Neutrino oscillations in Finite Time Path Out-of-equilibrium Thermal Field Theory

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Non-perturbative in strongly interacting quantum many-body systems

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Overview

Introduction

* While wishing to describe neutrinos phenomenologically as good as possible in the chosen model framework, this is secondary.

* Presently, the primary task is demonstrating that the **Finite-Time-Path Field Theory** (**FTPFT**) is an adequate tool for calculating neutrino oscillations.

* **FTPFT** is a variant of the **real time formalism**, which is suitable also for out-of-equilibrium problems, but is quite complicated
$\Longrightarrow$ **Neutrino modeling must be simple enough**

- **We apply FTPFT using a mass-mixing Lagrangian involving the correct Dirac spin and chirality structure and a Pontecorvo-Maki-Nakagawa-Sakata (PMNS)-like mixing matrix.**
- **The Dyson-Schwinger equations transform propagators of the input free (massless) neutrino flavor eigenstates into a linear combination of the propagators of oscillating (massive) neutrinos.**
- **The results are consistent with the predictions of the PMNS matrix**

PMNS Theory of Neutrino Oscillation

The three flavor neutrino states $|\nu_\alpha\rangle$ interact weakly and are mixed to three different superposition of the three neutrino states $|\nu_i\rangle$ of definite masses m_i,

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i}\,|\nu_i\rangle\ , \qquad |\nu_i\rangle = \sum_\alpha U^*_{\alpha i}\,|\nu_\alpha\rangle \qquad (i = 1, 2, 3)\,, \quad (1)$$

where $\alpha = e, \mu, \tau$ are the lepton flavor indices. Indices i are of the masses m_i.

$U_{\alpha i}$ = elements of the 3×3 PMNS-matrix, analogous to the CKM matrix describing the mixing of quark flavors.

The matrix elements $U_{\alpha i}$ contain, most notably, sines and cosines of the mixing angles, $c_{ij} = \cos\theta_{ij}$, and $s_{ij} = \sin\theta_{ij}$.
(Also, there are phase factors α_1 and α_2, relevant only if neutrinos are Majorana particles (*i.e.*, if the neutrino is identical to its antineutrino). Another phase factor δ would reflect the degree of CP symmetry violation with neutrinos, which has not yet been observed experimentally.)

To describe antineutrinos one uses the complex-conjugate matrices $U_{\alpha i} \leftrightarrow U^*_{\alpha i}$.

Through weak processes, neutrinos are emitted and absorbed in the flavor states $|\nu_\alpha\rangle$. But, they travel as the mass eigenstates $|\nu_i\rangle$.

Neutrino mixing

In the ultrarelativistic limit, $|\vec{p}_i| = p_i \gg m_i$. Thus, $p_i \approx E$, the neutrino energy in the limit $m_i \to 0$.

$$\text{Thus, } \forall\, i\,, \qquad E_i \simeq p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}\,. \qquad \text{and} \qquad t \approx L\,, \tag{2}$$

t is the time since the beginning of ν-evolution and L is the distance traveled.

In the process of measurement the neutrino is projected back to the flavor states $|\nu_\alpha\rangle$. The probability that the initial neutrino with flavor α will be later detected as having flavor β is defined as

$$P_{\alpha\to\beta} = |\langle \nu_\beta(L)|\,\nu_\alpha\rangle|^2 = \left|\sum_i U_{\alpha i}^* U_{\beta i} e^{-\mathrm{i}\frac{m_i^2 L}{2E}}\right|^2 .$$

Squaring, rearranging and using $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, this gives

$$\begin{aligned} P_{\alpha\to\beta} = \delta_{\alpha\beta} \quad &- \quad 4\sum_{i>j} \mathrm{Re}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right) \\ &+ \quad 2\sum_{i>j} \mathrm{Im}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right). \end{aligned} \tag{3}$$

Dynamical model for neutrino oscillation

We start with the free Lagrangian $\mathcal{L}_0$ of massless neutrinos with three flavors;

however, we mix them dynamically through an interaction Lagrangian $\mathcal{L}_I$,

since in addition to the standard weak interaction term $\mathcal{L}_W$, it also has the term $\mathcal{L}_{Mix}$ which contains a Pontecorvo-Maki-Nakagawa-Sakata (PMNS)-like matrix involving neutrino masses:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \qquad \text{where} \qquad \mathcal{L}_0 = \sum_{\alpha} \bar{\nu}_\alpha i \partial\!\!\!/ \nu_\alpha \qquad \text{and} \qquad \mathcal{L}_I = \mathcal{L}_W + \mathcal{L}_{Mix}$$

$$\mathcal{L}_{Mix} = \sum_{\alpha,i} \bar{\nu}_\alpha U^*_{\alpha i} M_{ij} U_{\beta j} \nu_\beta + antineutrinos \quad \text{with} \quad M = diag(m_1\ m_2\ m_3)$$

The term $\mathcal{L}_W$ is of course crucial for the processes of creation and detection of flavor neutrinos. But, we adopt the approximation where the neutrino mixing is fully due to the term $\mathcal{L}_{Mix}$. That is, due to the weakness of weak interactions, we neglect the residual influence of $\mathcal{L}_W$ on the neutrino masses and mixing.

Solving **nonperurbatively** the FTPFT Dyson-Schwinger equations (DSE) gives "dressed" propagators of massive, oscillating neutrinos – in the approximation where this dressing (i.e., their masses/self-energies) is due to $\mathcal{L}_{Mix}$ only.

FTPFT ∼ Schwinger-Keldysh Closed Time Path Formalism

F. Gelis, *A Stroll Through Quantum Fields*, p. 81: “ ... it is useful to imagine that the time axis is in fact a contour $\mathcal{C}$ made of two branches labeled $+$ and $-$ running parallel to the real axis, as illustrated in figure 1.4. This contour is oriented, with

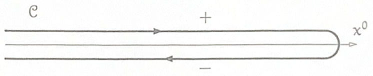

Figure 1.4: Time contour in the Schwinger-Keldysh formalism.

the $+$ branch running in the direction of increasing time, followed by the $-$ branch running in the direction of decreasing time. Then, it is convenient to introduce a *path ordering*, denoted by P and defined as a standard ordering along the contour $\mathcal{C}$. In more detail, one has

$$\mathrm{P}\,A(x)B(y) = \begin{cases} \mathrm{T}\,A(x)B(y) & \text{if} \quad x^0, y^0 \in \mathcal{C}_+ \,, \\ \overline{\mathrm{T}}\,A(x)B(y) & \text{if} \quad x^0, y^0 \in \mathcal{C}_- \,, \\ A(x)B(y) & \text{if} \quad x^0 \in \mathcal{C}_- \,,\; y^0 \in \mathcal{C}_+ \,, \\ B(y)A(x) & \text{if} \quad x^0 \in \mathcal{C}_+ \,,\; y^0 \in \mathcal{C}_- \,. \end{cases} \tag{1.362}$$

...”

Same with $G_{ab}(x,y)$, where $a,b = +,-$ $\Rightarrow$ 4 propagators $G_{++}, G_{+-}, G_{-+}, G_{--}$

Keldyish basis

These 4 propagators S_{ab} $(a, b = +, -)$, or rather, components of the 2×2 **matrix propagator** S, are not independent, as $S_{++} + S_{--} = S_{+-} + S_{-+}$.

Recouple into 3 independent ones, retarded S_R, advanced S_A and Keldysh S_K.

The Keldysh (or "statistical") component contains information about particle distributions $n_f(\vec{p})$, and even quantum/statistical fluctuations, and noise.

Keldysh propagator carries the information which, after the equal time limit, gives the number and momentum-distribution of all types of flavor neutrinos measured at the time t. After the time t, the number of oscillating neutrinos is expressed through the average equal-time limit of the resummed Keldysh component $\tilde{S}_K$ of the neutrino propagator

$$1 - \langle N_{\beta,\vec{p}}(t)\rangle = \frac{1}{2\pi}\,[\lim_{0<\Delta\to 0} + \lim_{0>\Delta\to 0}] \int dp_0\, e^{-ip_0\Delta}\; Tr[\,\frac{\gamma_0}{4}\tilde{S}_{\beta,K,t}(p)\frac{1-\gamma_5}{2}\,]$$

where $\Delta = s_{01} - s_{02}$, $X_0 = (s_{01} + s_{02})/2 = t$. This expression takes into account that the initial condition contains only flavor neutrinos of the type f, but not antineutrinos, and we calculate number of flavor neutrinos of the type β.

Solutions for R, A and K propagators of neutrinos

Solutions are the re-summed (through DSE) Green's functions $\tilde{S}$.
Schematically, $\tilde{S} = [S^{-1} - \Sigma]^{-1}$:

- the resummed retarded propagator component

$$\tilde{S}_{\beta,\alpha,R}(p) \;=\; (1 - i\Sigma * S_R)^{-1}_{\beta,\eta} S_{\eta,\alpha,R}(p) = \sum_i U^*_{\beta i} \frac{-i(m_i + \not{p})}{p^2 - m_i^2 + ip_0\epsilon} U_{\alpha i} \tag{4}$$

- the resummed advanced propagator component

$$\tilde{S}_{\beta,\alpha,A}(p) = (1 - i\Sigma * S_A)^{-1}_{\beta,\eta} S_{\eta,\alpha,A}(p) = \sum_i U^*_{\beta i} \frac{-i(m_i + \not{p})}{p^2 - m_i^2 - ip_0\epsilon} U_{\alpha i}\,. \tag{5}$$

Inserting these R & A components of dressed $\tilde{S}$ into DSE of dressed K-component enables finding the resummed Keldysh component.

The resumed Keldysh component, **just a part**, as too big for one slide

$$\tilde{S}_{\beta,\alpha,K} = -\delta_{f,\alpha}\sum_i U^*_{\beta i}U_{\alpha i}\frac{i}{p^2-m_i^2-ip_0\epsilon}$$

$$\times\{[1-2n_f(\vec{p})]\frac{p_0+|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|-\vec{\gamma}\vec{p})-[1-2n_{\bar{f}}(-\vec{p})]\frac{p_0-|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|+\vec{\gamma}\vec{p})\}\frac{1-\gamma_5}{2}$$

$$+\delta_{\beta,f}\sum_i U^*_{\beta,i}U_{\alpha,i}\frac{i}{p^2-m_i^2+ip_0\epsilon}$$

$$\times\{[1-2n_f(\vec{p})]\frac{p_0+|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|-\vec{\gamma}\vec{p})-[1-2n_{\bar{f}}(-\vec{p})]\frac{p_0-|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|+\vec{\gamma}\vec{p})\}\frac{1-\gamma_5}{2}$$

$$+i\delta_{f,\alpha}\sum_{i,j}U^*_{\beta,i}\frac{p^2+m_i\not{p}}{(p^2-m_i^2+ip_0\epsilon)(p^2+ip_0\epsilon)}U_{\alpha,i}[-im_i(-i\not{p})$$

$$*\{[1-2n_f(\vec{p})]\frac{p_0+|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|-\vec{\gamma}\vec{p})$$

$$-[1-2n_{\bar{f}}(-\vec{p})]\frac{p_0-|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|+\vec{\gamma}\vec{p})\}\frac{1-\gamma_5}{2}$$

$$+\{[1-2n_f(\vec{p})]\frac{p_0+|\vec{p}|}{2|\vec{p}|}(\gamma_0|\vec{p}|-\vec{\gamma}\vec{p}) \qquad + (\ldots\text{the rest}\ldots)$$

Final result of the FTPFT calculation

At time t, total number of ν's of $flavor$ β stemming from the initial $f = \alpha$, is

$$\langle N_{\alpha\to\beta,\vec{p}}(t)\rangle = \delta_{\alpha\beta}\, n_\alpha(\vec{p}) \; - \; 4\, n_\alpha(\vec{p}) \sum_{i>j} Re(U^*_{\beta i} U_{\alpha i} U^*_{\alpha j} U_{\beta j}) \sin^2 \frac{m_i^2 - m_j^2}{4|\vec{p}|} t$$

$$+ \; 2\, n_\alpha(\vec{p}) \sum_{i>j} Im(U^*_{\beta j} U_{\alpha j} U^*_{\alpha i} U_{\beta i}) \sin \frac{m_i^2 - m_j^2}{2|\vec{p}|} t\,. \quad (7)$$

* features of Eq. (7):

a) It is identical to the standard PMNS expression (3), since division by the initial distribution of the number of particles n_f recasts (7) in terms of probability, like in (3), and since $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, the arguments of sines are the same. $\Rightarrow$ With the same presently available inputs, our result (7) would give the same numerical results as, for example, Yasuda's review.

b) If we sum over β, the oscillating contribution vanishes, reflecting the conservation of the total neutrino number within the realm of chiral neutrinos; sterile neutrinos are not involved! Our conclusion is valid also for low energy neutrino beams, as can be seen from Eqs. (31)-(34) in the paper by I. Dadić and D. Klabučar, Symmetry **15**, no.11, 1970 (2023) doi:10.3390/sym15111970 [arXiv:2311.11875 [hep-ph]].

c) The results are also valid for moderate energies, but then one has to skip the ultrarelativistic simplification (2) and include all contributions.

Summary and Conclusions

The Finite-Time-Path Field Theory (FTPFT), originally designed to deal with out-of-equilibrium many body statistical ensembles, was applied to a problem in particle physics. We have demonstrated that FTPFT is an appropriate tool for the treatment of neutrino oscillations.

Within a simple model with the interaction Lagrangian containing the term $\mathcal{L}_{Mix}$ built as mass-mixing via PMNS-matrix, with built-in Dirac-spinor and chirality structure, - ve cal Z

The FTPFT approach successfully passes the test of neutrino oscillations in the sense that we demonstrate how one does calculations of these oscillations using this approach. It reproduces the standard PMNS results in the present model, which introduces the neutrino masses precisely through the PMNS matrix in the mixing part of the model Lagrangian. That is, the present approach does not address the issue of the origin of neutrino masses, and in general does not provide answers about dynamics and physics of processes, but can use the existing knowledge as an input in the form of masses, matrices, self-energies, *etc*.

Outlook

The application of the FTPFT approach to neutrino oscillations shows that it is an interesting candidate for a complementary tool to the S-matrix, which is formulated for infinite times and involves switching interactions on and off adiabatically.

In the case of phenomena where finite times are essential, such as the presently pertinent neutrino oscillations, but also the oscillation of kaons and B and D mesons, decays and symmetry violations, one has been using largely heuristic methods like elementary quantum mechanics in the PMNS approach and Gell-Mann–Pais approach for kaons. In such cases, the FTPFT approach is obviously a candidate for a more rigorous description.