Instantons and the Anomalous $U(1)_A$ Symmetry in High Temperature QCD

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Symmetries of QCD and their realization

- $m_u \approx m_d \approx 0$
- Symmetries: $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$
  - $U(1)_A$ anomalous
  - $SU(2)_L \times SU(2)_R \to SU(2)_V$ spontaneously broken below T_c
- Order parameter of the symmetry breaking (Banks-Casher formula):

$$\langle \bar{\psi}\psi \rangle \propto \frac{1}{V}\sum_i \frac{1}{\lambda_i + m} \propto \int_{-\Lambda}^{\Lambda} d\lambda \, \frac{m}{\lambda^2 + m^2} \rho(\lambda) \xrightarrow[m \to 0]{} \rho(0)$$

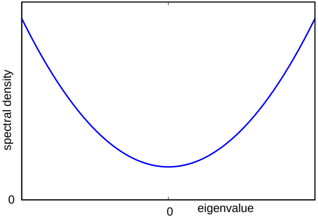
λ_i: eigenvalues of the Dirac operator, $\rho(\lambda)$: its spectral density

The finite temperature transition

Standard picture

Below T_c

- Chiral symmetry broken
- Order parameter:
  $\rho(0) \neq 0$



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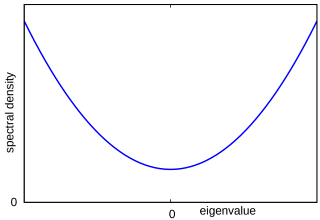
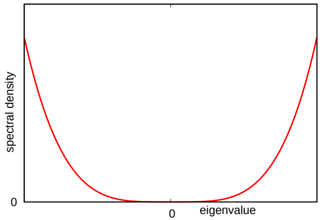


Above T_c

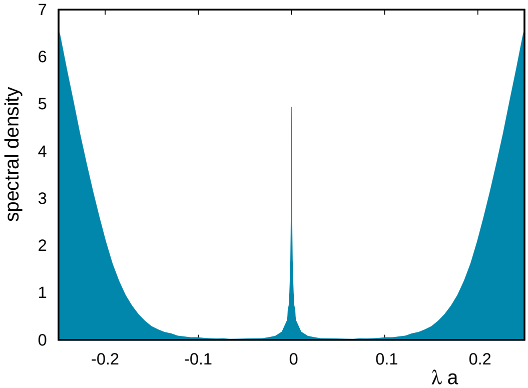
- Chiral symmetry restored
- Order parameter $\rho(0) = 0$
- (Pseudo)gap
  lowest Matsubara mode



spectral density at 0 $\Longleftrightarrow$ realization of chiral symmetry

Spectral density at $T = 1.1\,T_c$ on the lattice

quenched (quark back reaction omitted)



$$Z = \int \mathscr{D}U \prod_f \det(D[U] + m_f) \cdot e^{-S_g[U]}$$

Peak at zero in the spectral density!

Edwards et al. PRD 61 (2000); Alexandru & Horvath, PRD 92 (2015); 2404.12298; Kaczmarek, Mazur, Sharma, PRD 104 (2021) 2021

Questions

- Why is there a peak at zero?
- How is it suppressed if the quark determinant is included?
- How does the peak influence chiral symmetry as $m \to 0$?

Instantons $\rightarrow$ zero eigenvalues of $D(A)$

- (Anti)instanton
  $\rightarrow$ zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode
- High T:
  large instantons "squeezed out" in the temporal direction
  $\rightarrow$ dilute gas of instantons and antiinstantons
- Zero modes exponentially localized:

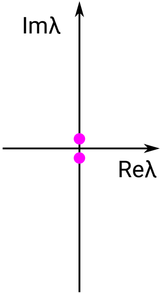
$$\psi(r) \propto e^{-\pi T r}$$

Instanton-antiinstanton pair

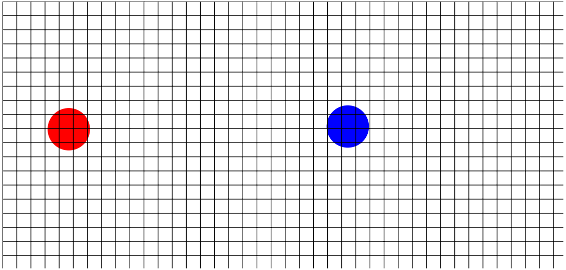
The Dirac operator in the subspace of zero modes

$$D(A) = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix} \qquad w \propto e^{-\pi Tr}$$

Spectrum of $D(A)$



Instanton and antiinstanton



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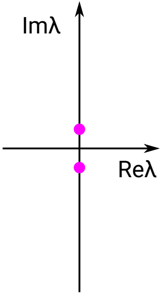

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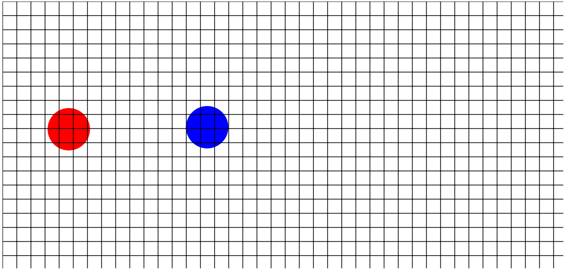

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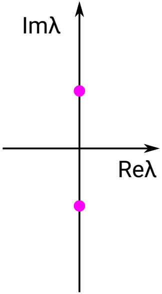

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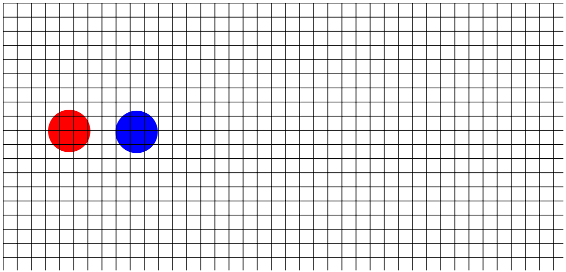

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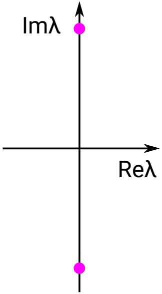

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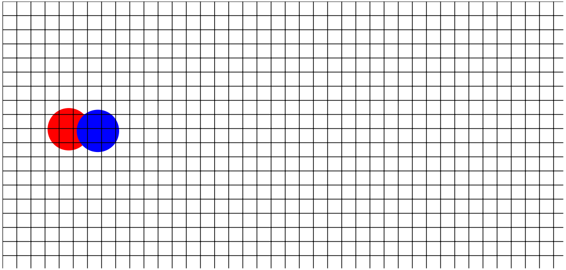
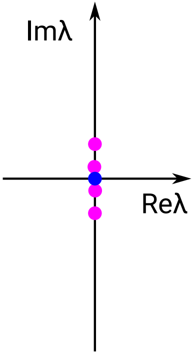


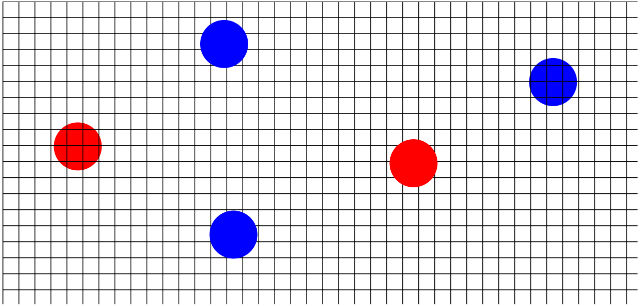
Spectrum of $D(A)$ in dilute gas of instantons

The Dirac operator in the subspace of zero modes

Spectrum of $D(A)$



Instantons and antiinstantons



n_i instantons n_a antiinstantons

$\rightarrow$ $|n_i - n_a|$ exact zero modes + mixing near zero modes

Dirac operator in the subspace of zero modes (ZMZ)

Work by E.V. Shuryak, J.J.M. Verbaarschot, T. Schäfer (1990-2000)...

- Given n_i instantons, n_a antiinstantons in 3d box of size L^3
- Construct $(n_\text{i}+n_\text{a})\times(n_\text{i}+n_\text{a})$ matrix:

$$D = \overbrace{}^{n_\text{i}}\overbrace{}^{n_\text{a}} \quad D=\left(\begin{array}{c|c} 0 & iW \\ \hline iW^\dagger & 0 \end{array}\right)$$

- $w_{ij} = A\cdot\exp(-\pi T\cdot r_{ij})$ $\qquad r_{ij}$ is the distance of instanton i and antiinstanton j

Random matrix model of $D(A)$ in the zero mode zone

- How to choose instanton numbers (n_i, n_a) and locations?

- Quenched lattice $T > 1.05\,T_c \quad \rightarrow \quad$ free instanton gas

  Bonati et al. PRL 110 (2013); Vig R. & TGK, PRD 103 (2021)

  - n_i and n_a independent identical Poisson-distributed

  $$p(n_i, n_a) = e^{-\chi V} \cdot \frac{(\chi V/2)^{n_i}}{n_i!} \cdot \frac{(\chi V/2)^{n_a}}{n_a!}$$

  χ is the topological susceptibility

  - Locations random (uniform)

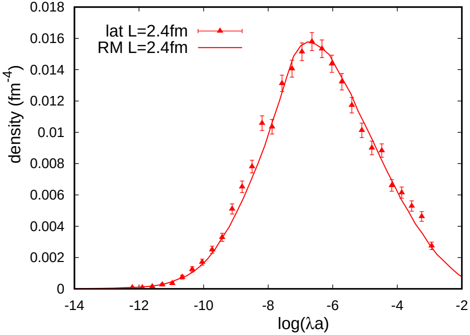
- $\rightarrow \quad$ D(A) in quenched QCD $\quad \Leftrightarrow \quad$ ensemble of random matrices

Fit parameters to quenched lattice Dirac spectrum

$T = 1.1\,T_c$ overlap Dirac spectrum

- Two parameters:
  - χ – topological susceptibility: from exact zero modes $\rightarrow \chi = \langle Q^2 \rangle / V$
  - A – prefactor of the exponential mixing between zero modes
- Fit A to distribution of Dirac eigenvalues (lowest eigenvalue)

$L = 2.4$fm fit



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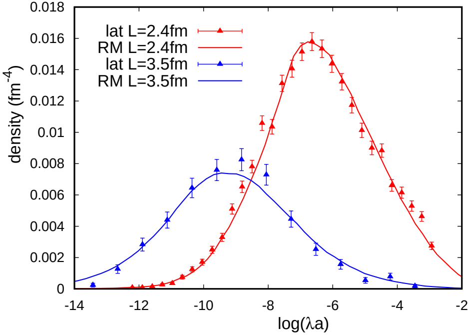
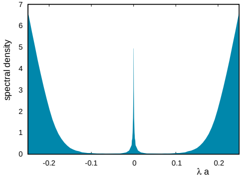
$L = 2.4$fm fit

$L = 3.5$fm prediction



Random matrix model of full QCD zero mode zone

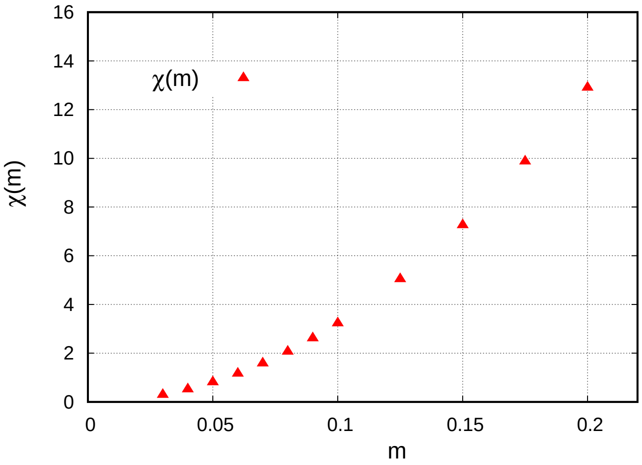
- Include $\det(D+m)^{N_f}$ in Boltzmann weight
- $\det(D+m) = \prod_{\text{zmz}}(\lambda_i+m) \times \prod_{\text{bulk}}(\lambda_i+m)$
- Bulk weakly correlated with zero mode zone
- Approximate det with $\prod_{\text{zmz}}(\lambda_i+m)$
- Consistently included in RM model:



$$P(n_i, n_a) = \underbrace{e^{-\chi_0 V} \frac{1}{n_i!} \frac{1}{n_a!} \left(\frac{\chi_0 V}{2}\right)^{n_i+n_a}}_{\text{free instanton gas with random locations}} \times \det(D+m)^{N_f}$$

Random matrix simulation: results for $N_f = 2$

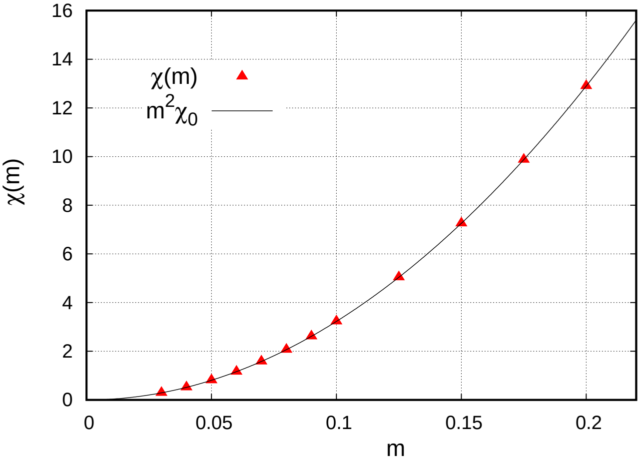
Topological susceptibility:



Random matrix simulation: results for $N_f = 2$

Topological susceptibility: $\chi(m) = m^2 \chi_0$ not a fit!

↑ quenched susceptibility



Explanation: free instanton gas

- Quark determinant for n_i instantons and n_a antiinstantons:

$$\det(D+m)^{N_f} = \prod_{n_\text{i},n_\text{a}} (\lambda_i + m)^{N_f} \approx m^{N_f(n_\text{i}+n_\text{a})}$$

$$\text{if } |\lambda_i| \ll m$$

- Reweighting depends on number of topological objects, not on their type or location

$$P(n_\text{i}, n_\text{a}) \propto \left(\frac{\chi_0 V}{2}\right)^{n_\text{i}+n_\text{a}} \times \det(D+m)^{N_\text{f}} \approx \left(\frac{m^{N_\text{f}}\chi_0 V}{2}\right)^{n_\text{i}+n_\text{a}}$$

- Free gas, but susceptibility suppressed as $\chi_0 \to m^{N_\text{f}} \chi_0$
- As $m \to 0$ instanton gas more dilute $\Rightarrow$ $|\lambda_i|$ smaller
- Even in the chiral limit $|\lambda_i| \ll m \implies$ free instanton gas & singular spike

“Banks-Casher” for singular spectral density

$$\langle\bar{\psi}\psi\rangle \;\propto\; \left\langle \sum_i \frac{m}{m^2+\lambda_i^2} \right\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m} \;=\; m^{N_\mathrm{f}-1}\chi_0 V$$

$$|\lambda_i| \ll m$$

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$$|\lambda_i| \ll m$$

$U(1)_A$ breaking susceptibility $\chi_\pi - \chi_\delta$

$$\langle\sum_i \frac{m^2}{(m^2+\lambda_i^2)^2}\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m^2} \;=\; m^{N_f-2}\chi_0 V$$

$$\rightarrow \quad \lim_{m\rightarrow 0}(\chi_\pi - \chi_\delta) \neq 0 \qquad \text{for } N_f = 2$$

Conclusions

- At high T non-interacting degrees of freedom: free "instantons"
- Dirac spectral density has singular peak at zero

  at any finite T, for any nonzero quark mass
- Chiral symmetry restoration nontrivial $\rightarrow$ $N_f \leq 2$: $U(1)_A$ anomaly remains
- Spontaneously broken $SU(2)$ restored
- Chiral limit with N_f degenerate light quarks:
  - $\langle \bar{\psi}\psi \rangle \propto m^{N_f - 1}$
  - $\chi_\pi - \chi_\delta \propto m^{N_f - 2}$

agrees with small m expansion of the free energy

Kanazawa and Yamamoto, PRD 91 (2015), JHEP 01 (2016)

also with exact constraints on the Dirac spectrum

$\rightarrow$ Matteo's talk