UNIVERSITÄT GRAZ

NAWI Graz
Natural Sciences

Hadronic Structure from Functional Methods

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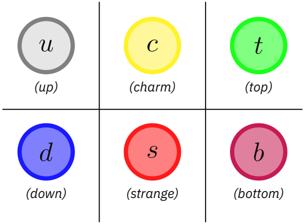
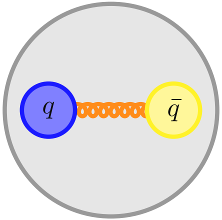
Hadron Physics



Quarks

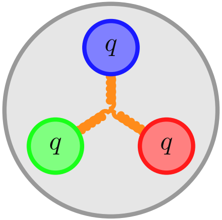
Gluons

Hadron Physics



Mesons

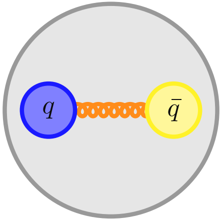
π, ρ, K, η, ...



Baryons

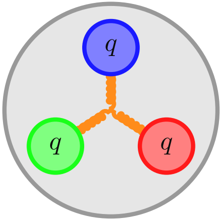
p, n, Δ, Σ, Λ, ...

Hadron Physics



Mesons

$\pi, \rho, K, \eta, \ldots$



Baryons

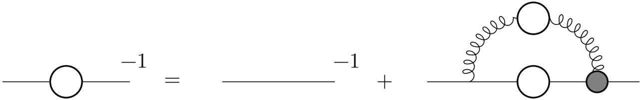
$p, n, \Delta, \Sigma, \Lambda, \ldots$

- Their dynamics and properties are determined by QCD
  - Atomic nuclei and nuclear stability, EMC effect
  - Radius; spin-, momentum-, charge distributions;
  - Interaction with external currents: $\{e^-, \nu, \ldots\}N$ scattering, $N\pi$ scattering, nucleon compton scattering, meson photoproduction

(EIC White Paper), (Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer; 2016), (European Muon Collaboration; 1983), ...

Dyson-Schwinger Equations

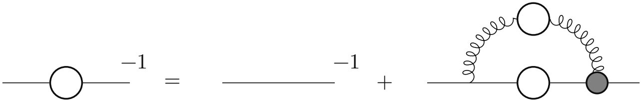
- Can relate n-point functions with each other:

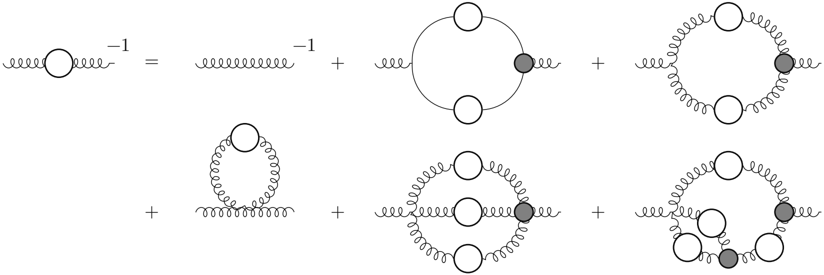


- The **full quark propagator** is the bare propagator plus self-energy Σ
- Σ depends on gluon and qqg vertex

Dyson-Schwinger Equations

- Can relate n-point functions with each other:

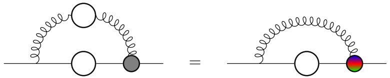




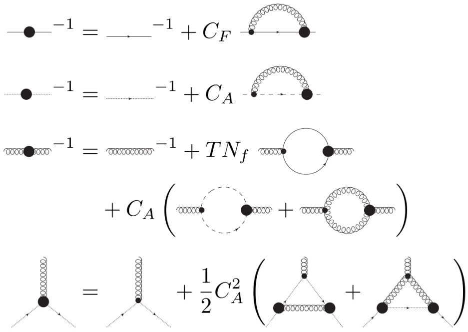
- Coupled equations: n-point function depends on higher order correlations.

Truncations

- Calculations only possible with truncations
  - Chosen to reproduce the interesting physics of the system.
  - Common truncation: Maris-Tandy model (Maris, Roberts; 1997), (Maris, Tandy; 1999), (Maris, Tandy; 2000)



$$\frac{\alpha(k^2)}{k^2} = \pi \eta^7 \left(\frac{k^2}{\Lambda^2}\right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}} + \frac{2\pi\gamma_m \left(1 - e^{-\frac{k^2}{\Lambda_t^2}}\right)}{\ln\left[e^2 - 1 + \left(1 + \frac{k^2}{\Lambda_{QCD}^2}\right)^2\right]}$$



(Alkofer, Zierler; 2023)

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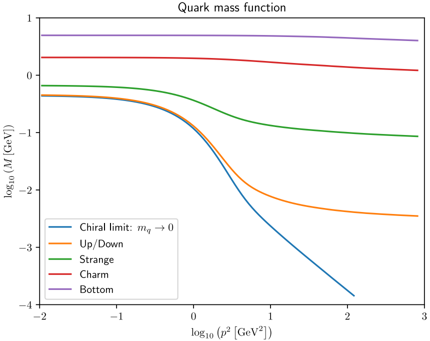
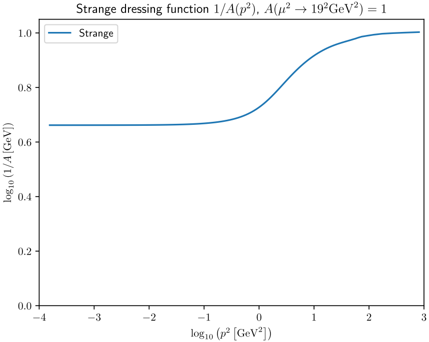


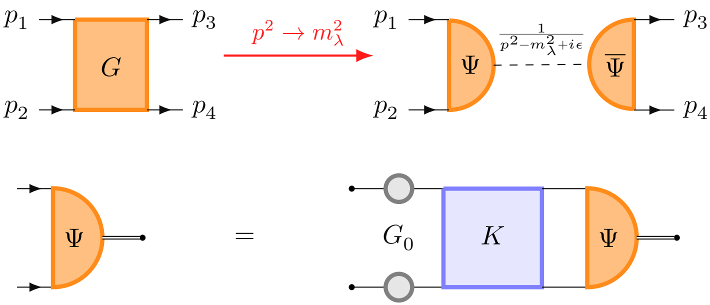


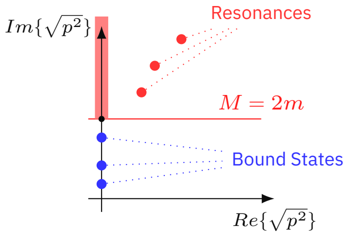
$$S(p) = \sigma_s(p^2)\mathbb{1} - i\sigma_v(p^2)\not{p} = \frac{1}{A(p^2)}\frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2}$$

(Numerical results by: Raúl)

Bethe-Salpeter Equation

- The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:





- Poles in correlation functions encode the theory's bound-state spectrum
- BSWF characterizes the bound-state

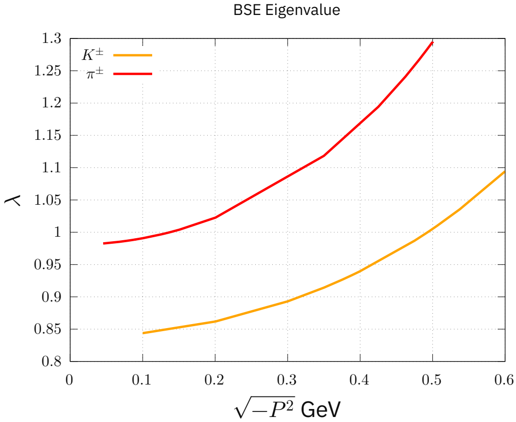
(Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer; 2016)

Bethe-Salpeter Equation *Eigenvalue/vector equation*

- The BSE is usually solved as an eigenvalue equation:

$$\lambda_i\left(P^2\right)\psi_i = \mathbf{K G_0}\psi_i$$

- States found when $\lambda_i(P^2) \equiv 1$.
- Mass is given by $P^2 = -M^2$
- Eigenvalue/vector spectrum gives ground state and excited states.
- Like DSE, analytic structure is important, and **contour deformations** may be needed!



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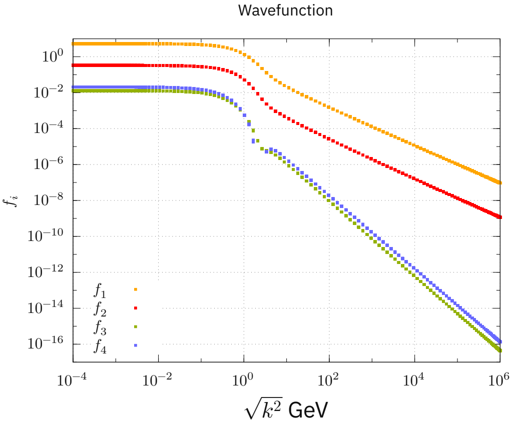
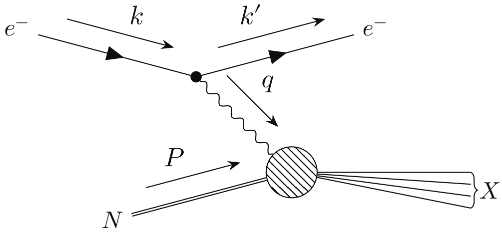


1 Functional Methods

2 Hadronic Structure

Parton model



(PDG Section 18 (Structure Functions))

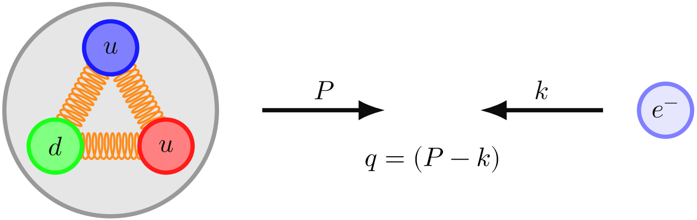
- Common experiment:
  Deep inelastic scattering.
- Electron (or neutrino) probe interacts with nucleon via $\gamma, Z, W^{\pm}$.
- Main idea:
  - Hadrons are *bags* of **partons** .
  - Scattering occurs between the exchanged boson and a parton from the nucleon
- Cross-section separates:

$$\frac{d^2\sigma}{dx dq^2} \propto L_{\mu\nu} W^{\mu\nu}$$

- $W^{\mu\nu}$ described via PDFs.

PDFs/TMDs

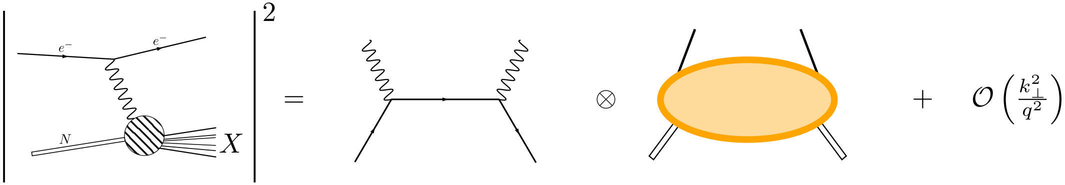
- How to they show up?



$$W_{\mu\nu}(p,q,\sigma) \propto \int d^4z \, \langle p,\sigma| \, J^\dagger_\mu(z) J_\nu(0) \, |p,\sigma\rangle = \sum_i \tau(p,q,\sigma)^i_{\mu\nu} f_i(p,q)$$

PDFs/TMDs

- How to they show up?
- Take the collinear, q^2 very large limit.



$$\sigma \propto \sum_f \int \frac{dx}{x} H(x, Q^2/\Lambda^2) q_f(x, \Lambda^2) + \mathcal{O}\left(\frac{k_\perp^2}{Q^2}\right)$$

$$q_f(x, \Lambda) \propto \int \frac{d^4\ell}{(2\pi)^4} \delta\left(x - \frac{\ell^+}{p^+}\right) \mathrm{Tr}\left[\langle p| \overline{\psi}_f(\ell)\gamma^+\psi_f(\ell) |p\rangle\right]$$

- Hard scale $Q^2 = -q^2$ introduced allows for factorization – power supression $\mathcal{O}(Q^{-2})$
- Works also on other processes, like Drell-Yan.

PDFs/TMDs

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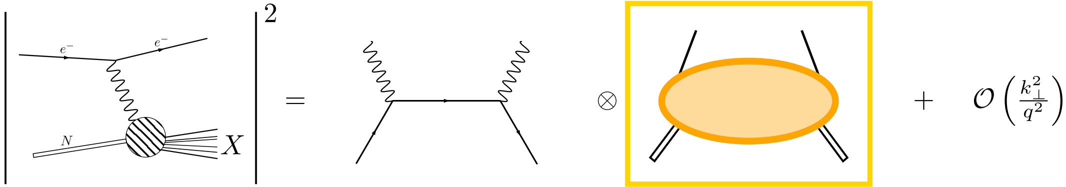

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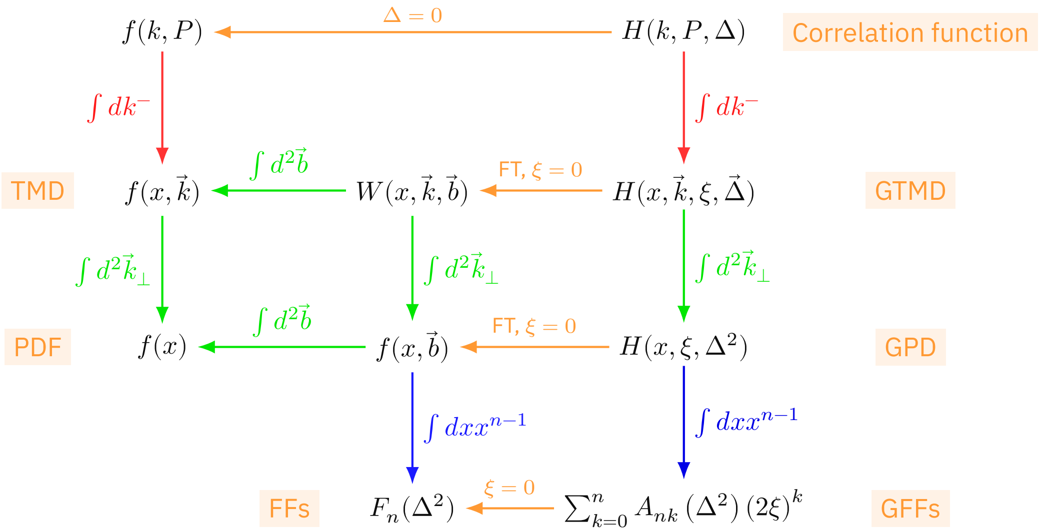
- Focus on the non-perturbative hadronic part.
- Can we build it from first-principles **FUNctional Methods** calculations?

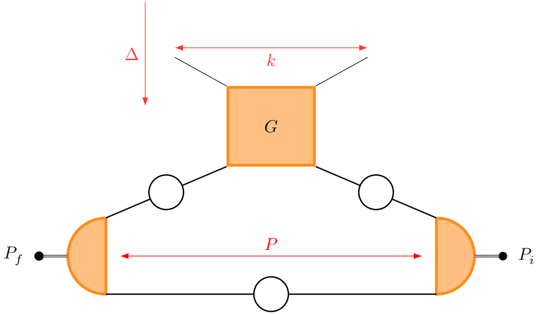
Hadronic quantities



(Lorce, Pasquini, Vanderhaeghen; 2011), (Picture adapted from: Diehl, 2016), (Diehl, 2003), (Meißner, Goeke, Metz, Schlegel; 2008), (Meißner, Metz, Schlegel; 2009)

Our Goal

- **Main Goal:** Get partonic distribution functions from hadron-hadron correlations via **FUN**ctional Methods



- G is the four-point quark correlation function, calculated with scattering equation.
- The BSWF is calculated via the meson BSE.
- First scalar toy model, then QCD.

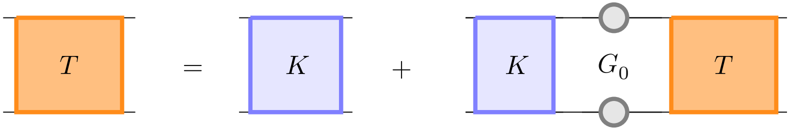
(Mezrag; 2015), (Diehl, Gousset; 1998), (Tiburzi, Miller; 2003), (Mezrag, Chang, Moutarde, Roberts, Rodríguez-Quintero, Sabatié, Schmidt; 2015), many others, ...

$$\mathcal{G}^{[\Gamma]}(P,k,\Delta) = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \int \frac{d^4z}{2\pi^4} e^{ik\cdot z} \langle P_f | \overline{\psi}(z)\mathcal{W}\Gamma\psi(0) | P_i \rangle\right]$$

- Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.

4-point function

- 4- point function determined from scattering equation:
  $\mathbf{G} = \mathbf{G_0} + \mathbf{G_0 T G_0} \implies \mathbf{T} = \mathbf{K} + \mathbf{K G_0 T} \implies \mathbf{T} = (\mathbb{1} - \mathbf{K G_0})^{-1}\,\mathbf{K}.$



(Eichmann, Duarte, Peña, Stadler; 2019)

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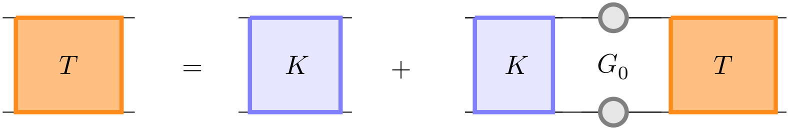
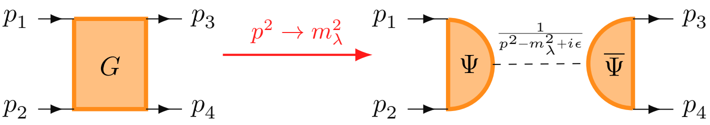


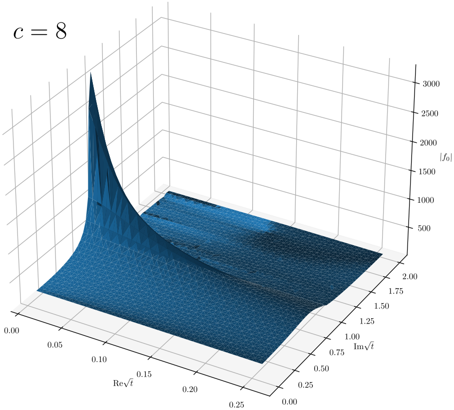


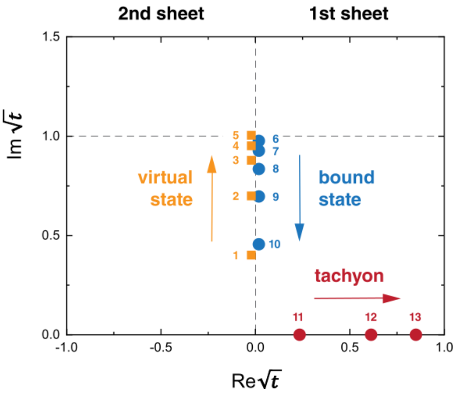
- Same G_0 and K as in the BSE
- *All* dynamics of 2 ϕ particles:
  - **Must produce bound state poles dynamically!**

(Eichmann, Duarte, Peña, Stadler; 2019)

4-point function

- Partial-wave expansion shows bound-state pole in the first Riemann sheet



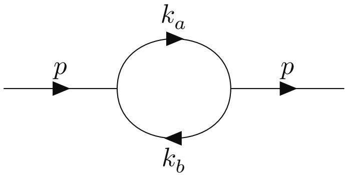


(Eichmann, Duarte, Peña, Stadler; 2019)

- Describes both long-range and short-range qq dynamics.

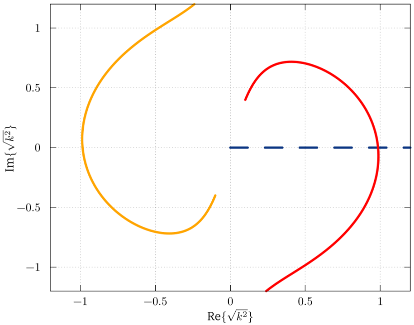
Analytic Structure is Important

- Calculating quantities in time-like momenta is **complicated** !
  - Analytic structure prevents naïve integration.
  - Poles and branch cuts are present.
- Calculate quantities in the P^2 complex plane.
- **Euclidean** $\Leftrightarrow$ Minkowski



$$I(p^2) = \int d^4k \frac{1}{k_a^2 - m^2 + i\epsilon} \frac{1}{k_b^2 - m^2 + i\epsilon}$$

$\Longrightarrow$

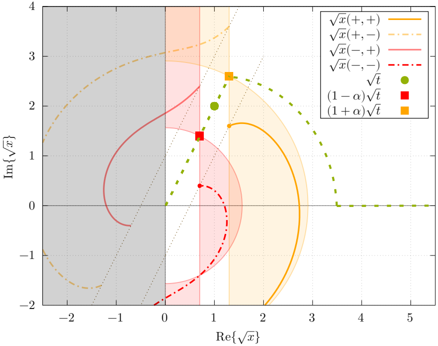


Analytic Structure of BSE $\mathrm{Im}\sqrt{t} > \min\left(\frac{1}{1\pm\alpha}\right)$

$$\mathbf{G_0}^{-1} = (l_+^2 + m^2)(l_-^2 + m^2)$$

- Branch cuts in complex $\sqrt{l}$ plane

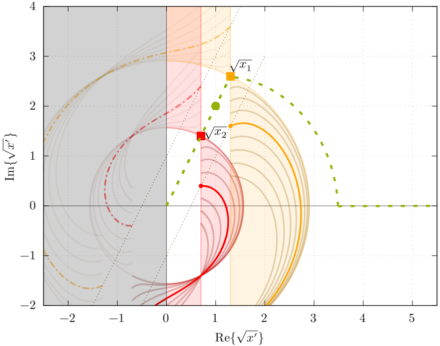
$$\sqrt{l}_\pm^\lambda = \mp(1\pm\alpha)\sqrt{t}\left[z_l + i\lambda\sqrt{1 - z_l^2 + \frac{1}{(1\pm\alpha)^2 t}}\right]$$



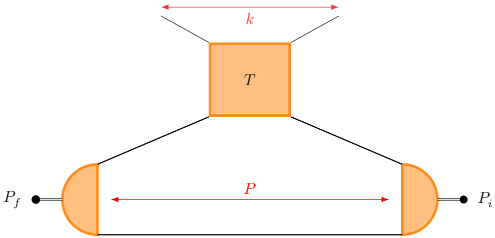
$$\mathbf{K}^{-1} = (q-l)^2 + \mu^2$$

- Branch cuts in complex $\sqrt{l}$ plane depend on path taken:

$$\sqrt{l} = \sqrt{\rho}\left(\Omega \pm i\sqrt{1 - \Omega^2 + \frac{\beta^2}{\rho}}\right)$$



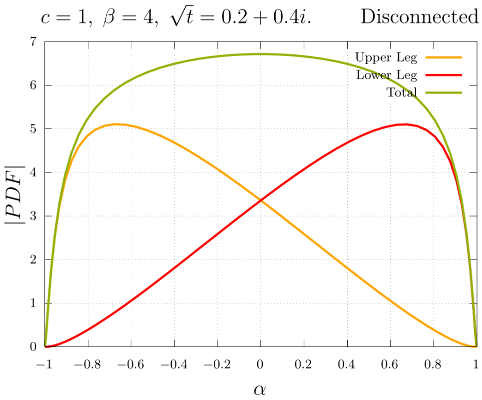
Put everything together

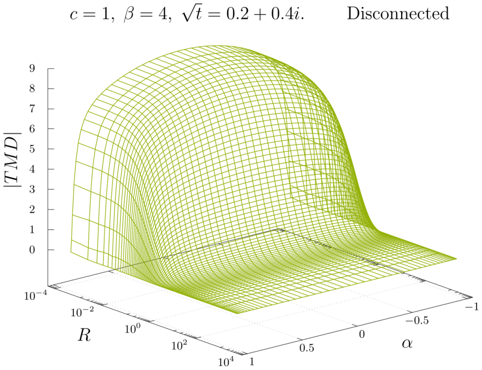


1. Calculate 4-point function
2. Calculate BSE
3. Do the loop integration
4. Project to LF

- We solve one triangle for each point in the R, α grid – **HPC Needed!**
  - $N_\alpha \times N_R \times N_Z$ 4-point functions!
- Obtained PDFs/TMDs for a system of two bound scalars ϕ.

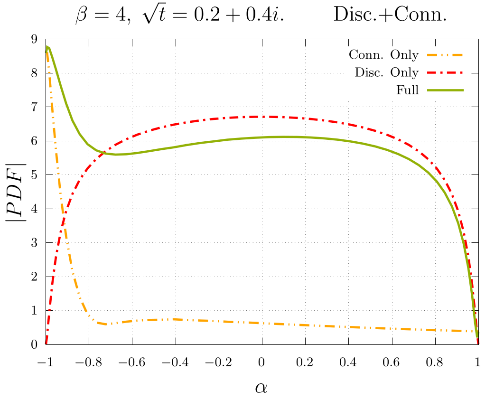
Results

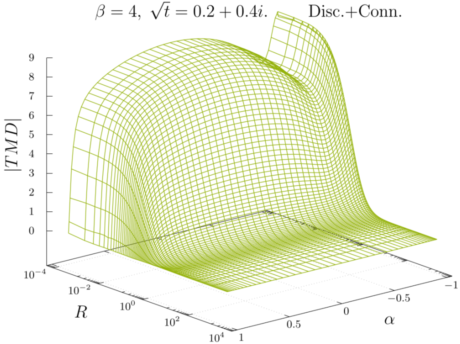




- Publication coming soon!

Results





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Conclusions

- Framework for calculation of PDFs/TMDs.
- Applications to hadronic structure calculations starting from self-consist first principles calculations.
- Calculation of $q\bar{q}$ scattering equation – rich dynamics possible.
- Future applications to realistic QCD models soon.