Dirac spectrum in the chirally symmetric phase of QCD and the fate of $\mathrm{U}(1)_A$ symmetry

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QCD in the chiral limit

Up and down quark very light $\Rightarrow$ QCD close to $N_f = 2$ chiral limit $m \to 0$, chiral symmetry $\mathrm{U}(2)_L \times \mathrm{U}(2)_R = \underbrace{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}_{\to \mathrm{SU}(2)_V \text{ @low T}} \times \underbrace{\mathrm{U}(1)_A}_{\text{anomalous}} \times \mathrm{U}(1)_V$

Open questions about the chiral limit: nature of symmetry-restoring transition and fate of $\mathrm{U}(1)_A$ in the symmetric phase

Effective-Lagrangian analysis: [Pisarski, Wilczek (1984), Pelissetto, Vicari (2013)]

$\mathrm{U}(1)_A$ broken $\Rightarrow$ 2nd order $\mathrm{O}(4)$ class
$\mathrm{U}(1)_A$ restored $\Rightarrow$ 1st order, or 2nd order $\mathrm{U}(2)_L \times \mathrm{U}(2)_R / \mathrm{U}(2)_V$ class

Other approaches lead to different scenarios [Fejős (2022), Fejős, Hatsuda (2024), Bernhardt and Fischer (2023), Pisarski, Rennecke (2024), Giacosa *et al.* (2025)]

Numerical lattice results contradictory [HotQCD (2019), JLQCD (2021)]

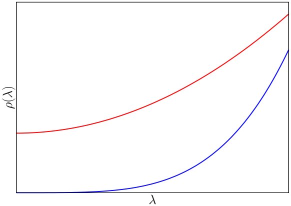
Spectrum and eigenvectors of the Dirac operator encode quark dynamics, constraints from $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ restoration can give us first-principles information about fate of $\mathrm{U}(1)_A$ [Cohen (1996), Aoki, Fukaya, Taniguchi (2012)]

Chiral symmetry restoration and the Dirac spectrum

$\mathrm{SU}\!\!\!/(2)_A$ if density of near-zero Dirac modes $\rho(0^+;0) \neq 0$ [Banks, Casher (1980)]

$$-\langle\bar{\psi}\psi\rangle = \int d\lambda \, \frac{m}{\lambda^2 + m^2} \, \rho(\lambda; m)$$

$$\rho(\lambda; m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \left\langle \sum_{n,\lambda_n \neq 0} \delta(\lambda - \lambda_n) \right\rangle$$



@ low T: $\langle\bar{\psi}\psi\rangle_{m\to 0} \neq 0$, expect $\rho(0^+; m) \neq 0$ – get it

(actually log divergence [Osborn, Toublan, Verbaarschot (1999)])

@ high T: $\langle\bar{\psi}\psi\rangle_{m\to 0} = 0$, expect $\rho(0^+; m) = 0$

[Edwards *et al.* (1999), Cossu *et al.* (2013), Alexandru, Horváth (2015), Dick *et al.* (2015), Brandt *et al.* (2016), Tomiya *et al.* (2017), HotQCD (2019), JLQCD (2021), Vig, Kovács (2021), Kaczmarek *et al.* (2021), Meng *et al.* (2023), Alexandru *et al.* (2024)]

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- What does it do to $\mathrm{U}(1)_A$?

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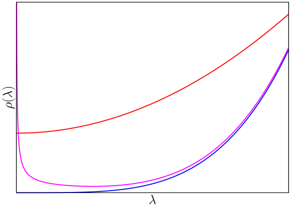

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Chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$

How does one characterise $\mathrm{SU}(2)_A$ restoration?

assumptions	conclusions
• observables analytic in m^2 $\langle \delta_A O \rangle = 0$ ρ power series near $\lambda = 0$, or $\sim \lambda^\alpha$, $\alpha > 0$	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration [Cohen (1996), Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]
• thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\not\Rightarrow$ $\mathrm{U}(1)_A$ restoration, $\mathrm{U}(1)_A$ broken by topological effects [Evans, Hsu, Schwetz (1996), Lee, Hatsuda (1996)]
• free energy analytic in m^2 thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration, unless $\rho \sim m^2\delta(\lambda)$ [Azcoiti (2023)]

What assumptions follow from first principles?

Symmetry restoration conditions

Local field theory: symmetry restored
$\Rightarrow$ local correlators related by symmetry become equal

$$\lim_{m\to 0} \left(\langle F_1'(x_1) \ldots F_n'(x_n) \rangle - \langle F_1(x_1) \ldots F_n(x_n) \rangle \right) = 0$$

No massless excitations, finite corr. length expected in restored phase
$\Rightarrow$ susceptibilities related by symmetry become equal

$$\lim_{m\to 0} \lim_{\mathrm{V}\to\infty} \left(\frac{1}{\mathrm{V}} \langle \mathcal{F}_1' \ldots \mathcal{F}_n' \rangle_c - \frac{1}{\mathrm{V}} \langle \mathcal{F}_1 \ldots \mathcal{F}_n \rangle_c \right) = 0 \qquad \mathcal{F}_i = \int d^4x \, F_i(x)$$

Gauge fields unaffected by chiral transformations
$\Rightarrow$ correlators involving nonlocal functionals $\mathcal{G}[A]$ (e.g., spectral density) become equal if related by symmetry

$$\lim_{m\to 0} \lim_{\mathrm{V}\to\infty} \left(\frac{1}{\mathrm{V}} \langle \mathcal{F}_1' \ldots \mathcal{F}_n' \, \mathcal{G}_1 \ldots \mathcal{G}_l \rangle_c - \frac{1}{\mathrm{V}} \langle \mathcal{F}_1 \ldots \mathcal{F}_n \, \mathcal{G}_1 \ldots \mathcal{G}_l \rangle_c \right) = 0$$

Scalar and pseudoscalar bilinears

Irreducible $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ multiplets

$$O_{\mathcal{S}} = \begin{pmatrix} S \\ i\vec{P} \end{pmatrix} \qquad \begin{array}{l} S = \bar{\psi}\psi \\ \vec{P} = \bar{\psi}\vec{\sigma}\gamma_5\psi \end{array} \qquad O_{\mathcal{P}} = \begin{pmatrix} iP \\ -\vec{S} \end{pmatrix} \qquad \begin{array}{l} P = \bar{\psi}\gamma_5\psi \\ \vec{S} = \bar{\psi}\vec{\sigma}\psi \end{array}$$

Under chiral transformations

$$O_{\mathcal{S},\mathcal{P}} \xrightarrow{\mathcal{U}} \mathcal{R}(\mathcal{U}) O_{\mathcal{S},\mathcal{P}} \qquad \mathcal{R} \in \mathrm{SO}(4)$$

Corresponding susceptibilities expressible in terms of the spectrum of D only, constraints result from symmetry restoration

Regularise theory on the lattice, chiral symmetry problematic but GW fermions [Ginsparg, Wilson (1982)] have exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ [Lüscher (1998)]

Generating function

Include source terms for bilinears in the partition function

$$\mathcal{Z}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) = \int DA \int D\psi D\bar{\psi}\, e^{-S_{\rm eff}[A] - \bar{\psi}(\not{D}[A]+m)\psi - K[\psi,\bar{\psi},A;\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}}]}$$

$$S_{\rm eff} = \text{gauge and massive fermion contributions}$$

$$K[\psi, \bar{\psi}, A; \mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}] = j_S S + i\vec{\jmath}_P \cdot \vec{P} + i j_P P - \vec{\jmath}_S \cdot \vec{S} = \mathrm{J}_{\mathcal{S}} \cdot O_{\mathcal{S}} + \mathrm{J}_{\mathcal{P}} \cdot O_{\mathcal{P}}$$

Generating function of scalar and pseudoscalar susceptibilities

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \ln \mathcal{Z}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m)$$

Under chiral transformations

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) \xrightarrow{\mathcal{U}^{-1}} \mathcal{W}(\mathcal{R}\mathrm{J}_{\mathcal{S}}, \mathcal{R}\mathrm{J}_{\mathcal{P}}; m)$$

Symmetry restoration condition:

$$\lim_{m\to 0} \Big[\mathcal{W}(\mathcal{R}\mathrm{J}_{\mathcal{S}}, \mathcal{R}\mathrm{J}_{\mathcal{P}}; m) - \mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m)\Big] = 0 \qquad \forall \mathcal{R} \in \mathrm{SO}(4)$$

Symmetry restoration in scalar/pseudoscalar sector

$\mathcal{Z}$ function of $j_S + m$ only + chiral symmetry of exactly massless theory $\Rightarrow$

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) = \hat{\mathcal{W}}(m^2 + \overbrace{2mj_S + \mathrm{J}_{\mathcal{S}}^2}^{u}, \overbrace{\mathrm{J}_{\mathcal{P}}^2}^{w}, \overbrace{2(mj_P + \mathrm{J}_{\mathcal{S}} \cdot \mathrm{J}_{\mathcal{P}})}^{\tilde{u}})$$

$$= \sum_{n_u, n_w, n_{\tilde{u}}} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$$

$$\partial_{m^2} \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2) = \mathcal{A}_{n_u+1\, n_w n_{\tilde{u}}}(m^2)$$

$\mathcal{A}_{n_u n_w n_{\tilde{u}}}$ equivalent to subset of susceptibilities

chiral symmetry restored at the level of susceptibilities

$\Longleftrightarrow$

$\mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$ non-divergent in the chiral limit $\Longleftrightarrow$ $\mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2) \in C^\infty$

$\Longleftrightarrow$

susceptibilities with even/odd no. of S, P are $C^\infty / m \cdot C^\infty$

χSR for *nonlocal* gauge functionals $\Rightarrow$ $\rho(\lambda; m) \in C^\infty(m^2)$

Lowest-order constraints

Requirement of finiteness of $\mathcal{A}_{100}$, $\mathcal{A}_{010}$, $\mathcal{A}_{002}$ $\Rightarrow$

$$\frac{\chi_\pi}{2} = \frac{n_0}{m^2} + \int d\lambda\, \frac{\rho(\lambda;m)}{\lambda^2+m^2} = O(m^0) \qquad n_0 = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}}\langle N_0\rangle$$

$$\frac{\chi_\pi - \chi_\delta}{4} = \int d\lambda\, \frac{m^2\rho(\lambda;m)}{(\lambda^2+m^2)^2} = \frac{\chi_t}{m^2} + O(m^2) \qquad \chi_t = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}}\langle Q^2\rangle$$

$N_0 \sim \sqrt{\mathrm{V}}$, zero-mode density $n_0 = 0$
$|\chi_\delta| \le \chi_\pi = O(m^0)$, $\mathcal{A}_{001} = 0$ by CP

$\mathrm{U}(1)_A$ order parameter: $\Delta \equiv \lim_{m\to 0} \frac{\chi_\pi - \chi_\delta}{4} = \lim_{m\to 0} \frac{\chi_t}{m^2}$

$\mathrm{SU}(2)_A$ restoration and $\mathrm{U}(1)_A$ breaking compatible at this stage, need technical assumptions on ρ to make progress

If ρ regular at the origin, $C^\infty(m^2) \Rightarrow \mathrm{U}(1)_A$ restored if $\mathrm{SU}(2)_A$ restored

$$\rho(\lambda;m) = \textstyle\sum_n \rho_n(m^2)\lambda^n \quad \text{or} \quad \rho(\lambda;m) \simeq c(m^2)\lambda^\alpha\,,\ \alpha > 0$$

Confirms [Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]

$\mathrm{U}(1)_A$ breaking by singular peak

Assume that near $\lambda = 0$

$$\rho(\lambda; m) \simeq C(m)\lambda^{\alpha(m)}$$

allowing $\alpha(m) < 0$, with $-1 \leq \alpha(0) < 1$ $\qquad$ $\alpha(0) \geq 1$ cannot break $\mathrm{U}(1)_A$

$\mathrm{SU}(2)_A$ restoration requires

$$C(m) = \frac{\cos\left(\alpha(m)\frac{\pi}{2}\right)}{(1-\alpha(0))\frac{\pi}{2}}|m|^{1-\alpha(0)}\hat{C}(m) \qquad |\hat{C}(0)| < \infty$$

$\mathrm{U}(1)_A$: order parameter $\Delta = \hat{C}(0)$, effectively broken if $\hat{C}(0) \neq 0$

If χSR nonlocal $\Rightarrow \rho \in C^\infty(m^2)$, possibilities strongly restricted:

$$\boxed{\alpha(0) = -1}$$

with $\alpha(m), \hat{C}(m) \in C^\infty(m^2)$

Singular peak

Singular peak compatible with χSR and $\mathrm{U}(1)_A$ breaking if

$$\rho_{\mathrm{peak}}(\lambda; m) = \left[\frac{\Delta}{2} + B(m^2)\right] \frac{m^2\gamma(m^2)}{\lambda^{1-\gamma(m^2)}} \qquad B, \gamma = O(m^2)$$

Peak becomes more singular as it vanishes away – at least $O(m^4)$

Connection with topology: $n_{\mathrm{peak}} \approx \chi_t$ for small m

$$\lim_{m\to 0} \frac{n_{\mathrm{peak}}}{m^2} = \lim_{m\to 0} \frac{1}{m^2} \int d\lambda\, \rho_{\mathrm{peak}}(\lambda; m) = \Delta = \lim_{m\to 0} \frac{\chi_t}{m^2}$$

Compatible with condition for commutativity of limits [Azcoiti (2023)]

$$\lim_{m\to 0} |m|^{-1} \rho_{\mathrm{peak}}(|m|z; m) = \frac{\Delta}{2}\delta(z) \qquad z \in [-1, 1]$$

Singular peak from topological fluctuations

Peak reproduced in weakly interacting, dilute (density $n_{\mathrm{inst}} = \chi_t \propto m^2$) instanton gas model: peak modes $\sim$ instanton zero modes [Kovács (2023)]

- m-dependent power α, peak "height" both decreasing as $m \to 0$
- $\alpha \to -1$ as disorder ($\sim 1/n_{\mathrm{inst}}$) increases in a similar cond-mat model [Evangelou and Katsanos (2003)]
- density of peak modes n_{peak} matches the required instanton density

$$n_{\mathrm{peak}} \approx \chi_t = n_{\mathrm{inst}}$$

Technically viable, physically motivated: ideal-gas-like behaviour of topological charge distribution, with $n_{\mathrm{inst}} = \chi_t$, *required* if chiral symmetry restored with broken $\mathrm{U}(1)_A$ in the chiral limit [Kanazawa, Yamamoto (2015)]

Summary and outlook

- Chiral symmetry is restored in the scalar/pseudoscalar sector in the $N_f = 2$ massless limit *if and only if* susceptibilities are non-divergent
- $\mathrm{U}(1)_A$ breaking compatible with χSR but requires singular near-zero spectral density, $\rho \sim O(m^4)/\lambda$ as $m \to 0$
- Second-order constraints require also singular two-point function, and near-zero modes *not* localised (near-zero mobility edge?)
- $\mathrm{U}(1)_A$ breaking requires ideal-gas-like topology, required spectral features occur naturally if gauge field configurations include non-interacting topological objects

Open issues:

- other sectors
- larger N_f
- test against numerical results

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