangle\langle 45\rangle [45]^2}{\langle 15\rangle\langle 34\rangle\langle 35\rangle [12] [15]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \langle 12\rangle\langle 34\rangle [13] [34]^2 [34] [34] [34] [34] [34] [34] [34] [34]$	
	(15) (24) (34) (35) [12] [14] [15] (15) (35) [12] [15] (15) (34) (35) [12] [15] (15) (24) (35) [12] [14] [15] [23]	-
+	$\frac{\left(12\right)\left\langle45\right\rangle\left[13\right]\left[34\right]\left[45\right]^{2}}{\left(15\right)\left\langle24\right\rangle\left\langle35\right\rangle\left[12\right]\left[14\right]\left[15\right]\left[23\right]} + \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[13\right]\left[24\right]\left[34\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[34\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left\langle45\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left\langle45\right\rangle\left\langle45\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle45\right\rangle\left\langle45\right\rangle\left(12\right)} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left\langle45\right\rangle\left(12\right)}{\left(13\right)\left\langle24\right\rangle\left\langle45\right\rangle\left\langle45\right\rangle\left(12\right)} - \frac{\left(12\right)\left\langle45\right\rangle\left\langle45\right\rangle\left(12\right)}{\left(13\right)\left\langle45\right\rangle\left\langle45\right\rangle\left(12\right)} - \frac{\left(12\right)\left\langle45\right\rangle\left(12\right)\left\langle45\right\rangle\left(12\right)}{\left(13\right)\left\langle45\right\rangle\left(12\right)} - \frac{\left(12\right)\left(14\right)\left\langle45\right\rangle\left(12\right)}{\left(13\right)\left\langle45\right\rangle\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)\left(12\right)}{\left(13\right)\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)\left(12\right)}{\left(13\right)\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(13\right)\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(12\right)}{\left(12\right)\left(12\right)} - \frac{\left(12\right)\left(12\right)}{\left(12\right)} - \frac{\left(12\right)\left(12\right)\left(12\right)}{\left(12\right)} - \frac{\left(12\right)\left(12\right)\left(12\right)}{\left(12\right)} - \frac{\left(12\right)\left(12\right)\left(12\right)}{\left(12\right)} $	
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+ -	$\frac{13}{(15)(24)(35)[12][14][15][23]} - \frac{1}{(13)(24)(35)(45)[12]^2} + \frac{1}{(13)(24)(35)(45)[12]^2} - \frac{1}{(13)(24)(35)(45)[12]^2} + \frac{1}{(13)(24)(35)(45)[$	
	$(12)(23)^{2}$ [15] [23] [24] [34] $(12)(23)^{2}$ [23] [25] $(12)(23)^{2}$ [15] [23] [24] $(12)(23)$ [23] [24]	
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+ -	$\frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle [13] [24] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} + \frac{\langle 12 \rangle \langle 23 \rangle [13] [24] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [25] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 13 \rangle \langle 13 \rangle [45]} + \frac{\langle 12 \rangle \langle 13 \rangle [45]}{\langle 13 \rangle [$	
($\frac{13}{24} \frac{13}{30} \frac{12}{40} \frac{13}{30} \frac{12}{24} \frac{13}{30} \frac{12}{24} \frac{13}{30} \frac{12}{21} \frac{13}{21} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{21} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{20} 12$	
+ -	$\frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [25] [45]}{\langle 13\rangle\langle 24\rangle\langle 35\rangle [12]^2} - \frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [15] [24] [45]}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12]^2 [14]} + \frac{\langle 13\rangle\langle 23\rangle [13] [34]^2 [35]}{\langle 24\rangle\langle 35\rangle\langle 45\rangle [12] [14] [23] [45]} + \frac{\langle 13\rangle\langle 23\rangle^2 [13] [34] [35]}{\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14] [15]} - \langle 13\rangle\langle 23\rangle [13\rangle [24\rangle [35] [12\rangle [12\rangle [12\rangle [12\rangle [12\rangle [12\rangle [12\rangle [12\rangle$	
($13 \langle 24 \rangle \langle 35 \rangle [12]^{2} \langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^{2} \\ [14] \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \\ [12] \\ [14] \\ [23] \\ [45] \langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle \\ [12] \\ [14] \\ [15] \\$	
+ -	$ \frac{\langle 13 \rangle \langle 23 \rangle [34] [35]}{15 \rangle \langle 34 \rangle \langle 35 \rangle [12] [15]} + \frac{\langle 13 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14] [15] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle^2 [13] [34] [45]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 24 \rangle \langle 35 \rangle \langle 15 \rangle [13] [34] [34]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34] [35]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34] [35]}{\langle 15 \rangle \langle 15 \rangle \langle$	
	$15 (34) (35) [12] [15] \stackrel{+}{\longrightarrow} (15) (24) (35) [12] [14] [15] [23] \stackrel{+}{\longrightarrow} (24) (35) (45) [12] [14] [23] \stackrel{+}{\longrightarrow} (15) (24) (34) (35) [12] [14] [15]$	
	$\frac{\langle 14 \rangle \langle 23 \rangle [34] [45]}{15 \rangle \langle 34 \rangle \langle 35 \rangle [12] [15]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [45]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14] [15] [23]} - \frac{\langle 15 \rangle \langle 23 \rangle [15] [35]}{\langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 23 \rangle^2 [13] [24] [34] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [14] [45]}$	
+ -	$\frac{15}{34} + \frac{15}{24} + \frac{12}{15} + \frac{12}{15} + \frac{12}{15} + \frac{12}{15} + \frac{12}{12} + \frac{12}{14} + \frac{15}{12} + \frac{12}{12} + \frac{12}{24} + 12$	
,	(23) ² [23] [34] [35] (23) ² [13] [24] [35] (23) [24] [35] (23) [34] [35] (14) (15) (23) [15] [45]	
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+ -	$ \frac{\langle 23\rangle^{2} [23] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12]^{2} [45]} + \frac{\langle 23\rangle^{2} [13] [24] [35]}{\langle 24\rangle \langle 34\rangle \langle 35\rangle [12]^{2} [14]} + \frac{\langle 23\rangle [24] [35]}{\langle 34\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 23\rangle [34] [35]}{\langle 24\rangle \langle 35\rangle [12]^{2}} - \frac{\langle 14\rangle \langle 15\rangle \langle 23\rangle [15] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 24\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 15\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 15\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [34]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 23\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [23] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 23\rangle $	2
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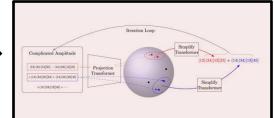


 $\overline{\mathcal{M}} = -\frac{\langle 12 \rangle^3}{\langle 15 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle}$



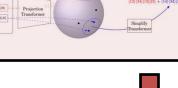
Example application: 5 gluon amplitude

$\mathcal{M} =$	$\frac{\langle 12\rangle^{3} [13]}{\langle 23\rangle \langle 24\rangle \langle 35\rangle \langle 45\rangle [23]} + \frac{\langle 12\rangle^{3} [14] [25]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [45]} - \frac{\langle 12\rangle^{3} [15] [24]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [45]} - \frac{\langle 12\rangle^{2} [13] [34]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [14] [23]} + \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [45]} - \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [14] [23]} + \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [45]} - \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [23] [41] [23]} + \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [41] [23]} + \frac{\langle 12\rangle^{2} [13] [24]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [23] [41] [23] [41] [41] [41] [41] [41] [41] [41] [41$	
	$ \begin{array}{c} (23)(24)(35)(45)[23] \\ (23)(24)(35)(45)[23] \\ (23)(24)(35)(45)[23] \\ (23)(24)(35)(45)[23] \\ (24)(35)$	
-	$\frac{\langle 12 \rangle^2 \left[13\right] \left[24\right] \left[35\right]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \left[12\right] \left[23\right] \left[45\right]} + \frac{\langle 12 \rangle^2 \left[13\right] \left[25\right] \left[34\right]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \left[12\right] \left[23\right] \left[45\right]} + \frac{\langle 12 \rangle^2 \left[14\right] \left[35\right]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[34\right]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[45\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 35 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \langle 24 \rangle \langle 25 \rangle \left[12\right] \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right] \left[23\right]}{\langle 23 \rangle \left[23\right]} - \frac{\langle 12 \rangle^2 \left[13\right]}{\langle 23 \rangle \left[23\right]} - \langle 12 \rangle^2 \left[13\right]$	
	(24)(35)(45)[12][23][45] = (24)(35)(45)[12][23][45] = (24)(35)(45)[12][45] = (24)(35)(45)[12][45] = (23)(24)(45)[12][45] = (23)(24)(45)[12][45] = (23)(24)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)[12][45] = (23)(24)(45)(45)(45)(45)(45)(45)(45)(45)(45)(4	
-	$\frac{\langle 12 \rangle^2 \langle 23 \rangle \left[13 \right] \left[45 \right]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle \left[14 \right] \left[15 \right]} - \frac{\langle 12 \rangle^2 \left[45 \right]}{\langle 15 \rangle \langle 34 \rangle \langle 35 \rangle \left[15 \right]} - \frac{\langle 12 \rangle^2 \left[13 \right] \left[34 \right] \left[45 \right]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle \left[14 \right] \left[15 \right] \left[23 \right]} + \frac{\langle 12 \rangle^2 \left[13 \right] \left[45 \right]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[45 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[15 \right]}{\langle 15 \rangle \langle 23 \rangle \langle 35 \rangle \left[12 \right] \left[15 \right]} + \frac{\langle 12 \rangle^2 \left[14 \right] \left[15 \right]}{\langle 15 \rangle \langle 15 \rangle \langle$	
	$ \begin{array}{c} (12)^{2}(34)(33)(14)(15) \\ (12)^{2}(34)(13)(34)(45) \\ (12)^{2}(34)(13)(34)(45) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15) \\ (12)^{2}(15)(15)(15)(15) \\ (12)^{2}(15)(15)(15)(15)(15)(15)(15)(15)(15)(15)$	
+	$\frac{(12)^{-}(34)}{(15)}\frac{(34)}{(45)}\frac{(45)}{(12)}\frac{(12)^{-}(15)}{(13)}\frac{(12)^{-}(15)}{(13)}\frac{(15)}{(13)}\frac{(12)^{-}(15)}{(13)}\frac{(12)^{-}(23)}{(13)}\frac{(23)}{(13)}\frac{(23)}{(13$	
	(10/23)(24)(30)[14][10][23] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14] (10)(24)(30)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)[14][40] (10)(40)(40)(40)[14][40] (10)(40)(40)(40)(40)(40)(40)(40)(40)(40)(4	
+	$\frac{\langle 12 \rangle^2 \langle 23 \rangle [15] [24] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [45]} + \frac{\langle 12 \rangle^2 \langle 23 \rangle [14] [23] [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [15] [23] [24]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [25]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [12]^2 [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [45]} - \frac{\langle 12 \rangle^2 \langle 23 \rangle [12]^2 [45]}{\langle 13 \rangle \langle 24 \rangle [45]} - \langle 12 \rangle^2 [45]$	
	$ \begin{array}{c} (12)^{2}(23)(15)[24] + (12)^{2}(24)(15)(12)(12)(12)(12)(12)(12)(12)(12)(12)(12$	
+	$\frac{\langle 12 / (22) / (12) / (24)}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14]} + \langle 12 / (24) / (24) / (24) / (24) / (24) / (24) / (24) / (25) / (26) $	
	$\frac{\langle 12 \rangle^2 [24] [45]}{\langle 12 \rangle^2 [34] [45]} = \frac{\langle 12 \rangle^2 (34) [13] [24] [34] [45]}{\langle 12 \rangle^2 [34] [45]} = \frac{\langle 12 \rangle (13) [13] [34] [35]}{\langle 12 \rangle (13) [13] [15] [34]^2} = \langle 12 \rangle (13) [13] [13] [13] [13] [13] [13] [13] [13]$	
+	$\frac{(12)(12)(12)(13)}{(13)(15)(35)(12)(15)} + \frac{(12)(12)(12)(12)(12)(12)(12)(12)}{(13)(15)(24)(35)(12)(12)(12)(12)(12)(12)(12)(12)(12)(12$	
+	$\frac{(12)\langle 13\rangle\langle 23\rangle [13] [34]}{(15)\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14]} + \frac{(12)\langle 13\rangle [34]}{(15)\langle 34\rangle\langle 35\rangle [12]} + \frac{\langle 12\rangle\langle 13\rangle [13] [34]^2}{(15)\langle 24\rangle\langle 35\rangle [12] [14] [23]} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle [13] [45]}{(15)\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [15]}$	_
	$\begin{array}{c} (12)(14) \left[14\right] \left[45\right] \qquad (12)(14) \left[13\right] \left[34\right] \left[45\right] \qquad (12)(15) \left[13\right] \left[15\right] \left[34\right] \qquad (12)(23) \left[13\right] \left[24\right] \left[34\right] \left[35\right] \end{array} \right]$	
+	$\frac{(12)\langle 14\rangle [14] [45]}{(15)\langle 34\rangle \langle 35\rangle [12] [15]} + \frac{(12)\langle 14\rangle [13] [34] [45]}{(15)\langle 24\rangle \langle 35\rangle [12] [15] [23]} - \frac{\langle 12\rangle \langle 15\rangle [13] [15] [34]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [23] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12] [14] [35] [35]} + \frac{\langle 12\rangle \langle 23\rangle [13] [15] [36]}{\langle 24\rangle \langle 35\rangle [12] [15] [36]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [35] [36]}{\langle 24\rangle \langle 35\rangle [12] [15] [36]} + \frac{\langle 12\rangle \langle 23\rangle [13] [24] [34] [34] [36] [36] [36]}{\langle 24\rangle \langle 35\rangle [36] [36]} + \langle 12\rangle \langle 23\rangle [36] [36] [36] [36] [36] [36] [36] [36]$	
	$\frac{\langle 12 \rangle \langle 23 \rangle [34] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [45]} + \frac{\langle 12 \rangle \langle 23 \rangle [15] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [45]} - \frac{\langle 12 \rangle \langle 23 \rangle [13] [24] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]} + \frac{\langle 12 \rangle \langle 23 \rangle [14] [23] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [45]}$	
_	$\frac{1}{(24)(35)(45)[12][45]} + \frac{1}{(24)(35)(45)[12][14][45]} - \frac{1}{(24)(35)(45)[12]^2[45]} + \frac{1}{(24)(35)(45)[12][45]} + \frac{1}{(24)(35)(45)[12][$	(
	$\frac{\langle 12 \rangle \langle 23 \rangle [35]}{\langle 24 \rangle \langle 35 \rangle [12]} + \frac{\langle 12 \rangle \langle 23 \rangle [15] [34]}{\langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14]} + \frac{\langle 12 \rangle [34]}{\langle 35 \rangle \langle 45 \rangle [12]} + \frac{\langle 12 \rangle \langle 34 \rangle [13] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} - \frac{\langle 12 \rangle \langle 23 \rangle^2 [13] [23] [45]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]}$	
_	$\frac{1}{(24)(34)(35)[12]} + \frac{1}{(24)(34)(35)[12][14]} + \frac{1}{(35)(45)[12]} + \frac{1}{(24)(35)(45)[12]} + \frac{1}{(24)(35)(45)[12][14][23]} - \frac{1}{(15)(24)(34)(35)[12][14][15]} + \frac{1}{(15)(24)(34)(35)[12][14][15][14][15][14][15]} + \frac{1}{(15)(24)(34)(35)[12][14][15][15][15][15]$	T+
+	$\frac{(12)(23)^2 [13] [24] [35]}{(15)(24)(34)(35) [12] [14] [15]} - \frac{(12)(23) [23] [45]}{(15)(34)(35) [12] [15]} + \frac{(12)(23) [24] [35]}{(15)(34)(35) [12] [15]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [13] [24] [34] [35]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [24] [36]}{(15)(24)(35) [12] [14] [15] [23]} + \frac{(12)(23) [24] [36]}{(15)(24)(35) [12] [14] [15] [23] [24] [34] [35] [24] [34] [35]} + \frac{(12)(23) [24] [36]}{(15)(24)(35) [12] [14] [15] [23] [24] [36] [36] [36] [36] [36] [36] [36] [36$	-
Ŧ	$\frac{1}{(15)(24)(35)(12)[14][15]} - \frac{1}{(15)(34)(35)[12][15]} + \frac{1}{(15)(34)(35)[12][15]} + \frac{1}{(15)(24)(35)[12][15]} + \frac{1}{(15)(24)(35)[12][14][15][23]} + \frac{1}{(15)(24)(35)[12][14][15][23]} + \frac{1}{(15)(24)(35)[12][15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)(24)(35)[15]} + \frac{1}{(15)($	
+	$\frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [13] [45]^2}{\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14] [15]} + \frac{\langle 12\rangle [34] [45]}{\langle 15\rangle\langle 35\rangle [12] [15]} + \frac{\langle 12\rangle\langle 45\rangle [45]^2}{\langle 15\rangle\langle 34\rangle\langle 35\rangle [12] [15]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \frac{\langle 12\rangle\langle 34\rangle [13] [34]^2 [45]}{\langle 15\rangle\langle 24\rangle\langle 35\rangle [12] [14] [15] [23]} + \langle 12\rangle\langle 34\rangle [13] [34]^2 [34] [34] [34] [34] [34] [34] [34] [34]$	
	(15) (24) (34) (35) [12] [14] [15] (15) (35) [12] [15] (15) (34) (35) [12] [15] (15) (24) (35) [12] [14] [15] [23]	-
+	$\frac{\left(12\right)\left\langle45\right\rangle\left[13\right]\left[34\right]\left[45\right]^{2}}{\left(15\right)\left\langle24\right\rangle\left\langle35\right\rangle\left[12\right]\left[14\right]\left[15\right]\left[23\right]} + \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[13\right]\left[24\right]\left[34\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[34\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left\langle45\right\rangle\left[12\right]} - \frac{\left(12\right)\left\langle14\right\rangle\left\langle23\right\rangle\left[45\right]}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left(12\right)} - \frac{\left(12\right)\left(14\right)\left\langle23\right\rangle\left(12\right)}{\left(13\right)\left\langle24\right\rangle\left\langle35\right\rangle\left(12\right)} - \frac{\left(12\right)\left(14\right)\left\langle23\right\rangle\left(12\right)}{\left(13\right)\left\langle24\right\rangle\left(13\right)\left\langle25\right\rangle\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(23\right)\left(12\right)}{\left(13\right)\left\langle24\right\rangle\left(13\right)\left(12\right)} - \frac{\left(12\right)\left(14\right)\left(23\right)\left(14\right)\left(23\right)\left(12\right)}{\left(13\right)\left(24\right)\left(13\right)\left(12\right)} - \left(12\right)\left(14\right)\left(13\right)\left(14\right)\left(13\right)\left(14\right)\left(13\right)\left(14\right)$	
	(15) (24) (35) (12] [14] [15] [23] (13) (24) (35) (45) [12] [14] [23] (13) (24) (35) (45) [12] (13) (24) (34) (35) [12]	
+	$\frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} - \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle^2}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}\frac{[12]^2}{[12]^2} + \langle 12\rangle\langle 14\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12$	
	$\frac{\langle 12\rangle\langle 14\rangle\langle 23\rangle}{13}\frac{[13]}{[24]}\frac{[34]}{[45]} - \frac{\langle 12\rangle\langle 15\rangle\langle 23\rangle}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[15]}{[23]} + \frac{\langle 12\rangle\langle 23\rangle^2}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[12]^2}{[23]^2} - \frac{\langle 12\rangle\langle 23\rangle^2}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[12]}{[23]}\frac{[23]}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[22]} + \frac{\langle 12\rangle\langle 23\rangle^2}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[23]^2} - \frac{\langle 12\rangle\langle 23\rangle^2}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[22]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{[23]}\frac{[23]}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[22]}\frac{[23]}{[23]}\frac{[23]}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[22]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{[23]}\frac{[23]}{(13)\langle 24\rangle\langle 35\rangle\langle 45\rangle}\frac{[23]}{[22]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{[23]}\frac{[23]}{[23]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{[23]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{[23]}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)\langle 23\rangle^2}\frac{[23]}{(12)$	
+ -	$\frac{13}{(15)(24)(35)[12][14][15][23]} - \frac{1}{(13)(24)(35)(45)[12]^2} + \frac{1}{(13)(24)(35)(45)[12]^2} - \frac{1}{(13)(24)(35)(45)[12]^2} + \frac{1}{(13)(24)(35)(45)[$	
	$(12)(23)^{2}$ [15] [23] [24] [34] $(12)(23)^{2}$ [23] [25] $(12)(23)^{2}$ [15] [23] [24] $(12)(23)$ [23] [24]	
+ -	$\frac{\langle 12\rangle\langle 23\rangle^2 \left[15\right] \left[23\right] \left[24\right] \left[34\right]}{13\rangle\langle 24\rangle\langle 35\rangle \left\{45\rangle \left[12\right]^2 \left[14\right] \left[45\right]} - \frac{\langle 12\rangle\langle 23\rangle^2 \left[23\right] \left[25\right]}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle \left[12\right]^2} + \frac{\langle 12\rangle\langle 23\rangle^2 \left[15\right] \left[23\right] \left[24\right]}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle \left[12\right]^2 \left[14\right]} + \frac{\langle 12\rangle\langle 23\rangle \left[23\right] \left[24\right]}{\langle 13\rangle\langle 35\rangle\langle 45\rangle \left[12\right]^2} + \langle 12\rangle\langle 23\rangle \left[23\rangle \left[24\right] \left[24\rangle \left[23\rangle \left[24\rangle \left[24$	
`	$\frac{1}{2} \frac{1}{2} \frac{1}$	
+ -	$\frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle [13] [24] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} + \frac{\langle 12 \rangle \langle 23 \rangle [13] [24] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [25] [34]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2 [14]} - \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 12 \rangle \langle 23 \rangle [23] [45]}{\langle 13 \rangle \langle 1$	
($\frac{13}{24} \frac{13}{30} \frac{12}{40} \frac{13}{30} \frac{12}{24} \frac{13}{30} \frac{12}{24} \frac{13}{30} \frac{12}{21} \frac{13}{21} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{21} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{20} \frac{13}{20} \frac{12}{20} 12$	
+ -	$\frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [25] [45]}{\langle 13\rangle\langle 24\rangle\langle 35\rangle [12]^2} - \frac{\langle 12\rangle\langle 23\rangle\langle 45\rangle [15] [24] [45]}{\langle 13\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12]^2 [14]} + \frac{\langle 13\rangle\langle 23\rangle [13] [34]^2 [35]}{\langle 24\rangle\langle 35\rangle\langle 45\rangle [12] [14] [23] [45]} + \frac{\langle 13\rangle\langle 23\rangle^2 [13] [34] [35]}{\langle 15\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle [12] [14] [15]} - \langle 13\rangle\langle 23\rangle [13\rangle [24\rangle [35] [24\rangle [35\rangle [12] [14] [35] [24\rangle [35\rangle [12] [14] [35] [34\rangle [35\rangle [12] [14] [35\rangle [34\rangle [35\rangle [12] [34\rangle [35\rangle [34\rangle [35\rangle [12] [34\rangle [35\rangle [34\rangle [35\rangle [12] [34\rangle [35\rangle [34\rangle [34\rangle [34\rangle [34\rangle [34\rangle [34\rangle [34\rangle [34$	
($13 \langle 24 \rangle \langle 35 \rangle [12]^{2} \langle 13 \rangle \langle 24 \rangle \langle 35 \rangle [12]^{2} \\ [14] \langle 24 \rangle \langle 35 \rangle \langle 45 \rangle \\ [12] \\ [14] \\ [23] \\ [45] \langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle \\ [12] \\ [14] \\ [15] \\$	
+ -	$ \frac{\langle 13 \rangle \langle 23 \rangle [34] [35]}{15 \rangle \langle 34 \rangle \langle 35 \rangle [12] [15]} + \frac{\langle 13 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14] [15] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle^2 [13] [34] [45]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12] [14] [23]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [35]}{\langle 24 \rangle \langle 35 \rangle \langle 15 \rangle [13] [34] [34]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34] [35]}{\langle 15 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12] [14] [15]} - \frac{\langle 14 \rangle \langle 23 \rangle [13] [34] [35]}{\langle 15 \rangle \langle 15 \rangle \langle$	
	$15 (34) (35) [12] [15] \stackrel{+}{\longrightarrow} (15) (24) (35) [12] [14] [15] [23] \stackrel{+}{\longrightarrow} (24) (35) (45) [12] [14] [23] \stackrel{+}{\longrightarrow} (15) (24) (34) (35) [12] [14] [15]$	
	$\frac{\langle 14 \rangle \langle 23 \rangle [34] [45]}{15 \rangle \langle 34 \rangle \langle 35 \rangle [12] [15]} + \frac{\langle 14 \rangle \langle 23 \rangle [13] [34]^2 [45]}{\langle 15 \rangle \langle 24 \rangle \langle 35 \rangle [12] [14] [15] [23]} - \frac{\langle 15 \rangle \langle 23 \rangle [15] [35]}{\langle 24 \rangle \langle 34 \rangle \langle 35 \rangle [12]^2} + \frac{\langle 23 \rangle^2 [13] [24] [34] [35]}{\langle 24 \rangle \langle 35 \rangle \langle 45 \rangle [12]^2 [14] [45]}$	
+ -	$\frac{15}{34} + \frac{15}{24} + \frac{12}{15} + \frac{12}{15} + \frac{12}{15} + \frac{12}{15} + \frac{12}{12} + \frac{12}{14} + \frac{15}{12} + \frac{12}{12} + \frac{12}{24} + 12$	
,	(23) ² [23] [34] [35] (23) ² [13] [24] [35] (23) [24] [35] (23) [34] [35] (14) (15) (23) [15] [45]	
	$\frac{1}{(24)(25)(12)(245)} + \frac{1}{(24)(25)(12)(245)(24)(25)(12)(245)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(24)(25)(12)(24)(24)(24)(25)(12)(24)(24)(24)(24)(25)(12)(24)(24)(24)(25)(12)(24)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(24)(24)(25)(12)(25)(12)(25)(12)(12)(12)(12)(12)(12)(12)(12)(12)(12$	
(
+ -	$ \frac{\langle 23\rangle^{2} [23] [34] [35]}{\langle 24\rangle \langle 35\rangle \langle 45\rangle [12]^{2} [45]} + \frac{\langle 23\rangle^{2} [13] [24] [35]}{\langle 24\rangle \langle 34\rangle \langle 35\rangle [12]^{2} [14]} + \frac{\langle 23\rangle [24] [35]}{\langle 34\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 23\rangle [34] [35]}{\langle 24\rangle \langle 35\rangle [12]^{2}} - \frac{\langle 14\rangle \langle 15\rangle \langle 23\rangle [15] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 24\rangle \langle 35\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 15\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [35]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 15\rangle [12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle^{2} [13] [24] [45]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [14]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [14]}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \frac{\langle 14\rangle \langle 23\rangle [12]^{2} [12]^{2}}{\langle 13\rangle \langle 24\rangle \langle 35\rangle \langle 12]^{2}} + \langle 12\rangle \langle $	2
($ \begin{array}{cccccccccccccccccccccccccccccccccccc$	-





 $\overline{\mathcal{M}} = -\frac{\langle 12 \rangle^3}{\langle 15 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle}$



Example application: graviton-scalar scattering

Iteration Loop

Projection

 $-\frac{(12)(13)(23)[12][34]}{(15)(25)(34)(45)^2}-\frac{(12)(13)[12][24][34]}{(15)(35)(45)^2[23]}-\frac{(12)(13)[12][34]^2}{(15)(25)(45)^2[23]}-\frac{(12)(13)(15)^2(34)[12][14][15][25][34]}{(14)(35)^3(45)^3[23][35][45]}$ $\begin{array}{c} (12) \langle 13 \rangle \langle 15 \rangle^2 \langle 34 \rangle \left[12 \right] \left[15 \right]^2 \left[34 \right] \\ + \left(12 \rangle \langle 13 \rangle \langle 15 \rangle \left(25 \rangle \left(34 \right) \right] \left[15 \right] \left[24 \right] \left[25 \right] \left[34 \right] \\ + \left(12 \rangle \langle 13 \rangle \left(15 \rangle \left(34 \right) \right] \left[15 \right] \left[34 \right] \\ + \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \right) \right) \right] \left[15 \right] \left[34 \right] \\ + \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \right) \right) \right] \left[15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(34 \rangle \left(12 \rangle \left(13 \rangle \left(15 \rangle \left(13 \rangle \left(15 \rangle \left(13 \rangle \left(15 \rangle \left(15 \rangle \left(13 \rangle \left(15 \rangle \left($ (14)(23)(35)2(45)3 [23] [35] (14)(35)³(45)³ [23] [35] [45] (14)(35)²(45)³ [35] (14)(35)²(45)³ [23] [35] [45] (14)(35)²(45)³ [23] [45] (14)(25)(35)(45)³ [23] [45] $=\frac{\left<12\right>\left<13\right>\left<15\right>\left<25\right>\left<34\right>^2\left<12\right>\left<15\right>\left<25\right>\left<34\right>^2\left<12\right>\left<15\right>\left<25\right>\left<34\right>^2\left<12\right>\left<15\right>\left<25\right>\left<34\right>^2\left<12\right>\left<15\right>\left<34\right>^2\left<12\right>\left<15\right>\left<34\right>^2\left<12\right>\left<15\right>\left<34\right>^2\left<12\right>\left<15\right>\left<34\right>^2\left<12\right>\left<12\right>\left<12\right>\left<12\right>\left<12\right>\left<12\right>^2\left<12\right>\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^2\left<12\right>^$ $(14)(23)(35)^2(45)^3$ [23][35] (14)(23)(35)³(45)³ [23] [35] (14)(23)(35)²(45)² [23] [35] $+\frac{\binom{12}{(13)}\binom{13}{(23)}\binom{12}{(25)}\binom{12}{(25)}\binom{13}{(25)}}{(14)\binom{13}{(25)}\binom{12}{(25)}\binom{13}{(25)}$ $-\frac{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 12|\langle 34|\langle 45\rangle^2}{\langle 14\rangle\langle 35\rangle^2\langle 46\rangle^2\langle 35\rangle}+\frac{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 12|\langle 24|\langle 34\rangle}{\langle 14\rangle\langle 36\rangle^2\langle 45\rangle^2\langle 23\rangle}+\frac{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 12|\langle 25|\langle 34\rangle^2}{\langle 14\rangle\langle 36\rangle^2\langle 45\rangle^2\langle 23\rangle}+\frac{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 12|\langle 34\rangle^2\langle 34\rangle^2\langle 34\rangle^2}{\langle 14\rangle\langle 36\rangle\langle 34\rangle\langle 34\rangle\langle 34\rangle\langle 34\rangle\langle 34\rangle^2\langle 34\rangle^$ $= \frac{(12)(13)(25)(24)^2 [12] [24] [34] (45)}{(14)(23)(35)^2 (45)^2 [23] [24] (24)} = \frac{(12)(13)(25)(24)^2 [12] [24] (24)}{(25)(35)^2 (45)^2 [23] [25]} + \frac{(12)(15)(23)[12] [14] (25) [34)}{(25)(35)^4 (45)^2 [23] [25]} = \frac{(12)(13)(25)(24)}{(25)(35)^4 (45)^2 [23] (25)} = \frac{(12)(13)(25)(24)}{(25)(25)^4 (45)^2 (25)^2 (45)^2 (25)} = \frac{(12)(13)(25)(24)}{(25)(25)^4 (45)^2 (25)^2 (45)^2 ($ $-\frac{(12)(23)[12][34]}{(25)(45)^3}+\frac{(12)[12][34][45]}{(25)(45)^2[23]}-\frac{(12)(15)^2(23)(34)[12][14][25]^2[34]}{(14)(35)^3[23][45]}-\frac{(12)(15)^2(23)(34)[12][14][25][23]}{(14)(25)(25)^2(45)^3[23][45]}$ $+ \frac{(12)(15)^2(34)^2[12][14][25][34]}{(12)^2(34)^2(21)^3(22)^3(21)^2(34)} - \frac{(12)(15)^2(34)[12][15][25][34]}{(12)^2(24)^2(21)^2(24)^2(21)^2(24)^2(21)^2(24)^2(2$ (14)(35)³(45)³ [23] [35] (14)(35)²(45)³ [23] [35] (14)(25)(35)(45)³[23] (14)(23)(35)2(45)3 [23] [35] (14)(35)²(45)³ [35] (14)(25)(35)(45)³ $(14)(35)^{2}(45)^{3}[35]$ $\frac{(13)^2}{(23)}(35)^2(45)^3[23][25][45]} + \frac{(13)^2}{(35)^2(45)^2[25]} + \frac{(13)^2}{(35)^2(45)^2[25]} + \frac{(13)^2}{(35)^2(45)^2[25]} + \frac{(13)^2}{(23)^2(45)^2[25]} + \frac{(13)^2}{(23)^2(45)^$ $\frac{(13)^2(24)}{(23)(25)^2(45)^2(23)} + \frac{(13)^2(24)}{(23)(25)^4(45)^3(23)} \frac{(15)^2(24)}{(23)(25)^4(45)^3(23)} + \frac{(13)^2(24)}{(23)(25)(35)^4(45)^2(23)} \frac{(15)^2(24)^2(24)}{(23)(25)^2(24)^3(23)} + \frac{(13)^2(25)^2(24)}{(23)(25)^3(24)^3(23)} \frac{(15)^2(24)^2(24)}{(23)(25)^3(24)^3(23)} \frac{(15)^2(24)^2(24)}{(23)(25)^3(24)^3(24)^3(23)} \frac{(15)^2(24)^2(24)}{(23)(25)^3(24)^3(24)} \frac{(15)^2(24)^2(24)}{(24)^2(24)^3(24)^2(24)} \frac{(15)^2(24)^2(24)}{(24)^2(24)^2(2$ $\frac{(13)^2 (25)^2 [15]^2 [24]^2}{(23) (35)^2 (45)^3 [23] [45]} + \frac{(13)^2 (25) (34) [15]^2 [24] [34]^2}{(23) (35)^2 (45)^3 [23] [35] [45]} + \frac{(13)^2 (25) (34) [14] [15] [24] [34]}{(23) (35)^3 (45)^2 [23] [35]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} + \frac{(13)^2 (25) (34) [14] [15] [24] [34]}{(23) (35)^3 (45)^2 [23] [35]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} + \frac{(13)^2 (25) (34) [15]^2 [24] [34]}{(23) (35)^4 (45)^2 [23] [35]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^3 [23] [45]} - \frac{2(13)^2 (25) [15]^2 [24] [34]}{(23) (35)^4 (45)^4 (25$ $\frac{(13)^2(25)^2[15][24]}{(15)(24)(35)^2(45)^2} + \frac{(13)^2(24)[15][24]^2[34]}{(15)(23)(35)(45)^2[23][25]} + \frac{(13)^2(24)[15][24][34]^2}{(15)(23)(25)(45)^2[23][25]} + \frac{(13)^2(25)^2(34)[15][24]^2[34]}{(15)(23)(35)^3(45)^2[23][35]} + \frac{(13)^2(25)^2(34)[15][24][34]}{(15)(23)(35)^3(45)^2[23][35]} + \frac{(13)^2(25)^2(34)[35]}{(15)(23)(35)^3(45)^2[33][35]} + \frac{(13)^2(25)^2(34)[35]}{(15)(23)(35)^3(45)^2[33][35]} + \frac{(13)^2(25)^2(34)[35]}{(15)(23)(35)^3(45)^2[33][35]} + \frac{(13)^2(35)^2(35)}{(15)(35)(35)(35)^2$ $\frac{(13)^2(25)^2\left[15\right]\left[24\right]^2}{(15)(23)(35)^2(45)^2\left[23\right]} + \frac{(13)^2(25)\left(34\right)\left[15\right]\left[24\right]\left[24\right]^2}{(15)(23)(35)^2(45)^2\left[23\right]\left[35\right]} - \frac{(13)^2\left[25\right]\left[15\right]\left[24\right]\left[34\right]}{(15)(23)(35)(45)^2\left[23\right]} - \frac{(13)^2\left[15\right]\left[34\right]^2}{(15)(23)(45)^2\left[23\right]} - \frac{(13)^2\left[15\right]^2}{(15)(23)(45)^2\left[23\right]} - \frac{(15)^2\left[15\right]^2}{(15)(23)(4$ $+\frac{(13)^2(15)(34)[15]^2[34]^2}{(14)(35)^2(45)^3[35][45]}+\frac{(13)^2(15)[15]^2[24]}{(14)(35)^2(45)^2[26]}+\frac{(13)^2(15)[15]^2[34]}{(14)(35)^2(45)^2[35]}-\frac{(13)^2(15)(24)(34)[13][15][24]^2}{(14)(23)(35)^2(45)^3[23][25]}$ $\frac{(13)^2(24)(34)}{(14)(23)(25)(35)(45)^2[23][24]} = \frac{(13)(15)^2(24)[14]^2[15][24]}{(25)(33)^2(45)^3[23][45]} = \frac{(13)(15)^2(34)[14]^2[15][25][34]}{(35)^3(45)^3[23][35][45]} = \frac{(13)(15)^2(24)[14][15]^2[24](14)[14)[15]^2[24](14)[15]^2[24](14)[15]^2[24](14)[15]^2[24](15)[16]^2[24](14)[15]^2[24](15)[16]^2[24](14)[15]^2[24](14)14[15]^2[24](14)14[15]^2[24](14)[14)[14]$ $=\frac{(33)(15)^2(45)^4[14](15)^2[44]($ $=\frac{\langle 13 \rangle \langle 15 \rangle \langle 24 \rangle \left[14 \right] \left[15 \right] \left[24 \right] \left[45 \right]}{\langle 15 \rangle \left[24 \right] \left[45 \right]}=\frac{\langle 13 \rangle \langle 15 \rangle \langle 25 \rangle \langle 34 \rangle ^2 \left[14 \right] \left[15 \right] \left[24 \right] \left[34 \right]}{\langle 15 \rangle \left[25 \rangle \langle 34 \rangle \left[15 \right] ^2 \left[24 \right] \left[34 \right] \right]}$ $+\frac{(13)(15)(25)\left[15\right]^{2}[24]}{(23)(35)(45)^{3}[23]}-\frac{(13)(15)(34)^{2}[14](15)\left[154\right]^{2}}{(23)(35)^{2}(45)^{3}[23][34]}-\frac{(13)(15)(34)\left[14\right][15][34](45)}{(23)(35)^{2}(45)^{3}[23]}+\frac{(13)(15)(15)\left[15\right]^{2}[34]}{(23)(35)^{2}(45)^{3}[23]}$ 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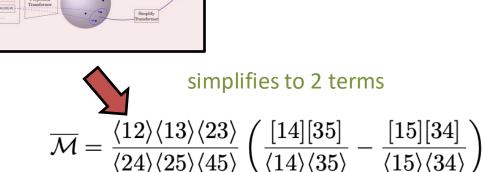
 $-\frac{(15)(24)(34)[14][24][45]}{(25)(33)(45)^3[25]}+\frac{(15)(24)(14)[14][24][34]}{(25)(33)(45)^3[23]}-\frac{(15)(24)[14][15][24](45]}{(25)(33)(45)^3[13][25]}+\frac{(15)(25)(34)^2[14][24][25][34]}{(33)^3(45)^3[23][35]}$ $-\frac{(15)(25)(34)(15)[24](25)[34]}{(35)^2(45)^3(23)(35)}-\frac{(15)(34)^2(14)[34](45)}{(35)^2(45)^3(35)}-\frac{(15)(34)(14)[25](34)}{(35)(45)^3(23)}-\frac{(15)(34)(14)[25](34)}{(35)^2(45)^2(23)(35)}$ $=\frac{(35)^2(45)^3}{(25)^2(24)^3}\frac{[25]}{[25]}\frac{[35]}{[25]}\frac{(35)^2(45)^3}{[25]}\frac{[35]}{[25]}\frac{(15)^2(4)^2(5)}{[25]^2[4]}\frac{[35]}{[25]}\frac{[24]}{[25]}\frac{[35]}{[24]}\frac{[35]}{[25]}\frac{[35]}{[24]}\frac{[35]}{[35]}\frac{$ 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Complicated Amplitude

(13) (24) [13] [25] - (14) (23) [13] [25]

+(14)(24)[12][45] -

298 terms



(12)(34)[13][25] + (14)(34)[13][45]

simplifies to 2 terms

2. S-Matrix bootstrap

The S-matrix is the fundamental object of Quantum Field Theory



- A lot is known about it from pertubation theory (Feynman diagrams)
- Some things are known/conjectured about it non-perturbatively
 - e.g. it should be *unitary* which implies the optical theorem

 $\frac{\text{Optical theorem}}{\text{Im}\langle a | \mathcal{M} | b \rangle} = \sum_{|c\rangle} \langle a | \mathcal{M}^{\dagger} | c \rangle \langle c | \mathcal{M} | b \rangle$

- Non-perturbative constraint
- Relates complex scattering amplitude M to real cross section $\sigma = M^{\dagger}M$

Given cross section $\sigma = |M|^2$ can the phase of M be uniquely determined?

Elastic scattering

Unitarity constraint simplifies in the ``elastic regime"

- 4 m² < s < 9 m²
- Only 2-particle states are relevant
- energy conserved, kinematics described by scattering angle $z = \cos \theta$

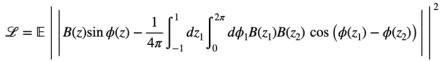
For what B(z) does
$$\phi(z)$$
 exist satisfying this equation?
For what B(z) is $\phi(z)$ is $\phi(z)$ unique or not-unique?

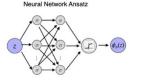
Can we find $\phi(z)$ given B(z) with ML? ... Yes!

[Dersy, MDS, Zhiboedov, 2308.09451]

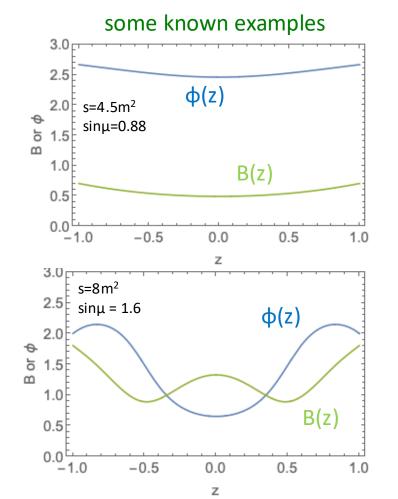
• Parametrize $\phi(z)$ as a neural network

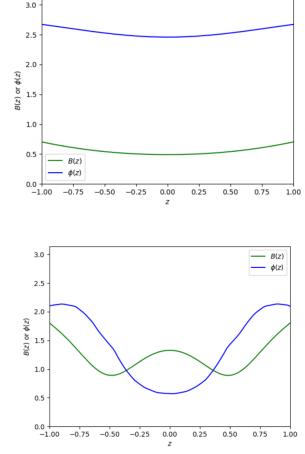
• Loss function is unitarity condition











excellent agreement with known results

When does a phase exist?

In 1967 Andre Martin proposed

$$\sin \mu = \max_{z} \frac{\int_{-1}^{1} dz_1 \int_{0}^{2\pi} d\phi_1 B(z_1) B(z_2)}{4\pi B(z)}$$

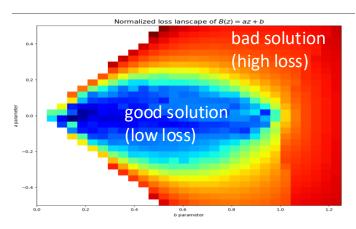
as an indicator of phase determination

- Martin proved:
 - If $sin\mu < 1$ for a given B(z) then there always exists a phase $\phi(z)$

What is special about sinµ?

Contours of sin μ

Loss landscape from ML search for φ



- Loss landscape correlates with sin $\boldsymbol{\mu}$
 - sin μ measures how hard these solutions are to find
 - Don't need exact solutions to learn this lesson

Can there be more than one ϕ given B?

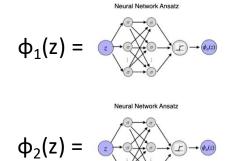
Crichton (1966): yes

Atkinson (1977) found two phases $\phi_1(z)$ and $\phi_2(z)$ for the same B(z) with sin μ > 2.15

Are there phase-ambiguous amplitudes with $sin\mu < 2.15$?

Open question since 1977

ML approach: two NNs for two phases $\phi_1(z)$ and $\phi_2(z)$



• Impose unitarity condition loss for each φ :

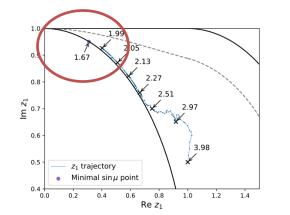
$$\mathscr{L} = \mathbb{E} \left| \left| B(z)\sin\phi(z) - \frac{1}{4\pi} \int_{-1}^{1} dz_1 \int_{0}^{2\pi} d\phi_1 B(z_1) B(z_2) \cos\left(\phi(z_1) - \phi(z_2)\right) \right| \right|^2$$

• Add repulsive loss to keep solutions apart

 $\mathcal{L}_{R}=\mathbb{E}_{z}\left|\left|d\left(\phi_{1}(z),\phi_{2}(z)
ight)
ight|
ight|^{-p}+\mathbb{E}_{z}\left|\left|d\left(\pi-\phi_{1}(z),\phi_{2}(z)
ight)
ight|
ight|^{-p}
ight.$

Gradient descent in sin $\boldsymbol{\mu}$ leads to new phase-ambiguous cross sections

First new phase-ambiguous solution in 50 years!



New lowest sinµ phase ambuity

sinµ = 1.67

3. The future



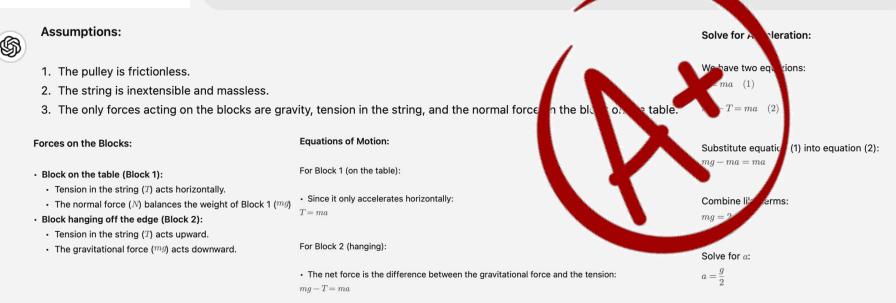
Markus Graf, July, 2023

Current AI models can do college-level physics

Ask Chat GPT 40

Two blocks with equal mass m are connected with a string. One of the blocks is placed on a table, the other hange over the edge. Determine the acceleration of the masses.

Correct!



Final Answer:

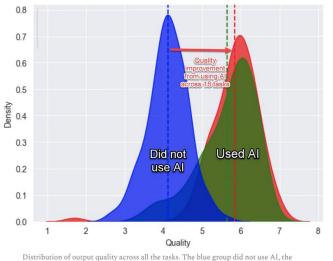
The acceleration of the masses is $\frac{g}{2}$, where g is the acceleration due to gravity (approximately 9.8 m/s^2).

Not only does it get the right answer, but it uses the right equations and shows its work

Augmented intelligence: LLMs help us thrive

Dell'Acqua et al, "Navigating the Jagged Technological Frontier" (Harvard Buisness School, 2023)

Can LLMs help consultants?



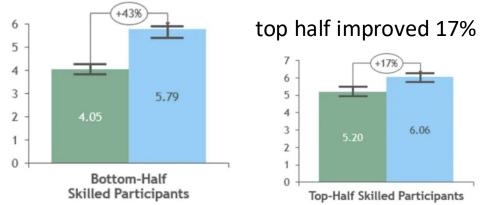
Distribution of output quality across all the tasks. The blue group did not use AI, the green and red groups used AI, the red group got some additional training on how to use AI.



- Mediocre consultants using AI become as good as the best consultants
- The best consultants also improve

Al is a "skill-leveler"

bottom half improved 43%



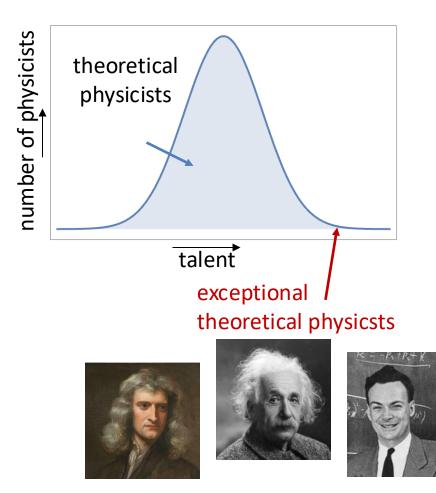
10,000 Einsteins

You

is progress in theoretical physics due to the outsized contribution of a small number of individuals?

S ChatGPT

Yes, progress in theoretical physics, like many fields of science, often appears to be driven by the outsized contributions of a relatively small number of individuals. These key figures often make



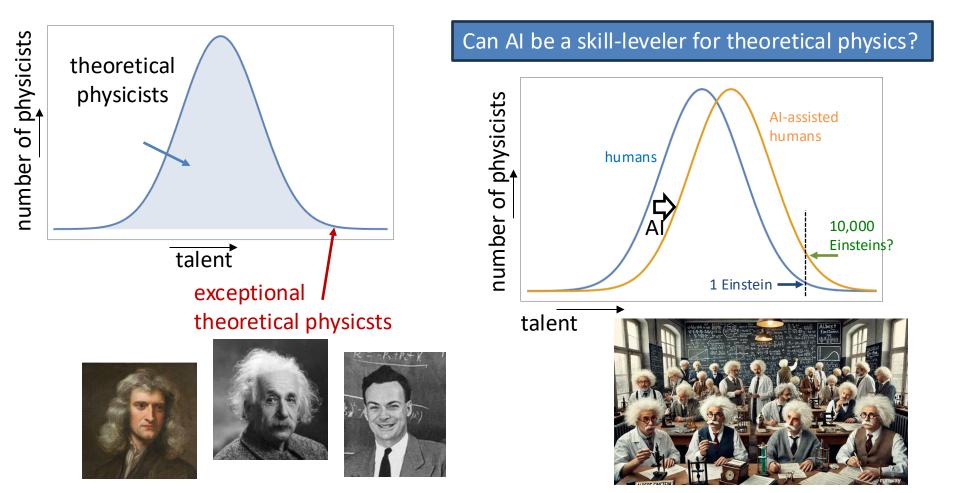
10,000 Einsteins

You

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S ChatGPT

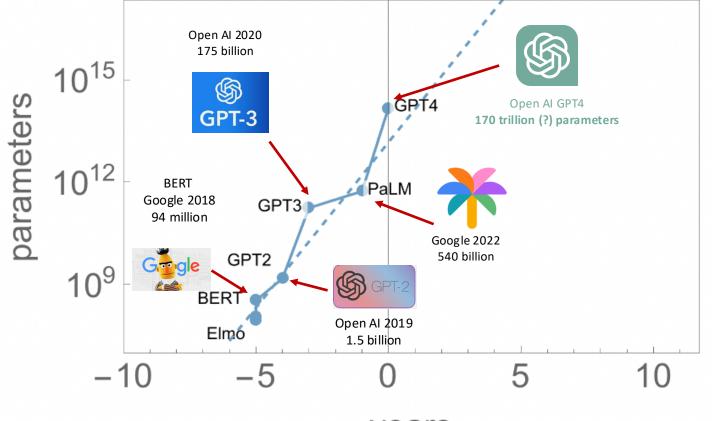
Yes, progress in theoretical physics, like many fields of science, often appears to be driven by the outsized contributions of a relatively small number of individuals. These key figures often make



Large Language Models are growing fast

What is a Large Language Model?

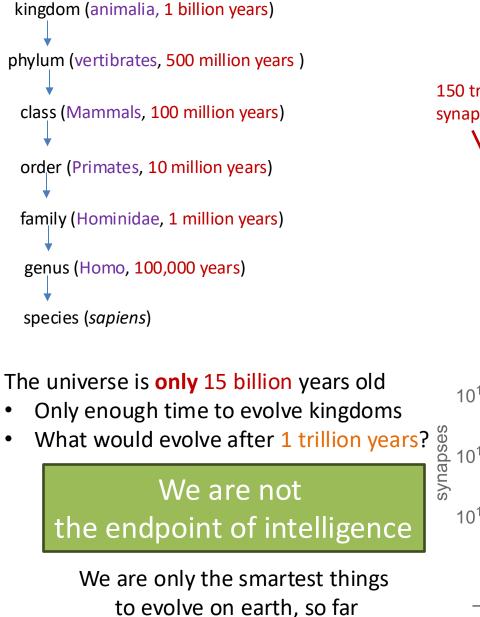
- Transformer-based artificial neural network
- Trained on vast amounts of textual data
- Designed to predict the next word (token)

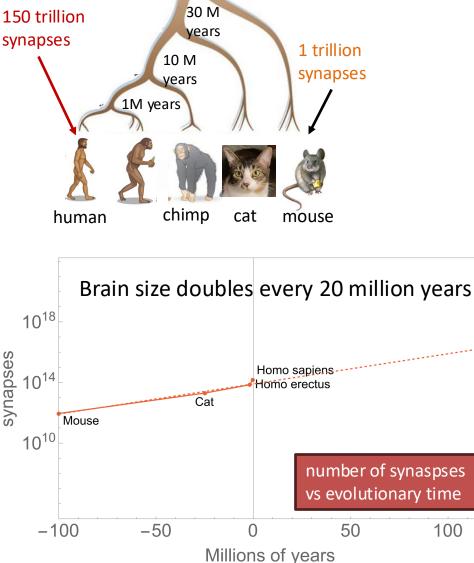


years

- Exponential growth: LLMs are 10 times more powerful each year!
- # parameters in current LLMs (170 trillion) ≈ # synapses in human brain (150 trillion)

Evolution is slow



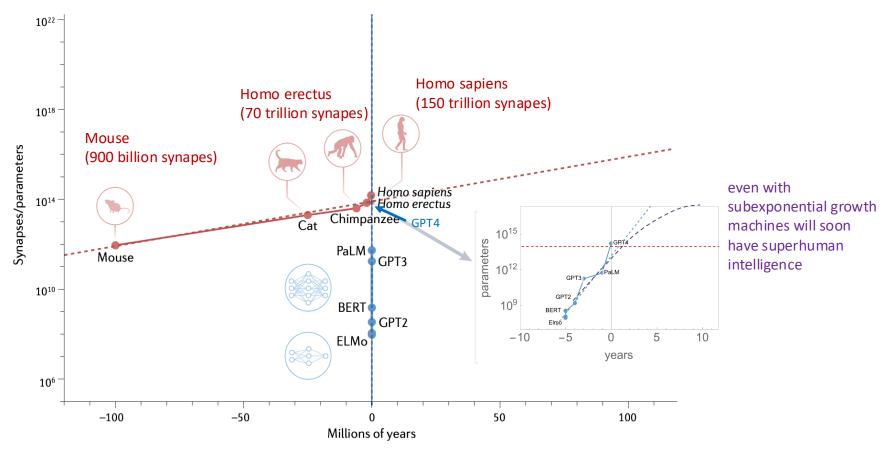


100 M years

Machine vs. Biological intelligence

- Machine intelligence grows by a factor of 10 in 1 year
- Biological intelligence grows by a factor of 2 in 20 million years

MDS, "Should artificial intelligence be interpretable to humans?" Nature reviews physics (2022)



- Both AI and biological intelligence grow exponentially
- Factor of 10⁷ difference in *exponent*
- Intersection time, when machines and biology have comparable "intellegence" is **now**

Can AI pass the turing test?

I.—COMPUTING MACHINERY AND INTELLIGENCE

BY A. M. TURING

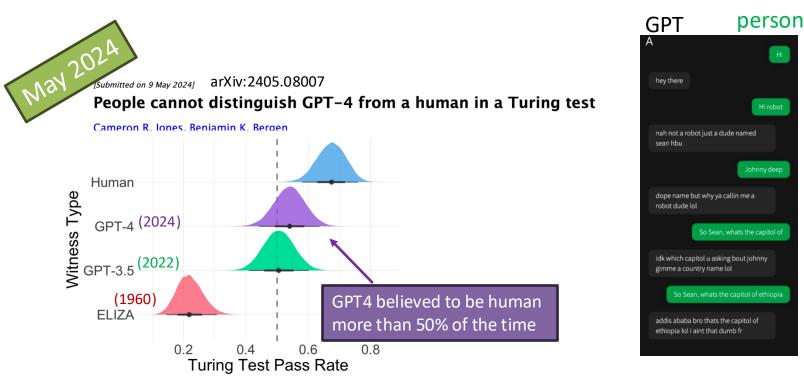
October, 1950 The Imitation Game.

1.

I PROPOSE to consider the question, 'Can machines think?' This should begin with definitions of the meaning of the terms 'machine' and 'think'. The definitions might be framed so as to reflect so far as



Turing Test: Using text only, can a person decide if it is talking with a machine or person?

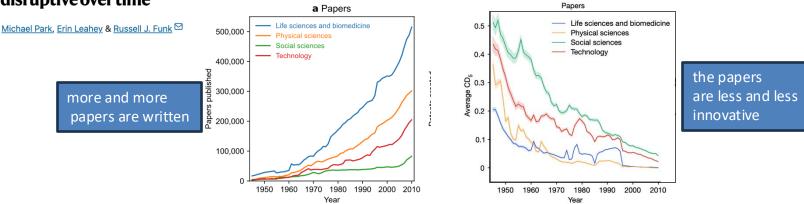


Theoretical particle physics may have stalled

nature

Article | Published: 04 January 2023

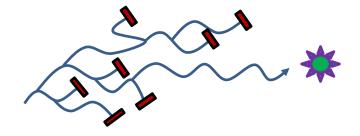
Papers and patents are becoming less disruptive over time

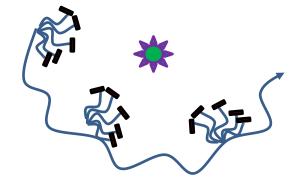


Maybe the problems are just too difficult

In the past, we made progress depsite many dead ends

Are we even making forward progress anymore?





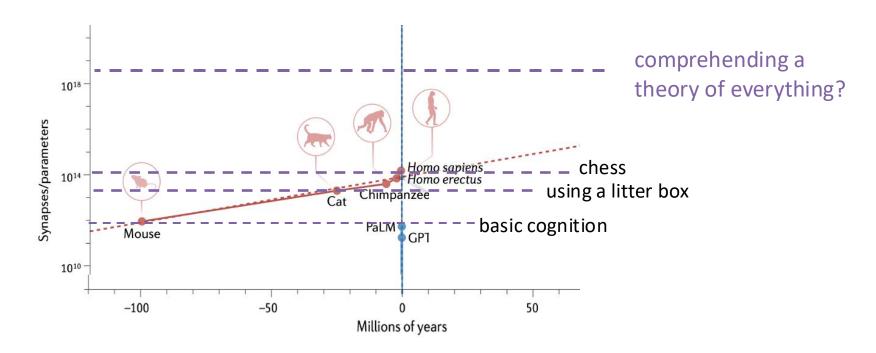
Maybe the problems are just too difficult (for us)



Could a cat ever learn to play chess?

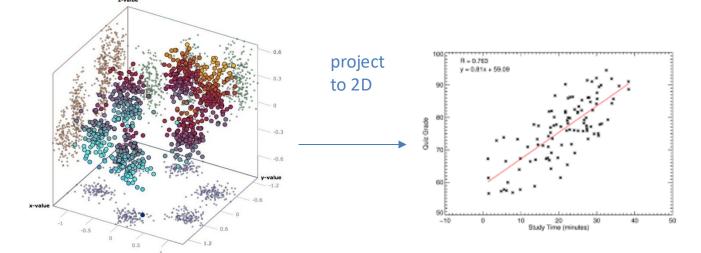
- Humans have limits too
- Why should Homo sapiens

be able to understand the theory of everything?



Humans are limited by biology

Humans like to "visualize"



Why do we do this? Because we have eyes

- 2D is not special to a machine.
- Machines can "visualize" in d dimensions

Eyes have **nothing to do** with fundamental physics!

Humans can only hold 5-9 concepts in working memory at once

• We like simple-looking equations

$$i\partial_t \psi = H\psi \quad i \not \partial \psi = m\psi \quad G_{\mu\nu} = \kappa T_{\mu\nu}$$

- Computer memory can handle much more than 5-9 concepts at once
- They can understand systems not governed by simple equations

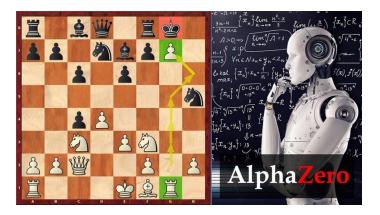
What do we need to progress further?

Current state-of-the art can solve textbook physics problems

- Trained on solved problems from books, physics.stackexchange, chegg etc.
- Books, chegg, etc. written by human beings who read books, chegg, etc.
- i.e. we generate our own training data

Next step: machines generate toy problems/training data

Alpha Zero: learns to solve chess problems by generating its own training data



Current LLMs

- can generate and solve problems
- user (human) feedback helps refine model
- LLMs can refine their own code!
 - GPT5 will be written largely by GPT 4

Language models are vey close to training themselves to be smarter

Superhuman intelligence

Suppose a machine understands the theory of everything but we don't

- e.g. can calculate electron mass from scratch
- e.g. can explain dark matter

Is this enough or do we need to understand it too?



• The authors of **Popular science books** understand the details; we just get the general idea

I don't understand the proof of Fermat's last theorem

- I'm glad that somebody does
- Does it matter that the person is human?

If a machine understands fundamental physics it can

- 1. Dumb it down so we can get the general idea
- 2. Find practical applications

Is this what we want? No. But maybe it's the best we will get. Because of AI, I am now optimistic for substantive progress in highenergy theory in my lifetime

Conclusions

- Machine learning is rapidly tranforming high energy physics
 - Current revolution in applications and advances are in "data science"
 - In hep-th and hep-ph problems are largely symbolic

1. How do we transition from data science to symbolic theoretical physics?

- It will get easier once we get started
 - Symbolic search problems (polylogarithms, spinor helicity)
 Dreparties of the Sametrix (unitarity)
 - Properties of the S-matrix (unitarity)
 - String Theory Vacuua

2. Generative AI is the future

- Short term: augmented intelligence
 - Machines help us organize information
 - Smooth transition to arXAIv: more and more AI input into arXiv papers
- Long term: artificial intelligence
 - Machines will suggest problems, solve problems: Chat Ph. D
 - Machines will dumb things down, so we can appreciate their work
 - Superhard problem in theoretical physics may finally be solved

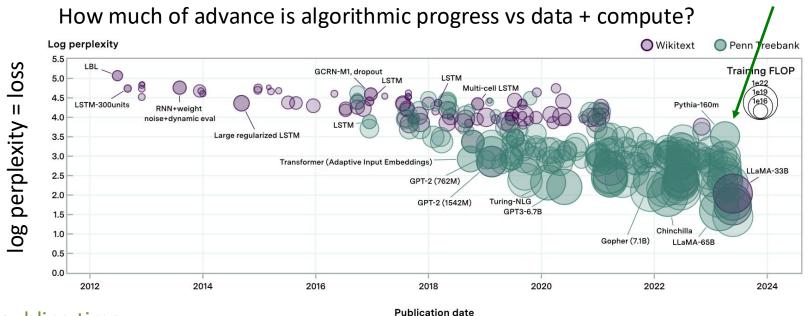
searching for simplicity

Will we run out of data/compute/energy?

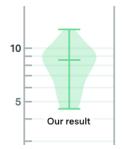
ALGORITHMIC PROGRESS IN LANGUAGE MODELS

Ho et al. arXiv:2403.05812

size is compute



doubling time



algorithmic doubling time = 6 to 14 months!

- controlling for data and network size
- algorithms will continue to get better, especially when written by AI