Five-body systems with Bethe-Salpeter equations

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G. E. and ***R. D. Torres****, Five-point functions and the permutation group S5, arXiv:2502.17225.*
G. E., M. T. P. and ***R.D. Torres****, Five-body systems with Bethe-Salpeter equations, arXiv:2502.17944.*

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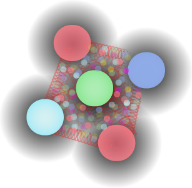
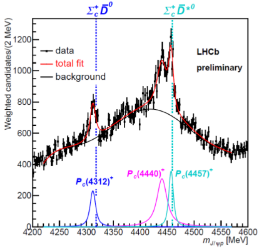
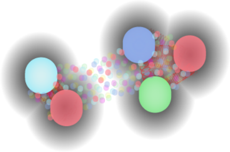
Motivation





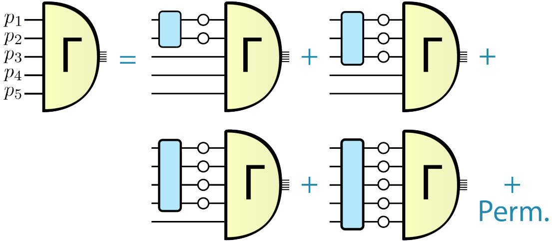


LHCb: Phys. Rev. Lett. 115, 072001 (2015).

Goal Solve the Bethe-Salpeter equation for a system of five scalar particles interacting by a scalar boson exchange, a massive Wick-Cutkosky model.

- We implement properties of the permutation group S_5
- We extract the ground and excited states along with the spectra of two-, three and four-body equations.
- Our study serves as a building block for the calculation of pentaquark properties using functional methods.

Five-body equation



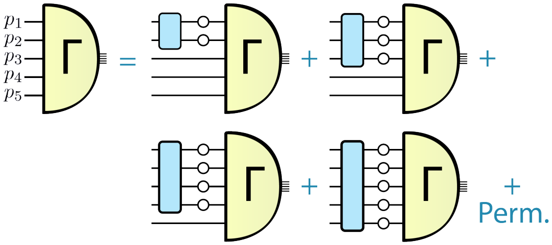
G.Eichmann, M.T.Peña and ***R. D. Torres****, arXiv:2502.17944.*

Starting point The homogeneous Bethe-Salpeter equation for a five body system:

$$\Gamma^{(5)} = K^{(5)} G_0^{(5)} \Gamma^{(5)} \quad (1)$$

$K^{(5)}$ is the five-body interaction kernel.
$G_0^{(5)}$ is the product of five dressed propagators.
$\Gamma^{(5)}$ is the five-body Bethe-Salpeter amplitude.

Five-body equation



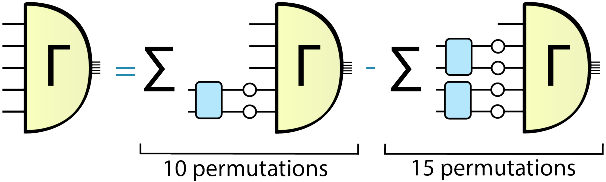
The five-body BSE can be derived from the pole behavior of the 5-body scattering matrix $T^{(5)}$, which is a ten-point correlation function

$$T^{(5)} = K^{(5)} + K^{(5)} G_0^{(5)} T^{(5)} \qquad (2)$$

At a given bound-state or resonance pole with mass M,

$$T^{(5)} \longrightarrow \frac{\Gamma^{(5)} \overline{\Gamma}^{(5)}}{P^2 + M^2}, \qquad (3)$$

Subtraction diagrams



A naive summation of two body kernels leads to **overcounting.**

Phys. Lett. B 718, 545 (2012).

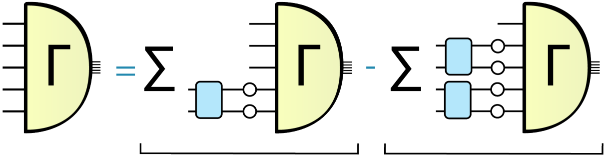
In a five-body system there are ten possible two-body kernels

$$K_a \in \{K_{12},\, K_{13},\, K_{14},\, K_{15},\, K_{23},\, K_{24},\, K_{25},\, K_{34},\, K_{45}\}, \tag{4}$$

and 15 independent double-kernel

$$\begin{aligned} K_aK_b \in \{&K_{12}K_{34},\, K_{12}K_{35},\, K_{12}K_{45},\\ &K_{13}K_{24},\, K_{13}K_{25},\, K_{13}K_{45},\\ &K_{14}K_{23},\, K_{14}K_{25},\, K_{14}K_{35},\\ &K_{15}K_{23},\, K_{15}K_{24},\, K_{15}K_{34},\\ &K_{23}K_{45},\, K_{24}K_{35},\, K_{25}K_{34}\}. \end{aligned} \tag{5}$$

Explicit form of the BSE

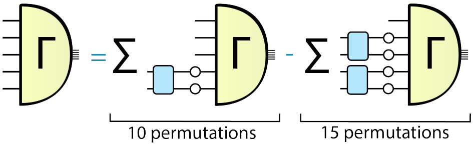


The BSE depends on five momenta $p_1, \cdots p_5$, whose sum is the total onshell momentum P with $P^2 = -M^2$.

$$\Gamma\left(\{p_i\}\right) = \sum_{a}^{10} \Gamma_{(a)}\left(\{p_i\}\right) - \sum_{a \neq b}^{15} \Gamma_{(a,b)}\left(\{p_i\}\right) \quad (6)$$

$$\begin{aligned}
\Gamma_{(a)}\left(\{p_i\}\right) &= \int \frac{d^4 r}{(2\pi)^4} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) \\
&\quad \times D(q_{a_1}) D(q_{a_2}) \Gamma(\{p_i\}, a), \\
\Gamma_{(a,b,)}\left(\{p_i\}\right) &= \int \frac{d^4 r}{(2\pi)^4} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) \\
&\quad \times D(q_{a_1}) D(q_{a_2}) \Gamma_{(b)}\left(\{p_i\}, a\right).
\end{aligned}$$

Lorentz invariants



In practice we work with the total momentum P and four relative momenta p, q, k, l instead of the five particle momenta.

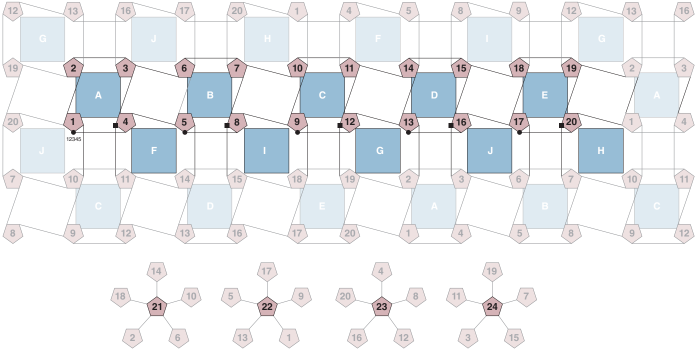
$$q = \frac{p_1 - p_5}{2}, \quad p = \frac{p_2 - p_5}{2}, \quad k = \frac{p_3 - p_5}{2}, \quad l = \frac{p_4 - p_5}{2}. \tag{7}$$

The amplitude $\Gamma(q, p, k, l, P)$ depends on 15 Lorentz invariants

$$\begin{aligned}
&q^2\,, && \omega_1 = q\cdot p\,, && \\
&p^2\,, && \omega_2 = q\cdot k\,, && \eta_1 = q\cdot P\,,\\
&k^2\,, && \omega_3 = q\cdot l\,, && \eta_2 = p\cdot P\,,\\
&l^2\,, && \omega_4 = p\cdot k\,, && \eta_3 = k\cdot P\,,\\
&P^2\,, && \omega_5 = p\cdot l\,, && \eta_4 = l\cdot P\,.\\
& && \omega_6 = k\cdot l\,, &&
\end{aligned} \tag{8}$$

These can be arranged into two singlets, two quartets and a quintet in S_5.

Permutation Group S_5

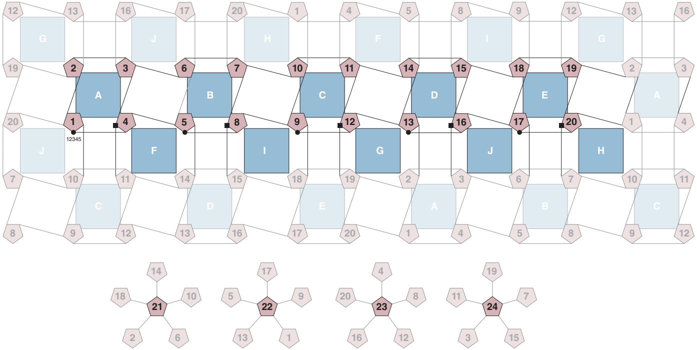


S_5 consist of $5! = 120$ elements and each permutation of an object f_{12345} can be reconstructed from two group elements, a transposition T and a $5 - cycle$ C.

*G.E. and **R. D. T.**, arXiv:2502.17225.*

The Cayley graph can be visualized by 24 pentagons, which are connected to each other by transpositions. The members are obtained by two successive chains of the form $(\mathrm{C}^2, \mathrm{T}^2, \mathrm{C}^2, \mathrm{T}, \mathrm{CT}, \mathrm{T})$.

Permutation Group S_5

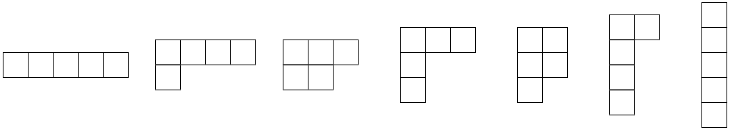


For later use, we collect the elements in the first five squares (A, B, C, D, E) into five vectors $f^{(n)}$ with $n = 1, \cdots, 5$. Together with the transpositions $\mathsf{T} f^{(n)}$, this yields twelve elements for each vector.

The remaining five squares (F, G, H, I, J) define the vectors $\tilde{f}^{(n)}$ with $n = 1, \cdots, 5$. Again, combined with the transpositions $\mathsf{T} \tilde{f}^{(n)}$, each vector $\tilde{f}^{(n)}$ defines a closed path with twelve elements.

Permutation Group S_5

$\mathbf{1}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	$\bar{\mathbf{5}}$	$\bar{\mathbf{4}}$	$\bar{\mathbf{1}}$
$\mathcal{S}$	$\mathcal{Q}_i^+$	$\mathcal{V}_j^+$	$\mathcal{W}_k$	$\mathcal{V}_j^-$	$\mathcal{Q}_i^-$	$\mathcal{A}$



$$\mathsf{T}\begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathcal{Q}_i^\lambda \\ \mathcal{V}_j^\lambda \\ \mathcal{W}_k \end{bmatrix} = \begin{bmatrix} \mathcal{S} \\ -\mathcal{A} \\ \lambda\, \mathbf{T}_4^T\, \mathcal{Q}_i^\lambda \\ \lambda\, \mathbf{T}_5^T\, \mathcal{V}_j^\lambda \\ \mathbf{T}_6^T\, \mathcal{W}_k \end{bmatrix}, \quad \mathsf{C}\begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathcal{Q}_i^\lambda \\ \mathcal{V}_j^\lambda \\ \mathcal{W}_k \end{bmatrix} = \begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathbf{C}_4^T\, \mathcal{Q}_i^\lambda \\ \mathbf{C}_5^T\, \mathcal{V}_j^\lambda \\ \mathbf{C}_6^T\, \mathcal{W}_k \end{bmatrix}. \tag{9}$$

The 120 permutations transform under irreducible representations of S_5 and are given by Young diagrams of S_5.

Permutation Group S_5 and five-body systems

One singlet is P^2 and the other

$$\mathcal{S}_0 = \frac{1}{5}\left(q^2 + p^2 + k^2 + l^2 - \frac{1}{2}\sum_{i=1}^{6}\omega_i\right). \tag{10}$$

The quintet is given by

$$\mathcal{V} = \begin{bmatrix} \sqrt{\frac{2}{3}}\,(\omega_1 + \omega_2 - \omega_3 + \omega_4 - \omega_5 - \omega_6) \\ 2\omega_1 - \omega_2 - \omega_3 - \omega_4 - \omega_5 + 2\omega_6 \\ \sqrt{3}\,(\omega_2 - \omega_3 - \omega_4 + \omega_5) \\ \omega_2 + \omega_3 - \omega_4 - \omega_5 \\ \frac{1}{\sqrt{3}}\,(2\omega_1 - \omega_2 + \omega_3 - \omega_4 + \omega_5 - 2\omega_6) \end{bmatrix}.$$

The two quartets are given by

$$\mathcal{Q}_1 = \begin{bmatrix} q^2 - p^2 - \frac{2}{3}\,v_4 \\ \frac{1}{\sqrt{3}}\,(q^2 + p^2 - 2k^2) - \frac{2}{3}\,v_5 \\ \frac{1}{\sqrt{6}}\,(q^2 + p^2 + k^2 - 3l^2) - \frac{2}{3}\,v_1 \\ -\frac{1}{\sqrt{10}}\,\frac{5}{3}\,(q^2 + p^2 + k^2 + l^2 - 8\mathcal{S}_0) \end{bmatrix},$$

$$\mathcal{Q}_2 = \begin{bmatrix} \eta_1 - \eta_2 \\ \frac{1}{\sqrt{3}}\,(\eta_1 + \eta_2 - 2\eta_3) \\ \frac{1}{\sqrt{6}}\,(\eta_1 + \eta_2 + \eta_3 - 3\eta_4) \\ \frac{1}{\sqrt{10}}\,(\eta_1 + \eta_2 + \eta_3 + \eta_4) \end{bmatrix}, \tag{11}$$

Permutation Group S_n and n-body systems

n	$\mathcal{S}_0$	P^2	η_i	$p_1^2 \dots p_n^2$	$\mathcal{M}$	Total	Indep.
2	1	1	1	–	–	3	3
3	1	1	2	2	–	6	6
4	1	1	3	3	2	10	10
5	1	1	4	4	5	15	14
6	1	1	5	5	9	21	18
n	1	1	$n-1$	$n-1$	$n(n-3)/2$	$n(n+1)/2$	$4n-6$

- P^2 is a singlet, and one singlet as the sum $p_1^2 + \cdots + p_n^2$.
- $n-1$ angular variables η_i form an $(n-1)-$plet.
- The variables $p_1^2 \cdots p_n^2$ form another $(n-1)-$plet.
- $n(n-3)/2$ gives another multiplet $\mathcal{M}$.

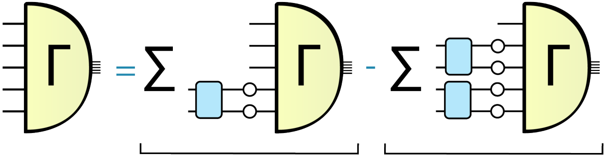
Five-body system - dimensional constraint

$$P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix},\ q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix},\ p = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \end{bmatrix},\ k = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix},\ l = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \tag{12}$$

- n four-vectors depend on $4n-6$ independent variables.
- The five vectors only have 14 independent entries and thus only 14 independent Lorentz invariants.
- $n-$body system with $n \geq 5$ independent four-momenta gives $4n-6$ independent Lorentz invariants (opposed to $n(n+1)/2$).
- In $n=5$ yields one constraint equation.
- The constraint relates all variables.

$$\begin{aligned}
&\left[(\omega_{12}\,\omega_{34} + \omega_{13}\,\omega_{24} + \omega_{14}\,\omega_{23})^2 - q_1^2\,q_2^2\,q_3^2\,q_4^2\right] P^2 \\
&+ \sum_i \eta_i^2 \left[q_j^2\,q_k^2\,q_l^2 - q_j^2\,\omega_{kl}^2 - q_k^2\,\omega_{jl}^2 - q_l^2\,\omega_{jk}^2\right] \\
&+ 2\sum_i \left(\eta_i^2 - q_i^2\,P^2\right)\omega_{jk}\,\omega_{jl}\,\omega_{kl} \\
&+ \sum_{i<j} \eta_i\,\eta_j \left[2q_k^2\left(\omega_{il}\,\omega_{jl} - q_l^2\,\omega_{ij}\right) + q_l^2\,\omega_{ik}\,\omega_{jk}\right. \\
&\qquad\qquad \left. + \,\omega_{kl}\left(\omega_{ij}\,\omega_{kl} - \omega_{ik}\,\omega_{jl} - \omega_{il}\,\omega_{jk}\right)\right] \\
&+ P^2 \sum_{i<j} \omega_{ij}^2\ \left(q_k^2\,q_l^2 - \omega_{kl}^2\right) = 0\,,
\end{aligned} \tag{13}$$

Explicit form of the BSE



• Solutions of two-, three- and four-body systems show that the dependence on the angular variables η_i is usually small or negligible.

• Four-body BSE dynamically generates intermediate two-body poles in the solution process.

• In a five body there are 10 possible two- and three-body configurations.

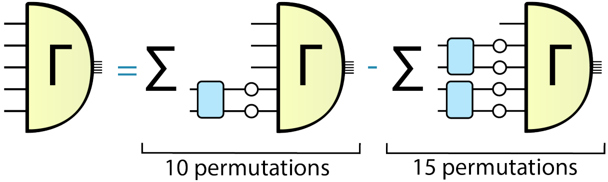
We reduce the momentum dependence to $\mathcal{S}_0$ but include two- and three-body poles:

$$\Gamma(p,q,k,l,P) \approx f(\mathcal{S}_0) \sum_{aa'} \mathcal{P}_{aa'}, \qquad (14)$$

with

$$\mathcal{P}_{(12)(345)} = \frac{1}{(p_1+p_2)^2 + M_M^2} \, \frac{1}{(p_3+p_4+p_5)^2 + M_B^2} \, .$$

Explicit form of the BSE



- Procedure requires knowledge of bound-state masses M_M, M_B of the two- and three-body equations in the same approach.
- We employ a dimensionless coupling constant c and mass ratio β via:

$$c = \frac{g^2}{(4\pi m)^2}, \quad \beta = \frac{\mu}{m} \tag{15}$$

- Employing tree-level propagators for scalar particles with mass m and a ladder approximation for a boson exchange with mass μ:

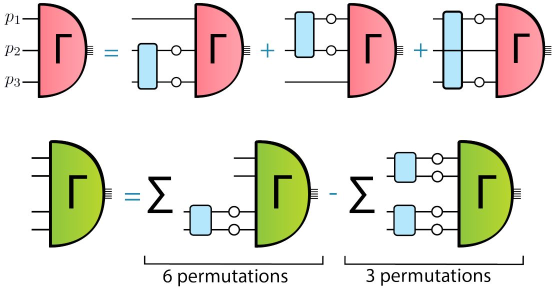
$$\begin{aligned} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) &= \frac{g^2}{r^2 + \mu^2}, \\ D(p) &= \frac{1}{p^2 + m^2}, \end{aligned} \tag{16}$$

Explicit form of the BSEs

The BSEs turn into eigenvalue equations of the form

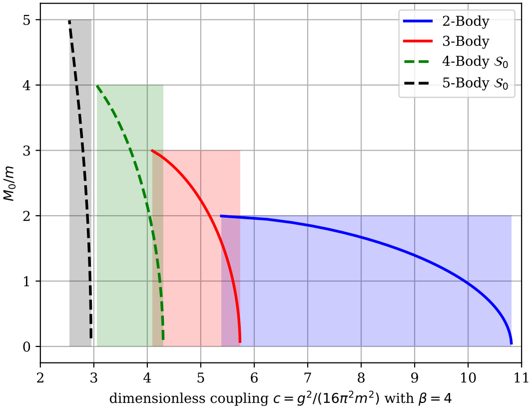
$$\lambda_i(P^2)\Psi_i(P^2) = \mathcal{K}(P^2)\Psi_i(P^2), \quad (17)$$

where $P^2 \in \mathbb{C}$ is the five-body momentum squared, $\mathcal{K}(P^2)$ is the kernel and the $\Psi_i(P^2)$ are its eigenvectors with eigenvalues $\lambda_i(P^2)$ for the ground ($i = 0$) and excited states ($i > 0$).

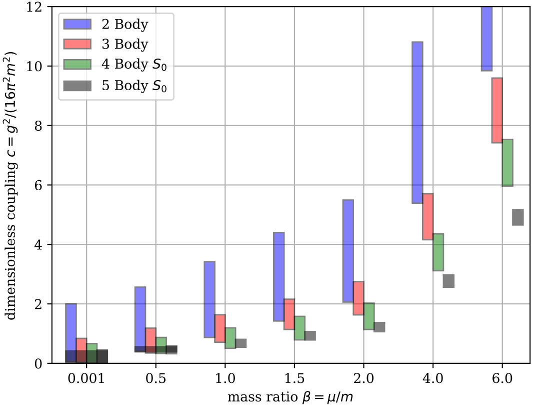


$$\Gamma(\{p_i\}) = \sum_a^3 \Gamma_{(a)}(\{p_i\}),$$
$$\Gamma(\{p_i\}) = \sum_a^6 \Gamma_{(a)}(\{p_i\}) - \sum_{a\neq b}^3 \Gamma_{(a,b)}(\{p_i\}). \quad (18)$$

Results

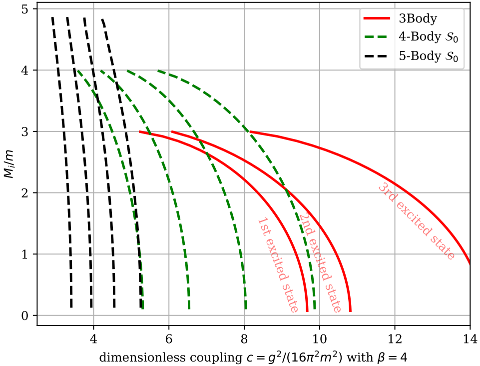


n-body BSEs for $\beta = 4$ and different values of coupling c.

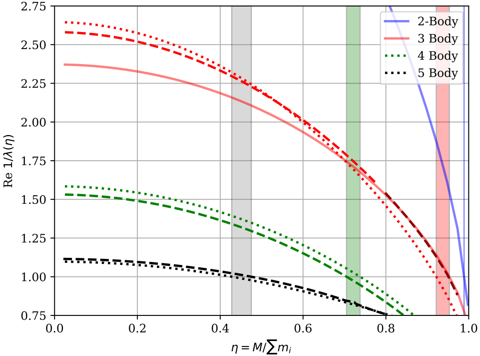


Coupling ranges where $n-$body ground states are possible for different values of β.

Results

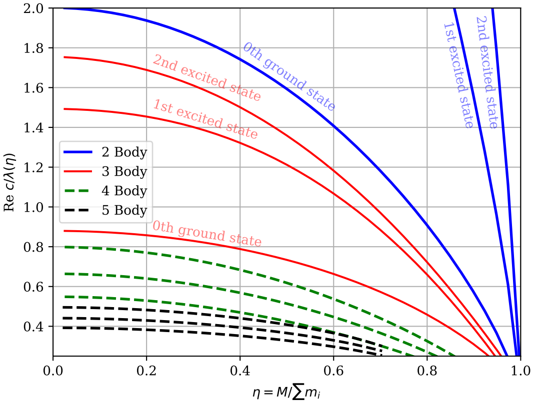


Radially excited state masses $M_{i>0}$ for $\beta = 4$ and different values of the coupling c.



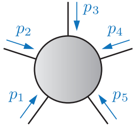
Eigenvalues of the $n-$body BSEs $\beta = c = 0.5$. Solid curves are full solutions, dotted singlet approx. and dashed singles $\times$ pole approx.

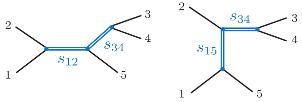
Results

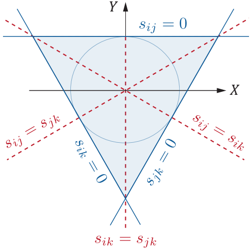


BSE eigenvalues for $\beta = 0.001$, with $c = 1/4$ to ensure coexisting solutions for the two-, three-, four- and five-body BSEs.

Results - extra applications with S_5 for five-point functions







- Five-point function depends on five incoming momenta with the sum $p_1^\mu + \cdots + p_5^\mu = 0$. Singlet vanishes and remaining four independent momenta form a quartet:

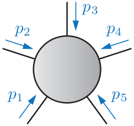
$$\mathcal{Q} = \begin{bmatrix} p_1^\mu - p_2^\mu \\ \frac{1}{\sqrt{3}}\left(p_1^\mu + p_2^\mu - 2p_3^\mu\right) \\ \frac{1}{\sqrt{6}}\left(p_1^\mu + p_2^\mu + p_3^\mu - 3p_4^\mu\right) \\ \frac{1}{\sqrt{10}}\left(p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu - 4p_5^\mu\right) \end{bmatrix} = \begin{bmatrix} p^\mu \\ q^\mu \\ k^\mu \\ l^\mu \end{bmatrix}. \qquad (19)$$

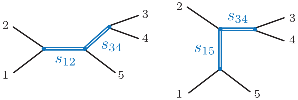
- One obtains a singlet $\mathcal{Q}\cdot\mathcal{Q}$, a quartet $\mathcal{Q}\wedge\mathcal{Q}$, a quintet $\mathcal{Q}\cup\mathcal{Q}$ and a sextet $\mathcal{Q}\star\mathcal{Q}$.
- Five-point functions can define ten Mandelstam variables

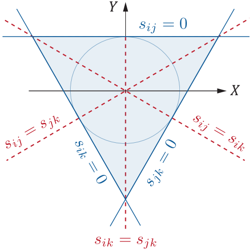
$$s_{ij} = (p_i + p_j)^2, \quad X = \frac{\sqrt{3}\,(s_{ik} - s_{jk})}{s_{ik} + s_{jk} + s_{ij}}, \quad Y = \frac{s_{ik} + s_{jk} - 2s_{ij}}{s_{ik} + s_{jk} + s_{ij}}$$

Intermediate resonances appear at fixed s_{ij}.

Results - extra applications with S_5 for five-point functions







- Five-gluon vertex general form reads

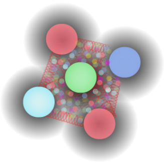
$$\Gamma^{\mu\nu\alpha\beta\gamma}(p_1 \ldots p_5) = \sum_{ij} F_{ij}(\ldots)\, \tau_i^{\mu\nu\alpha\beta\gamma}\, \mathsf{c}^{(j)}_{abcde} \,. \qquad (20)$$

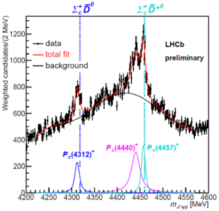
There are four types of color structures as seed elements

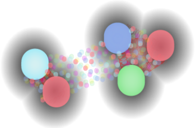
$$\begin{array}{llll} \delta_{ab}\, f_{cde}\,, & f_{abr}\, f_{cds}\, f_{ers}\,, & f_{ij} = \delta_{ij}\, f_{klm}\,, & \\ & & & k < l < m\,, \\ \delta_{ab}\, d_{cde}\,, & f_{abr}\, f_{cds}\, d_{ers}\,. & d_{ij} = \delta_{ij}\, d_{klm}\,, & \end{array} \qquad (21)$$

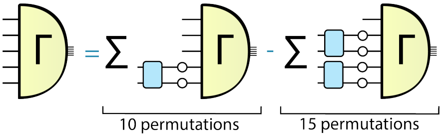
- Ten permutations of d_{ij} can be grouped into a singlet, a quartet and a quintent.
- Ten of f_{ij} return an antiquartet and a sextet.
- The seed $f_{abr}\, f_{cds}\, f_{ers}$ form a sextet.
- The seed $f_{abr}\, f_{cds}\, d_{ers}$ yields an antisinglet and an anti-quartet.

Summary





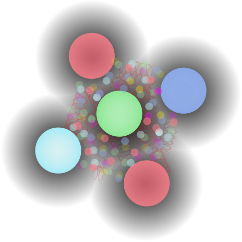


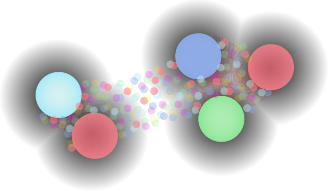


- We developed the five-body Bethe-Salpeter formalism and solve the five-body equation for a scalar model in a ladder truncation.
- We discussed applications of the permutation group S_5 for five-point functions and five-body wave functions.
- The approach developed in this work can be extended to QCD in view of investigating pentaquarks, and work in this direction is underway.

ACKNOWLEDGEMENTS

Thank you!





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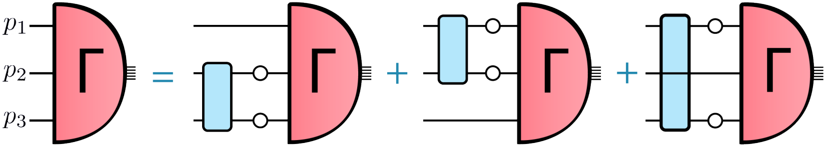
Backup slides - Three-body equation

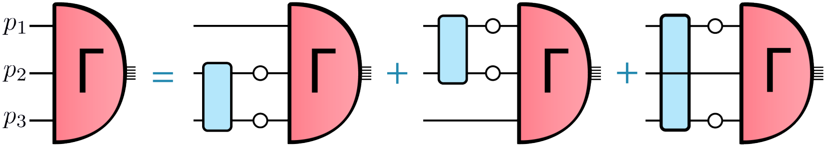


The Bethe-Salpeter amplitude $\Gamma(\{p_i\})$ for a three-body system depends on three momenta p_1, p_2, p_3, whose sum is the total onshell momentum P with $P^2 = -M_B^2$.

$$\Gamma(\{p_i\}) = \sum_a^3 \Gamma_{(a)}(\{p_i\})\,.$$
$$\Gamma_{(a)}(\{p_i\}) = \int \frac{d^4 r}{(2\pi)^4} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) D(q_{a_1})\, D(q_{a_2})\, \Gamma(\{p_i\}, a)\,. \tag{22}$$

In this care the are three possible two-body kernels $K_a \in \{K_{12}, K_{13}, K_{23}\}$ whose sum does not lead top overcounting in the three-body scattering matrix kernel $T^{(3)}$.

Backup slides - Three-body equation



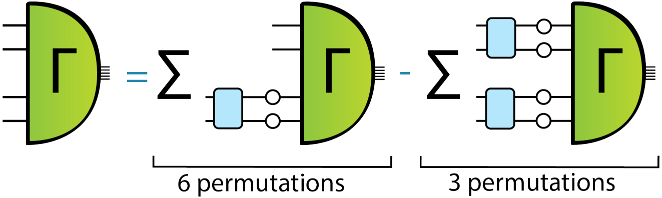
We furthermore employ a single $\times$ pole approximation

$$\Gamma(q,p,P) \approx f(\mathcal{S}_0) \sum_a \mathcal{P}_a, \qquad \mathcal{P}_{12} = \frac{1}{(p_1+p_2)^2 + M_M^2}, \tag{23}$$

with M_M the mass of the two-body subsystem or 'diquark', and the singlet variable $\mathcal{S}_0$ is

$$\mathcal{S}_0 = \frac{q^2}{3} + \frac{p^2}{4}. \tag{24}$$

Backup slides - Four-body equation



In this case there are six possible two-body kernels

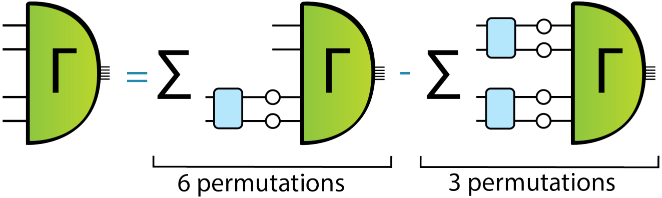
$$K_a \in \{K_{12}\,,\ K_{13}\,,\ K_{14}\,,\ K_{23}\,,\ K_{24}\,,\ K_{34}\} \tag{25}$$

and three independent double-kernel configurations of the form

$$K_a\,K_b \in \{K_{12}\,K_{34}\,,\ K_{13}\,K_{24}\,,\ K_{14}\,K_{23}\}\,. \tag{26}$$

The four-body Bethe-Salpeter amplitude $\Gamma(\{p_i\})$ depends on four-momenta $p_1 \ldots p_4$, whose sum is the total onshell momentum P with $P^2 = -M_T^2$ (T for 'tetra').

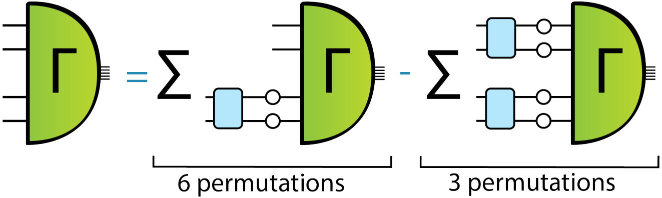
Backup slides - Four-body equation



The four-body equation is the Faddeev-Yakubowsku equation and can be written as

$$
\begin{aligned}
\Gamma(\{p_i\}) &= \sum_{a}^{6} \Gamma_{(a)}(\{p_i\}) - \sum_{a \neq b}^{3} \Gamma_{(a,b)}(\{p_i\}) \,, \\
\Gamma_{(a)}(\{p_i\}) &= \int \frac{d^4 r}{(2\pi)^4} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) D(q_{a_1})\, D(q_{a_2})\, \Gamma(\{p_i\}, a) \,, \\
\Gamma_{(a,b)}(\{p_i\}) &= \int \frac{d^4 r}{(2\pi)^4} K(p_{a_1}, q_{a_1}, p_{a_2}, q_{a_2}) D(q_{a_1})\, D(q_{a_2})\, \Gamma_{(b)}(\{p_i\}, a) \,,
\end{aligned}
\tag{27}
$$

Backup slides - Four-body equation



In the four-body case we employ the singlet $\times$ pole approximation

$$\Gamma(q,p,k,P) \approx f(\mathcal{S}_0) \sum_a \mathcal{P}_{aa'} \,, \tag{28}$$

where the two-body poles for $aa' = (12)(34)$ are given by

$$\mathcal{P}_{(12)(34)} = \frac{1}{(p_1+p_2)^2 + M_M^2} \frac{1}{(p_3+p_4)^2 + M_M^2} \tag{29}$$

and M_M is the mass of the two-body subsystem. The singlet variable $\mathcal{S}_0$ is

$$\mathcal{S}_0 = \frac{k^2+q^2+p^2}{4} \,. \tag{30}$$