Anomalous $U(1)_A$ couplings, the Columbia plot, and CP violation

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Phys. Rev. D 111, 016014

Outline

Global symmetries of QCD

If the quark masses are zero (chiral limit) $\Longrightarrow$ QCD invariant under the following global symmetry transformation :

$$\mathcal{G}_{cl} \equiv U(3)_L \times U(3)_R = U(1)_L \times U(1)_R \times \underbrace{SU(3)_L \times SU(3)_R}_{\mathcal{G}_{\text{chiral}}}$$

$$= \underbrace{U(1)_V}_{\text{baryon number}} \times \underbrace{U(1)_A}_{\text{axial}} \times \underbrace{SU(3)_L \times SU(3)_R}_{\mathcal{G}_{\text{chiral}}}$$

$U(1)_V \longrightarrow$ baryon number conservation (exact symmetry of nature)

$U(1)_A \longrightarrow$ axial symmetry

$SU(3)_L \times SU(3)_R \longrightarrow$ broken down to $SU(3)_V$ if $m_u = m_d = m_s \neq 0$;
or to $SU(2)_V$ if $m_u = m_d \neq m_s \neq 0$;
or to $U(1)_V$ if $m_u \neq m_d \neq m_s \neq 0$

Axial anomaly

From $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$ the $U(1)_A$ is anomalously broken

Physical consequences: e.g. mass of η and η' (*Phys.Rept. 142 (1986) 357-387*)

The $U(1)_A$ symmetry may be restored around the chiral transition temperature (but above or below?)

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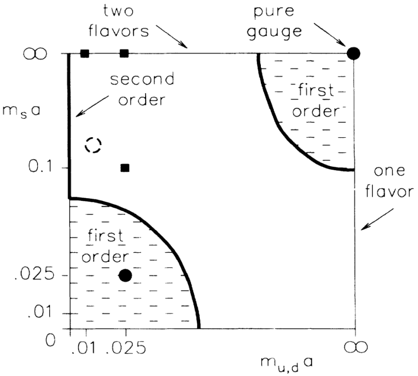
Axial anomaly and the order of the phase transition in the chiral limit:

From ϵ expansion (*Phys. Rev. D 29 (1984) 338*)

In the chiral limit:

- $N_f = 0$: Pure Yang-Mills, first-order.
- $N_f = 1$: Crossover transition with the anomaly. (can be second order without)
- $N_f = 2$: Without anomaly the transition is second order, with anomaly it might be first or second order depending on the restoration of the anomaly.
- $N_f = 3$: First-order transition.

Columbia plot

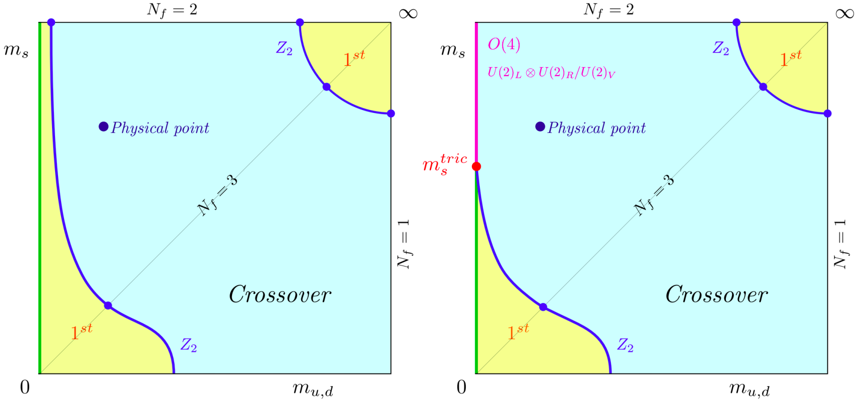


Columbia plot:

the order of the phase transition in the limits $m_{u,d} \to 0$ and $m_s \to 0$

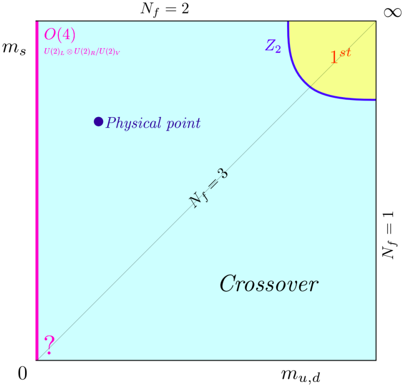
Phys Rev Lett 65 (1990) 2491

Usual scenarios for the Columbia plot



Symmetry 13 (2021) 11, 2079

1. Recent result from lattice

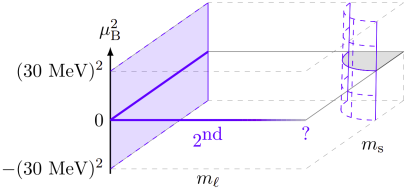


Lattice calculation to find the tricritical point.

Starting from a coarse lattice a finite m_s^{tric} cannot be reached in the continuum limit

Symmetry 13 (2021) 11, 2079

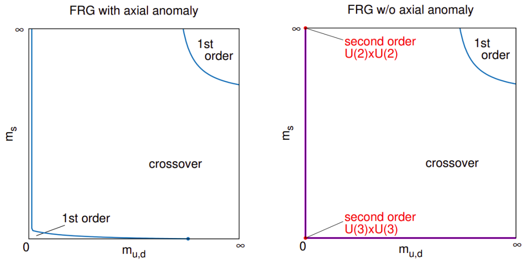
2. Recent result from DS approach



Dyson-Schwinger approach:
$U(1)_A$ anomaly through η meson (using 2-point functions only)

Phys.Rev.D 108 (2023) 11, 114018

3. Recent result fromFRG calculations



FRG calculation: multiple fixed points at $N_f = 2$ and $N_f = 3$ that gives first or second order transition in presence of axial anomaly

Phys.Rev.D 110 (2024) 1, 016021

In 3D the only relevant operator is $(\det\Phi + \det\Phi^\dagger)$!

Possible operators for $U(1)_A$ anomaly

$U(1)_A$ symmetry can be broken by higher order terms than the 't Hooft determinant

Topological charge $|Q| = 1,\ 2,\ \ldots$
Terms allowed by symmetry considerations (breaking only $U(1)_A$)

$$\xi_1\left(\det\Phi + \det\Phi^\dagger\right), \quad \xi_1^1\,\mathrm{tr}\left(\Phi^\dagger\Phi\right)\left(\det\Phi + \det\Phi^\dagger\right),$$
$$\xi_2\left[\left(\det\Phi\right)^2 + \left(\det\Phi^\dagger\right)^2\right], \quad \xi_2^{\pm}\left(\det\Phi \pm \det\Phi^\dagger\right)^2 \tag{1}$$

- ξ_1 : simplest, 't Hooft determinant, corresponding to $|Q| = 1$
- ξ_1^1 : the 't Hooft determinant, multiplied by an invariant term $|Q| = 1$
- ξ_2 : corresponds to $|Q| = 2$
- ξ_2^- : deviates from ξ_2 only in an invariant term, but do not affect the scalar mass terms $|Q| = 2$
- ξ_2^+ : deviates from ξ_2 only in an invariant term $|Q| = 2$

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Lagrangian of the eLSM model

$\mathcal{L}$ constructed based on linearly realized global $U(3)_L \times U(3)_R$ symmetry and its explicit breaking

$$
\begin{aligned}
\mathcal{L} = {} & \mathrm{Tr}[(D_\mu\Phi)^\dagger(D_\mu\Phi)] - m_0^2\mathrm{Tr}(\Phi^\dagger\Phi) - \lambda_1[\mathrm{Tr}(\Phi^\dagger\Phi)]^2 - \lambda_2\mathrm{Tr}(\Phi^\dagger\Phi)^2 \\
& + \mathcal{L}_{qu} + \mathrm{Tr}[H(\Phi+\Phi^\dagger)] - \frac{1}{4}\mathrm{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) \\
& + \mathrm{Tr}\left[\left(\frac{m_1^2}{2}\mathbb{1} + \Delta\right)(L_\mu^2 + R_\mu^2)\right] + i\frac{g_2}{2}(\mathrm{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \mathrm{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}) \\
& + \frac{h_1}{2}\mathrm{Tr}(\Phi^\dagger\Phi)\mathrm{Tr}(L_\mu^2 + R_\mu^2) + h_2\mathrm{Tr}[(L_\mu\Phi)^2 + (\Phi R_\mu)^2] + 2h_3\mathrm{Tr}(L_\mu\Phi R^\mu\Phi^\dagger) \\
& + \bar{\Psi}\left(i\gamma^\mu D_\mu - g_F(S - i\gamma_5 P)\right)\Psi - g_V\bar{\Psi}\left(\gamma^\mu(V_\mu + \gamma_5 A_\mu)\right)\Psi,
\end{aligned}
$$

$+$ Polyakov loop potential (for $T > 0$)

with

$$
\begin{aligned}
\mathcal{L}_{\mathrm{qu}} = {} & \xi_1\left(\det\Phi + \det\Phi^\dagger\right) + \xi_1^1\,\mathrm{tr}\left(\Phi^\dagger\Phi\right)\left(\det\Phi + \det\Phi^\dagger\right) \\
& + \xi_2\left[\left(\det\Phi\right)^2 + \left(\det\Phi^\dagger\right)^2\right]
\end{aligned}
$$

, and assume $\xi_1,\ \xi_1^1,\ \xi_2 > 0$

The grand potential

The grand potential at the mean-field level:

$$\Omega(T, \mu_q) = U_{Cl} + \mathrm{Tr}\int_K \log\left(iS_0^{-1}\right) + U(\Phi, \bar{\Phi})$$

- 1^{st} : Classical potential.
- 2^{nd} : Fermionic one-loop correction with vanishing fluctuating mesonic fields.

  $$\bar{\psi}\left(i\gamma^\mu\partial_\mu - \mathrm{diag}(m_u, m_d, m_s)\right)\psi$$

  Functional integration over the fermionic fields.
  The momentum integrals are renormalized.
- 3^{rd} : Polyakov loop potential.

Field equations (FE):

$$\frac{\partial\Omega}{\partial\phi_N} = \frac{\partial\Omega}{\partial\phi_S} = \frac{\partial\Omega}{\partial\bar{\Phi}} = \frac{\partial\Omega}{\partial\Phi} = 0$$

Curvature meson masses:

$$M^2_{ab} = \left.\frac{\partial^2\Omega}{\partial\varphi_a\partial\varphi_b}\right|_{\{\varphi_i\}=0}$$

Determination of the parameters

14 unknown parameters $(m_0, \lambda_1, \lambda_2, c_1, m_1, g_1, g_2, h_1, h_2, h_3, \delta_S, \Phi_N, \Phi_S, g_F) \longrightarrow$ determined by the min. of χ^2:

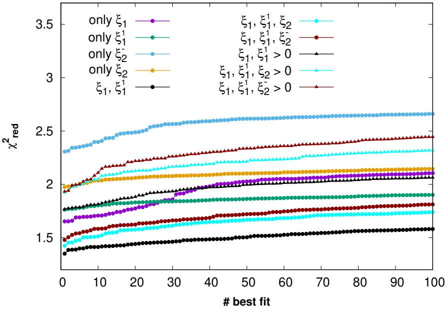
$$\chi^2(x_1, \ldots, x_N) = \sum_{i=1}^{M} \left[\frac{Q_i(x_1, \ldots, x_N) - Q_i^{\exp}}{\delta Q_i} \right]^2 ,$$

$(x_1, \ldots, x_N) = (m_0, \lambda_1, \lambda_2, \ldots)$, $Q_i(x_1, \ldots, x_N) \longrightarrow$ from the model, $Q_i^{\exp} \longrightarrow$ PDG value, $\delta Q_i = \max\{5\%, \text{PDG value}\}$
multiparametric minimalization $\longrightarrow$ MINUIT

- PCAC $\to$ 2 physical quantities: f_π, f_K
- Curvature masses $\to$ 16 physical quantities: $m_{u/d}, m_s, m_\pi, m_\eta, m_{\eta'}, m_K, m_\rho, m_\Phi, m_{K^\star}, m_{a_1}, m_{f_1^H}, m_{K_1}, m_{a_0}, m_{K_s}, m_{f_0^L}, m_{f_0^H}$
- Decay widths $\to$ 12 physical quantities: $\Gamma_{\rho\to\pi\pi}, \Gamma_{\Phi\to KK}, \Gamma_{K^\star\to K\pi}, \Gamma_{a_1\to\pi\gamma}, \Gamma_{a_1\to\rho\pi}, \Gamma_{f_1\to KK^\star}, \Gamma_{a_0}, \Gamma_{K_S\to K\pi}, \Gamma_{f_0^L\to\pi\pi}, \Gamma_{f_0^L\to KK}, \Gamma_{f_0^H\to\pi\pi}, \Gamma_{f_0^H\to KK}$
- Pseudocritical temperature T_c at $\mu_B = 0$

Parametrization and effective coupling

Can the different ξ_i terms or their combinations be distinguished in the parameterization?



- The parameterization with minimal $\chi^2_{red} = \chi^2/N_{dof}$ is more favorable.
- Each anomaly term must cover the same effect in the physical quantities caused by the axial anomaly.

$$\xi_{\text{eff}} = \xi_1 + \alpha\xi_1^1 + \beta\xi_2 + \beta^-\xi_2^- + \beta^-\xi_2^- + \ldots$$

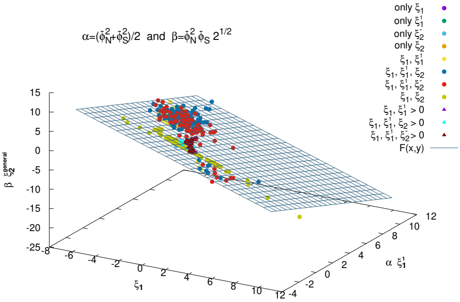
Phys.Rev.Lett. 132 (2024) 25, 251903

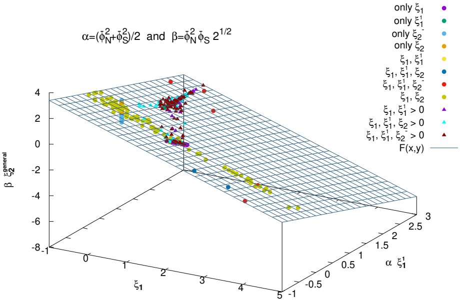
Thus many-many phenomenologically similar parameterizations

Visualization of ξ_{eff}

Comparing m_π^2 and m_η^2: $$\xi_{\text{eff}} \equiv \xi_1 + \xi_1^1\left(\phi_N^2+\phi_S^2\right)/2 + \xi_2\phi_N^2\phi_S/\sqrt{2}$$

(So in ELSM this corresponds to ξ_{eff})





Left: full range, right: zoom in

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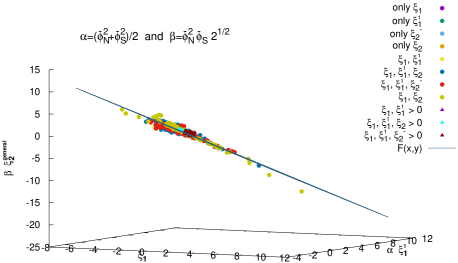

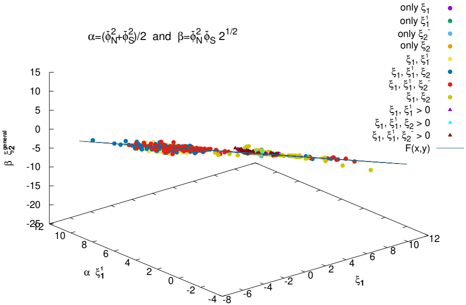

Since points are basically on a plane $\longrightarrow$ ξ_{eff} is a constant

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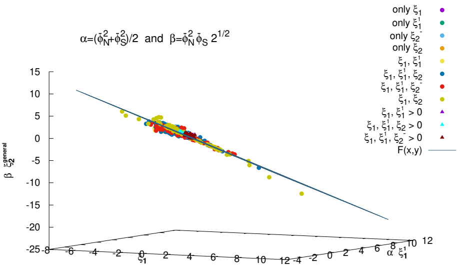
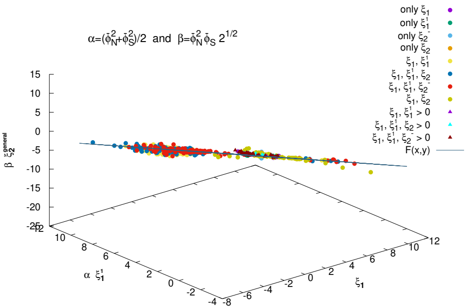




At finite temperature $\xi_{\text{eff}} = \xi_{\text{eff}}(T)$ due to $\phi_N(T)$ and $\phi_S(T)$

External fields and m_π, m_K

Keep only ξ_1 and ξ_2 and compare to $\bar{\xi}_1 = 1.50$ GeV.

Through the PCAC relation $h_{N,S} \propto m_{n,s}$ (with $m_n = (m_u + m_d)/2$) can be transformed into m_π and m_K

$$f_\pi\, m_\pi^2 = Z_\pi h_N$$

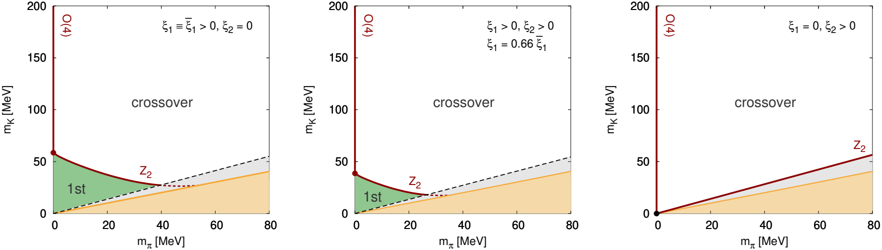
$$f_K\, m_K^2 = Z_K \left(\frac{1}{2} h_N + \frac{1}{\sqrt{2}} h_S\right)$$

We choose parameter sets that $f_\pi \gtrsim 65$ MeV, in this way we are roughly compatible with ChPT. For a given $h_{N/S}$ $\phi_{N/S}$ is determined at $T = 0$ from:

$$h_N = m_0^2 \phi_N + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^3 + \lambda_1 \phi_N \phi_S^2 - \frac{\xi_2}{4}\phi_N^3\phi_S^2$$
$$- \frac{\phi_N \phi_S}{\sqrt{2}}\left(\xi_1 + \xi_1^1 (\phi_N^2 + \frac{\phi_S^2}{2})\right) + \frac{g_F}{2}\sum_{l=u,d}\langle \bar{q}_l q_l\rangle$$

$$h_S = m_0^2 \phi_S + (\lambda_1 + \lambda_2)\,\phi_S^3 + \lambda_1 \phi_N^2 \phi_S - \frac{\xi_2}{8}\phi_N^4 \phi_S$$
$$- \frac{\phi_N^2}{2\sqrt{2}}\left(\xi_1 + \frac{\xi_1^1}{2}(\phi_N^2 + 3\phi_S^2)\right) + \frac{g_F}{\sqrt{2}}\langle \bar{q}_s q_s\rangle$$

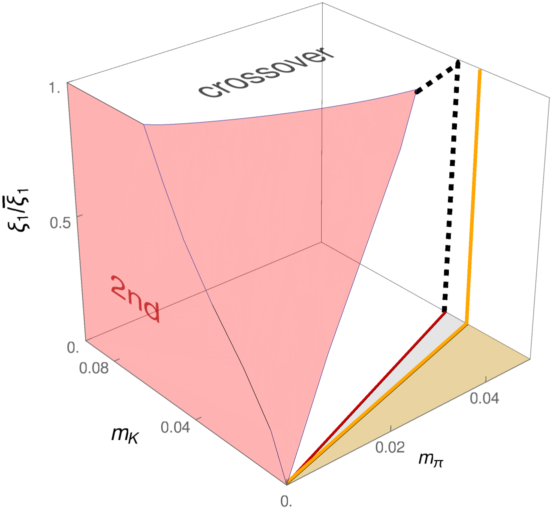
Columba plots for different anomaly terms



The Columbia plot in the m_π-m_K plane for three independent parameterizations: ξ_1 (left), ξ_2 (right), or both anomaly terms (center).

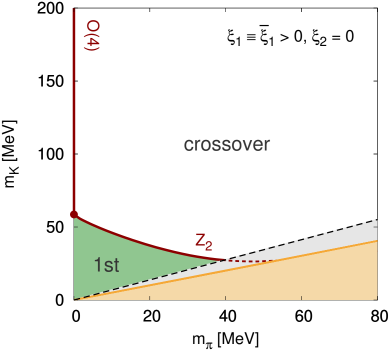
Red line → second order phase transition, the black dashed line → $h_S = 0$, the gray area → $h_S < 0$, where $m_s < 0$. Below the golden line CP symmetry may be spontaneously broken.

Columbia plot in 3D



$\xi_{\rm eff} = \bar{\xi}_1 = 1.50$ GeV is fixed, $\xi_1^1 = 0$ and ϕ_N, $\phi_S > 0$. On the bottom $m_\pi - m_K$ plane $\xi_1 = 0$, while on the top $\xi_2 = 0$.

CP breaking phases



Golden region:

Dashed line corresponds to $h_S = 0$. If we also introduce $\eta_{N/S}$ condensates, it can be shown that

- If there is no anomaly, there is no η condensation
- If $\xi_1 > 0$ there is a η condensation for some $h_S < 0$ ($m^2_{\eta'} > 0$)
- If $\xi_2 > 0$ there is no η condensation in the global minimum, only in the local one

Still under investigation ...

Conclusion and outlook

- Several terms to describe the $U(1)_A$ anomaly (almost) equally well.
- The general effective coupling can be parameterized as given by the combination of the individual couplings.
- In the $N_f = 1$ chiral limit, the transition is either crossover ($\xi_1 > 0$) or second order, depending on the anomaly term used.
- In the $N_f = 2$ chiral limit the transition is of second order for any anomaly term.
- In the $N_f = 3$ chiral limit the transition is of first order only for $\xi_1 > 0$.
- In addition, the relative size of ξ_1 determines the size of the first order region in the lower left corner of the Columbia pot.
- There is a CP violating phase in the nonphysical region of the Columbia plot where m_s (or h_S) is negative if $\xi_1 > 0$.

Thank you for your attention!

Field equations (FE)

Four coupled field equations are obtained by extremizing the grand potential

$$\Omega(T,\mu_q) = U^{\text{tree}}_{\text{meson}}(\langle M\rangle) + \Omega^{(0)\text{vac}}_{\bar{q}q} + \Omega^{(0)T}_{\bar{q}q}(T,\mu_q) + \mathcal{U}_{\log}(\Phi,\bar{\Phi})$$

using $\dfrac{\partial\Omega}{\partial\phi_N} = \dfrac{\partial\Omega}{\partial\phi_S} = \dfrac{\partial\Omega}{\partial\Phi} = \dfrac{\partial\Omega}{\partial\bar{\Phi}} = 0$ $\qquad E^{\pm}_f(p) = E_f(p) \mp \mu_q, \quad E^2_f(p) = p^2 + m^2_f$

1) $$-\frac{1}{T^4}\frac{d\,U(\Phi,\bar{\Phi})}{d\Phi} + \frac{6}{T^3}\sum_{f=u,d,s}\int\frac{d^3p}{(2\pi)^3}\left(\frac{e^{-\beta E^-_f(p)}}{g^-_f(p)} + \frac{e^{-2\beta E^+_f(p)}}{g^+_f(p)}\right) = 0$$

2) $$-\frac{1}{T^4}\frac{dU(\Phi,\bar{\Phi})}{d\bar{\Phi}} + \frac{6}{T^3}\sum_{f=u,d,s}\int\frac{d^3p}{(2\pi)^3}\left(\frac{e^{-\beta E^+_f(p)}}{g^+_f(p)} + \frac{e^{-2\beta E^-_f(p)}}{g^-_f(p)}\right) = 0$$

3) $$m^2_0\phi_N + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi^3_N + \lambda_1\phi_N\phi^2_S - \frac{c_1}{\sqrt{2}}\phi_N\phi_S - h_{0N} + \frac{3}{2}g_F\left(\langle\bar{q}_u q_u\rangle_T + \langle\bar{q}_d q_d\rangle_T\right) = 0$$

4) $$m^2_0\phi_S + (\lambda_1+\lambda_2)\phi^3_S + \lambda_1\phi^2_N\phi_S - \frac{\sqrt{2}}{4}c_1\phi^2_N - h_{0S} + \frac{3}{\sqrt{2}}g_F\langle\bar{q}_s q_s\rangle_T = 0$$

renormalized fermion tadpole: $\qquad m_{u,d} = \frac{g_F}{2}\phi_N$ and $m_s = \frac{g_F}{\sqrt{2}}\phi_S$

$$\langle\bar{q}_f q_f\rangle_T = 4m_f\left[-\frac{m^2_f}{16\pi^2}\left(\frac{1}{2} + \ln\frac{m^2_f}{M^2_0}\right) + \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_f(p)}\left(f^-_f(p) + f^+_f(p)\right)\right]$$

Parameter sets at $N_c = 3$

Parameter	Set A	Set B
ϕ_N [GeV]	0.1411	0.1290
ϕ_S [GeV]	0.1416	0.1406
m_0^2 [GeV^2]	2.3925_{E-4}	-1.2370_{E-2}
m_1^2 [GeV^2]	6.3298_{E-8}	0.5600
λ_1	-1.6738	-1.0096
λ_2	23.5078	25.7328
c_1 [GeV]	1.3086	1.4700
δ_S [GeV^2]	0.1133	0.2305
g_1	5.6156	5.3295
g_2	3.0467	-1.0579
h_1	37.4617	5.8467
h_2	4.2281	-12.3456
h_3	2.9839	3.5755
g_F	4.5708	4.9571
M_0 [GeV]	0.3511	0.3935

- Set A: $m_\sigma = 290$ MeV from *Phys.Rev.D 93 (2016) 11, 114014*
- Set B: similar m_σ mass and additional constraint, $3h_1 + 2h_2 + 2h_3 < 0$ from *Phys.Rev.D 105 (2022) 10, 103014*