Unified theory of temporal causal discover

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HUN-REN Wigner RCP

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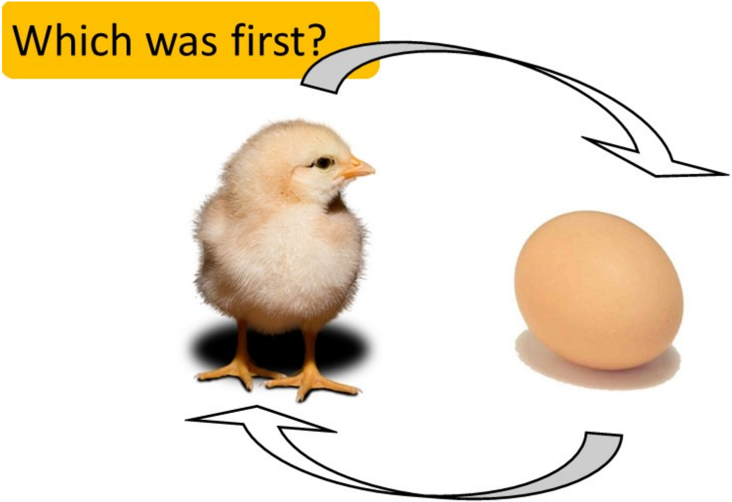
The question



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The unified theory

Fact

The unified theory handles deterministic and stochastic systems as well.

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related fork from us

Dr. Stippinger M., Dr. Zlatniczki Á, Dr. Benkő Zs., Dr. Stippinger M.,
Dr. Dr. Med. Fabó D., Dr. Dr. Med. Halász P., Dr. Dr. Med. Erőss L.,
Dr. Somogyvári Z, T.A.

The unified theory (Jakovác, T.)

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Deterministic dynamic systems

$$\frac{d^{o_x}}{dt^{o_x}}x = f(x)$$
$$\frac{d^{o_y}}{dt^{o_y}}y = g(x, y)$$

e.g.

$$f(x) = a_0 + a_1 x + a_2 \frac{dx}{dt} + \ldots + a_{o_X} \frac{d^{o_X - 1}x}{dt^{o_X - 1}}.$$

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Driving and observation

observation with τ time steps

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\begin{aligned}
x_{n+1} &= f(X^{n}_{n-o_X+1}) \\
y_{n+1} &= g(X^{n}_{n-o_X+1}, Y^{n}_{n-o_Y+1})
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$A_c = A_b^c = (A_c, A_{c-\tau}, A_{c-2\tau}...A_b)^T$ embedding.

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Stochastic dynamic systems

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where ξ, η i.i.d. independent from the whole past $\left(X_0^{n-o_X-1}, Y_0^{n-o_Y-1}\right)$ and from each other.

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When a DS is well defined (stoch.. or det.) ?

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$$\begin{aligned} x_{o_x+1} &= f\left(A_1^{o_X}, \xi_{o_x+1}\right) = f_1\left(A_{o_X}\right) \\ x_{n+1} &= f_n\left(X_n\right) = ... = f_n \circ f_{n-1} \circ \ldots \circ f_1(A_1^{o_x}) = F_n(A_{o_X}). \end{aligned}$$

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And the driven?

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Task: let us determine the df of the system.

How can we test? Let it be DDS or SDS

$$X_{k-o_x}^{k-1} = (x_{k-1}, \dots . x_{k-o_x})^T \mapsto x_k$$

fix o_x number of values of $X_{k-o_x}^{k-1}$ that determines the distribution, the evolution of the system, (DSD and SDS as well) (picture as recursion). Values past to the fixed elements do not influence the evolution.

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Summary (but not the end)

$$\max\{df_X, df_Y\} \leq df_{(X,Y)} \leq df_X + df_Y$$

Comparison

	WGC SDS	CCM DDS	DC DDAS	*df* both
$\nexists z, x \rightarrow y$	✓	✓	✓	✓ ✓
$\nexists z, x \leftarrow y$	✓	✓	✓	✓ ✓
$\nexists z, x \longleftrightarrow y$	✓	✓	✓	✓ ✓
$\exists z \left\{ x \nleftrightarrow y \right\}$	$\longleftrightarrow$	$\ldots$	✓	✓ ✓

Toy example

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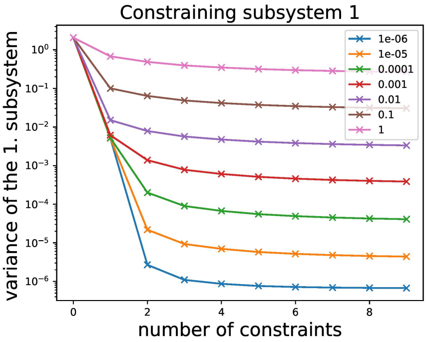
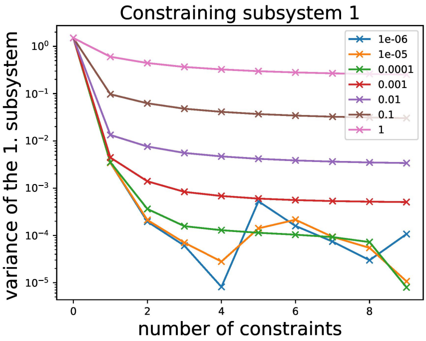
Let us fix x

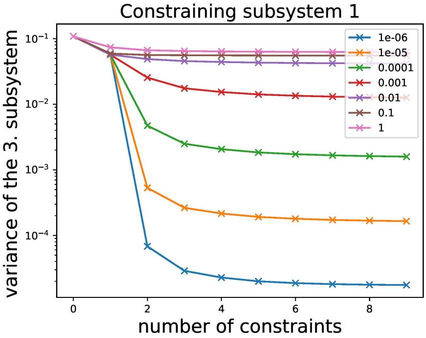




less stochastic more stochastic

remember, we set pairs of x!

We fix x and examine z



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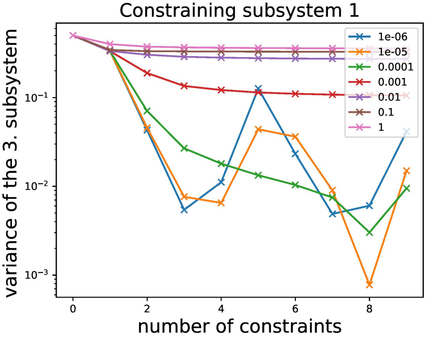

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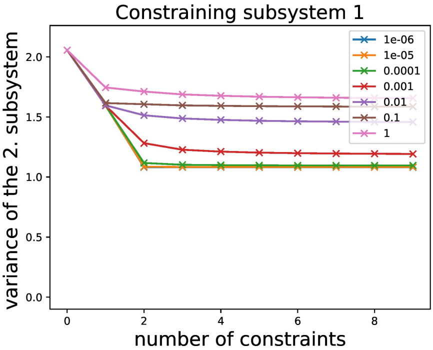

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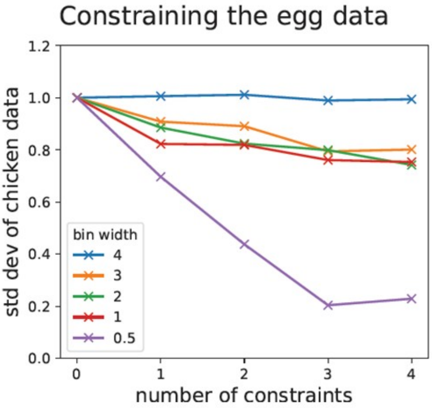
Chicken or egg?



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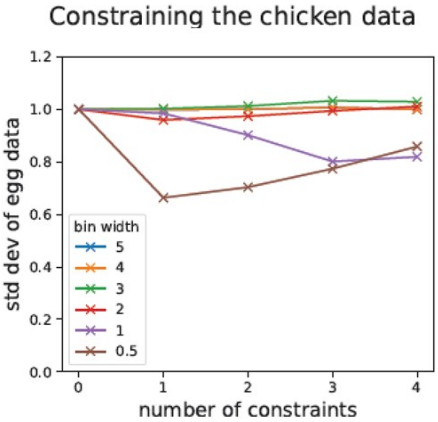

So we find assymmetry. $EGG \rightarrow Chicken$

Thanks
Q&A?