Topological features of an effective three-flavor meson model

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Outline

- Quick overview of QCD topological susceptibility studies
- Topological charge density (TCD) in three-flavor linear sigma model
- Composite field representation of TCD
- $\eta - \eta'$ mass difference and the range of topological fluctuations
- Conclusion

The subject of the talk is part of an ongoing investigation in collaboration with Gergely Fejős

Status of QCD topological susceptibility studies I

Non-abelian gluon-dynamics with CP-odd completion

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu.$$

$$\Delta L_{anomaly} = \Theta Q(x) \equiv \Theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}, \qquad \tilde{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}.$$

Measure of topological fluctuations $\chi = \int d^4x \langle T(Q(x)Q(0))\rangle$

$$\langle T(Q(x)Q(0))\rangle = \frac{\delta^2}{\delta\Theta(x)\delta\Theta(0)} \int \mathcal{D}\mu(A_\nu) e^{-S_{QCD} - \Delta S_{anomaly}}|_{\Theta(x)=0}$$

Evaluation on space-time lattice: S. Borsányi *et al.* 2016

$$\chi^{1/4}(T=0) = 75.6(1.8)(0.9)\mathrm{MeV}.$$

Status of QCD topological susceptibility studies II

Anomalous effect on $\eta - \eta'$ spectroscopy
Without anomaly (assuming singlet condensate):

$$U(\eta_0, \eta_8) = m_{00}^2 \eta_0^2 + 2 m_{08}^2 \eta_0 \eta_8 + m_{88}^2 \eta_8^2$$

$$m_{00}^2 = \frac{1}{3}(2m_K^2 + m_\pi^2), \quad m_{08}^2 = -\frac{2\sqrt{2}}{3}(m_K^2 - m_\pi^2), \quad m_{88}^2 = \frac{1}{3}(4m_K^2 - m_\pi^2)$$

Instanton contribution via $\eta_0 - Q$ coupling: $\frac{\sqrt{2N_f}}{f_\pi}\eta_0 Q$ produces

$$\Delta m_{00}^2 = \frac{N_f}{f_\pi^2} K_{instanton}.$$

Achievements: good $\eta - \eta'$ mass splitting

$$\chi = \frac{f_\pi^2 m_{top}^2}{2N_f}, \qquad m_{top}^2 = m_\eta^2 + m_{\eta'}^2 - 2m_K^2$$

G. Veneziano (1979), E. Witten (1979),
P. di Vecchia, G. Veneziano (1980),
R. Alkofer *et al.* (1989), E. Shuryak, J.J.M. Verbaarschot (1994)

Status of QCD topological susceptibility studies III

Perturbative computations of $\langle T(Q(x)Q(0))\rangle$ from effective quark/meson theories

Nambu–Jona-Lasinio model

$$\Delta L_{anomaly} = -K[\mathrm{det}\Phi + \mathrm{det}\Phi^\dagger], \qquad \Phi_{ij} = \bar{q}_i(1-\gamma_5)q_j$$

$$Q(x) = 2K\ Im[\mathrm{det}\Phi]$$

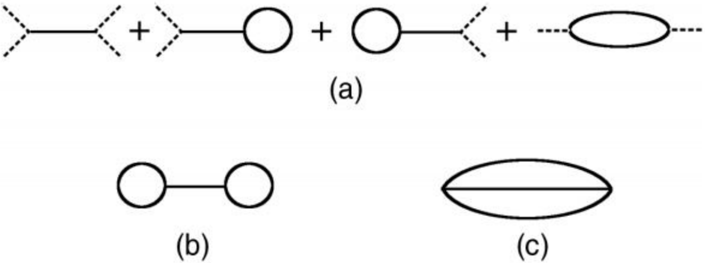
K. Fukushima *et al* (2001), P. Costa *et al.* (2008)

Linear three-flavor sigma-model with quartic potential

$$\Delta L_{anomaly} = c(\mathrm{det}M + \mathrm{det}M^\dagger), \qquad M = (\sigma_a + i\pi_a)\frac{\lambda^a}{2}$$

$$Q(x) = -2c\ Im[\mathrm{det}M(x)]$$

Y. Jiang *et al.* (2016)



Perturbative saturation of $\langle T(Q(x)Q(0))\rangle$

Our aim:
$\eta - \eta'$ mass matrix and topological susceptibility
self-contained determination within the linear sigma model via composite field representing topological charge density

TCD in three-flavor linear sigma model

$$S_M = \int_x \left(\mathrm{Tr}(\partial_m M^\dagger \partial_m M) + U(I_2, I_4, I_6) \right),$$

$$I_2 = \mathrm{Tr} M^\dagger M, \qquad I_{2l} = \mathrm{Tr}[(M^\dagger M) - I_2]^l \quad l = 2, 3$$

General functional of 't Hooft determinant breaking $U_A(1)$:

$$\Delta S_{M,top} = \sum_{n,i,j,k} A_n^{(i,j,k)} \int_x \left(I_2^i I_4^j I_6^k \right) (\det M + \det M^\dagger)^n$$

$$Q_M(x) = i\tilde{A}_{eff}(\det M(x) - \det M^\dagger)$$

$$\tilde{A}_{eff} = \sum_{n,i,j,k} A_n^{(i,j,k)} \left(I_2^i I_4^j I_6^k \right) n \left(\det M + \det M^\dagger \right)^{n-1} \Big|_{v_s v_{ns}}$$

$$= \sum_{n,i,j,k} A_n^{(i,j,k)} \left(I_2^i I_4^j I_6^k \right) n \left(\frac{v_{ns}^2 v_s}{2\sqrt{2}} \right)^{n-1}$$

$$\chi_{M,top} = \tilde{A}_{eff} \langle \det M + \det M^\dagger \rangle + \int d^4x \langle Q_M(x) Q_M(0) \rangle.$$

TCD as composite field via HS-transformation

Multiplying the partition function $\int \mathcal{D}M(x)e^{-S_M}$ by a "constant"

$$\int \mathcal{D}q_M e^{-\int d^4x \left(q_M(x) - \frac{Q_M(x)}{m_c^3}\right)\frac{m_c^2}{2}\left(q_M(x) - \frac{Q_M(x)}{m_c^3}\right)}$$

introduces $q_M - \eta_s, q_M - \eta_{ns}$ couplings:

$$iq_M \frac{\tilde{A}_{eff}}{m_c^3}(\det M - \det M^\dagger) \to -\frac{\tilde{A}_{eff}}{2\sqrt{2}m_c^3} q_M \left(v_{ns}^2 \eta_s + 2 v_{ns} v_s \eta_{ns}\right)$$

and shifts the mass-matrix in the $\eta_s - \eta_{ns}$

$$\begin{bmatrix} m^2_{\eta_{nsns}} & m^2_{\eta_{sns}} \\ m^2_{\eta_{sns}} & m^2_{\eta_{ss}} \end{bmatrix} \to \begin{bmatrix} m^2_{\eta_{nsns}} + \frac{\tilde{A}^2_{eff}}{2m_c^4} v_{ns}^2 v_s^2 & m^2_{\eta_{sns}} + \frac{\tilde{A}^2_{eff}}{4m_c^4} v_{ns}^3 v_s \\ m^2_{\eta_{sns}} + \frac{\tilde{A}^2_{eff}}{4m_c^4} v_{ns}^3 v_s & m^2_{\eta_{ss}} + \frac{\tilde{A}^2_{eff}}{8m_c^4} v_{ns}^4 \end{bmatrix}$$

$\eta - \eta'$ mass splitting

Repeat Veneziano's calculation assuming $v_{ns} = \sqrt{2} v_s$ (singlet condensate):

$$m^2_{nsns} = m^2_\pi, \qquad m^2_{ss} = 2m^2_K - m^2_\pi, \qquad m^2_{sns} = 0$$

Mass-scale setting by the **observed** trace of the mass-matrix

$$\frac{3}{8} v^4_{ns} \frac{\tilde{A}^2_{eff}}{m^4_c} + m^2_{\eta_{nsns}} + m^2_{\eta_{ss}} = M^2_\eta + M^2_{\eta'}$$

A rotation by angle $\phi \approx -\pi/4$ diagonalizes the matrix:

$$m_\eta = 562\mathrm{MeV}, \qquad m_{\eta'} = 949\mathrm{MeV}$$

Observed: $M_\eta = 549\mathrm{MeV}, M_{\eta'} = 958\mathrm{MeV}.$

Range of topological fluctuations

$q_M - \eta_s/\eta_{ns}$ couplings expressed in terms of mass eigenmodes

$$\eta' = \frac{1}{\sqrt{2}}(\eta_{ns} + \eta_s), \qquad \eta = \frac{1}{\sqrt{2}}(\eta_s - \eta_{ns})$$

allows integration over physical η-fields slightly modifying m_c^2

$$m_{c,eff}^2 = m_c^2 \left[1 - \frac{\tilde{A}_{eff}^2 v_{ns}^4}{16 m_c^4} \left(\frac{(1+\sqrt{2})^2}{m_{\eta'}^2} + \frac{(1-\sqrt{2})^2}{m_\eta^2} \right) \right]$$

Free q_M propagation leads to

$$\chi_{top} = \int d^4x \langle Q_M(x) Q_M(0) \rangle = m_c^6 \int d^4x \langle q_M(x) q_M(0) \rangle = \frac{m_c^6}{m_{c,eff}^2} \approx m_c^4$$

Compatibility with Witten-Veneziano formula

$$?? \quad m_c^4 = \frac{1}{6} v_{ns}^2 (M_\eta^2 + M_{\eta'}^2 - M_K^2) \quad ??$$

Using $\dfrac{\tilde{A}_{eff}^2}{m_c^4}$ one derives formula for the effective 't Hooft coupling

$$|\tilde{A}_{eff}| = \frac{2}{3} \frac{M_\eta^2 + M_{\eta'}^2 - M_K^2}{v_{ns}}$$

Substituting physical mass values + $v_{ns} = 100\mathrm{M}eV$ one finds $|\tilde{A}_{eff}| = 4.86\mathrm{GeV}$
Independent FRG computation of scalar+pseudoscalar meson spectra with

$$\Delta L_M = A_1(I_1)(\det M + \det M^\dagger) + A_2(I_1)(\det M + \det M^\dagger)^2$$

(Fejős,Patkós, 2024) leads to

$$\tilde{A}_{eff} \approx A_1(I_1(ground)) + 2A_2(I_1(ground)) = 4.75\mathrm{GeV}$$

Conclusion

A physically transparent,
technically simple,
self-contained,
internally consistent
treatment of topological effects
is realized for the three-flavor linear sigma model.