Scale dependence of non-perturbative parton distributions

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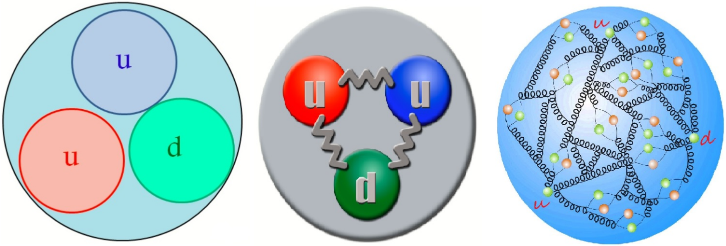
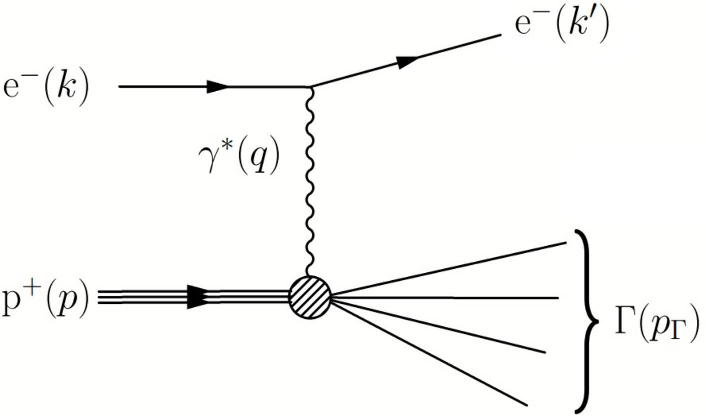
How to gain insight into the structure of hadrons

- Hadrons such as the proton are a mess of many interacting quarks/gluons!



- Nevertheless, protons have well-defined physical properties such as mass, spin etc.
  $\Rightarrow$ How can we explain these in terms of the properties of the constituent partons?
- Experimentally: Perform high-energy **scattering experiments** that can resolve the inner hadron structure (e.g. scatter electrons off a proton)

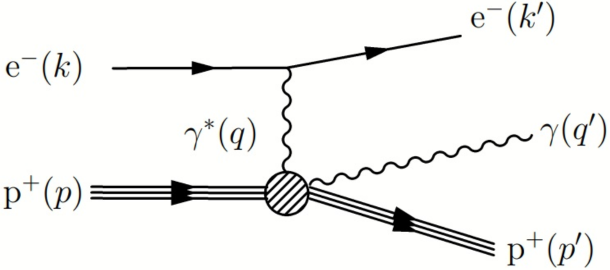
Scattering experiments: Deeply-inelastic scattering



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

Scattering experiments: Deeply-virtual Compton scattering



* Virtuality: $Q^2 = -q^2$
* Bjorken-x: $x_B = \frac{Q^2}{2p\cdot q}$
* Momentum transfer on hadronic target: $t = (p - p')^2 \equiv \delta^2$
* Skewedness: $\xi = \frac{(p-p')^+}{(p+p')^+}$ [lightcone coordinates: $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^3)$]

Description of scattering experiments

- Hard scale $\Rightarrow$ Factorization between short-range and long-range physics

$$\hat{\sigma}(p_A, p_B) = \sum_{a,b} \int_0^1 \mathrm{d}x_a\, f_{a/A}(x_a, \mu_F^2) \int_0^1 \mathrm{d}x_b\, f_{b/B}(x_b, \mu_F^2)\, \sigma_{ab}(p_a, p_b; \mu_F^2)$$

- Short-range physics characterized by the perturbative partonic cross section σ_{ab}
- Long-range physics described by non-perturbative parton distributions like **PDFs** and **GPDs**
- Through application of the **OPE**, these distributions are related to hadronic matrix elements of composite QCD operators

Leading-twist operators

The OPE is dominated by leading-twist operators, where *twist = dimension - spin*. We can distinguish 2 sets of leading-twist operators based on their representation in the QCD flavour group.

- Flavour non-singlet quark operator

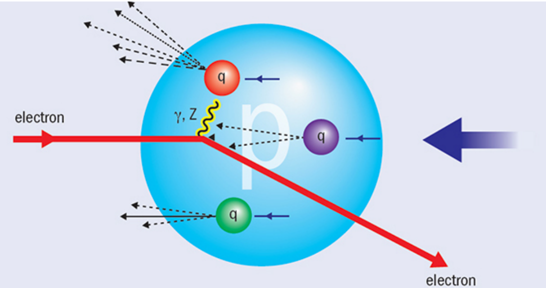
$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\lambda^{\alpha}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

- Flavour singlet quark operator + gluon operator

$$\mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\,F^{a_1}{}_{\mu\mu_1}\,D^{a_1a_2}_{\mu_2}\ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}\,F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$
$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

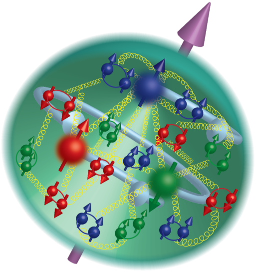
PDFs

- PDFs are defined in terms of forward matrix elements of the operators, $f_i(x) \sim \langle p^+(p)| \mathcal{O}^i_{\mu_1 \dots \mu_N} |p^+(p)\rangle$
- Probability to find a parton inside the proton with momentum xp $(0 \leq x \leq 1)$
- Encode the longitudinal momentum/polarization carried by partons
- Accessible in inclusive processes (e.g. DIS)



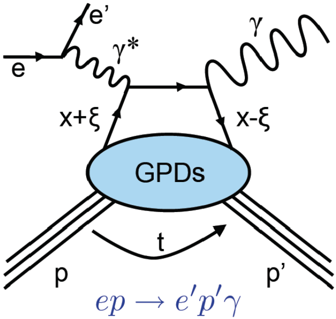
GPDs

- GPDs [Müller et al., 1994],[Radyushkin, 1996],[Ji, 1997] correspond to non-forward matrix elements of composite operators, $\langle p^+(p)|\,\mathcal{O}^i_{\mu_1\ldots\mu_N}\,|p^+(p')\rangle$. They generalize other types of non-perturbative QCD quantities like PDFs, form factors and distribution amplitudes.
- Transverse structure of target from GPDs, can be combined with longitudinal information $\Rightarrow$ 3D-description of hadron structure!
- If polarization of target changes during scattering: GPDs encode rich spin structure



GPDs

- Accessible in hard exclusive scattering processes (e.g. DVCS)
  $\rightarrow$ Very precise measurements to come in (near) future! (EIC [Boer et al., 2011],[Abdul Khalek et al., 2022]/EicC [Anderle et al., 2021], LHeC [Abelleira Fernandez et al., 2012], JLab22 upgrade [Accardi et al., 2024], ...)



Parton distributions and their scale dependence

As the distributions themselves are non-perturbative, we cannot use our standard perturbative techniques to derive them from first principles
$\rightarrow$ Fit from experimental data (see e.g. [Brock et al., 1995]) or use lattice techniques (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021], [Karthik and Sufian, 2021], [Gao et al., 2023])

However, the energy scale dependence of the distributions can be calculated perturbatively!

Forward case (DGLAP [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]):

$$\frac{\mathrm{d} f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

Non-forward case ([Müller et al., 1994],[Radyushkin, 1996],[Ji, 1997]):

$$\frac{\mathrm{d}\mathcal{G}(x,\xi,t;\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} \mathcal{P}\left(\frac{x}{y},\frac{\xi}{y}\right) \mathcal{G}(y,\xi,t;\mu^2)$$

Scale dependence of parton distributions

Because of the direct relation between the distributions and QCD operators, the scale dependence of the distributions is determined by the scale dependence of the operators, characterized by their anomalous dimension

$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}].$$

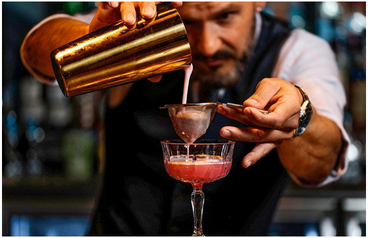
These anomalous dimensions can be computed perturbatively in QCD by renormalizing the (off-shell) partonic matrix elements of the operators.

$$\gamma^{ij}_{N,N} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x).$$

$$\underbrace{\sum_{k=0}^{N} \gamma_{N,k} y^k}_{\text{mixing!}} = -\int_0^1 \mathrm{d}x\, x^N\, \mathcal{P}(x,y).$$

Operator mixing: A cocktail of anomalous dimensions

- ◇ Without mixing: $1/\varepsilon$-pole of matrix element $\Rightarrow$ anomalous dimension
- ◇ With mixing: $1/\varepsilon$-pole gets multiple contributions $\Rightarrow$ How to disentangle?



- Non-forward kinematics: Mixing with total-derivative operators
- In flavour singlet case: Mixing with non-gauge-invariant or (👽) operators

For specific choices of operator bases, both sources of mixing can be analyzed using conjugation relations

Conjugations

Suppose we have a function f of some discrete variable N. A conjugation is then a specific sum over f that, when applied twice, gives back the **original** function. For example, if we have some function $f(N)$, then its **binomial transform**,

$$[f(N)]^C = \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(i)$$

is a conjugation since

$$\Rightarrow \left([f(N)]^C\right)^C = \sum_{j=0}^{N} (-1)^j \binom{N}{j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} f(i) = f(N)\,.$$

Relations based on such conjugations typically have a significantly reduced **function space**!

Solving conjugation relations

- To take full advantage of conjugation relations, one needs to be able to evaluate them analytically
- Use principles of **symbolic summation**!
- **Creative telescoping** [Zeilberger, 1991]: evaluate the sum of interest by rewriting it as a recursion relation using **Gosper's algorithm** [Gosper, 1978]
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

$\rightarrow$ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

$\rightarrow$ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Non-forward anomalous dimensions

To treat the mixing of operators with total-derivative ones in non-forward kinematics, we select the following basis (focus on flavour-non-singlet case)

$$\mathcal{O}^{\mathcal{D}}_{k,N-k} = (\Delta \cdot \partial)^k \{\overline{\psi'}(\Delta \cdot \Gamma)(\Delta \cdot D)^{N-k}\psi\} \qquad [\Delta^2 = 0]$$

$\rightarrow$ Based on counting derivatives, used e.g. for hadronic studies on the lattice, see [Gockeler et al., 2005] and [Gracey, 2009]

By also considering operators in which the covariant derivative acts on $\overline{\psi'}$, one can construct recursion relations between the operators which lead to consistency relations between the anomalous dimensions

$$\forall k: \qquad \sum_{j=k}^{N} \left\{ (-1)^k \binom{j}{k} \gamma^{qq,\mathrm{NS}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{qq,\mathrm{NS}}_{j,k} \right\} = 0\,.$$

Note that, for $k = 0$, this reduces to a conjugation as defined above

$$\left[\gamma^{qq,\mathrm{NS}}_{N,0}\right]^C = \sum_{j=0}^{N} \gamma^{qq,\mathrm{NS}}_{N,j}$$

Non-forward anomalous dimensions

- These relations were used in [Moch and Van Thurenhout, 2021] to determine the anomalous dimensions in the leading-n_f limit to 5-loop accuracy and in the planar limit to 2 loops
- Relations **independent** of the Dirac structure
  $\rightarrow$ 4-loop transversity ($\Gamma \sim [\gamma_\mu, \gamma_\nu]$) anomalous dimensions in leading-n_f limit [Van Thurenhout, 2022]
- Similar relations can be derived for **different types** of operators; e.g. in the flavour-singlet sector

$$\forall k > 0 : \sum_{j=k}^{N} \left\{ (-1)^k \binom{j-1}{k-1} \gamma_{N,j}^{gg} - (-1)^j \binom{N-1}{j-1} \gamma_{j,k}^{gg} \right\} = 0$$

$\rightarrow$ Derived at 1-loop level in which mixing with aliens can be ignored
$\rightarrow$ Hints that it nevertheless stays valid beyond 1-loop accuracy [needs further investigation!]

Forward singlet anomalous dimensions

In the flavour-singlet sector, one needs to take into account alien operators

- Non-gauge-invariant operators (👻)
- EOM operators

Recently, G. Falcioni and F. Herzog derived a method to consistently construct the aliens to any loop-order [Falcioni and Herzog, 2022].

- In their approach, the aliens are derived using **generalized gauge symmetry** of the QCD Lagrangian.
- Each alien operator features a coupling constant which obeys certain **constraint relations**, which were solved for fixed $N \leq 20$ in

  [Falcioni and Herzog, 2022, Falcioni et al., 2024a]
- The couplings of the bare alien operators can be interpreted as the renormalization constants that mix the physical operators into the aliens

Forward singlet anomalous dimensions

$$\mathcal{O}_{\text{EOM}}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) f^{abc} \sum_{i+j=N-3} \kappa_{ij} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_c^{(N),II} = -g_s f^{abc} \sum_{i+j=N-3} \eta_{ij} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0,$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i},$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0$$

NOTE: Bottom relation = conjugation!

- Using the techniques employed in [Moch and Van Thurenhout, 2021, Van Thurenhout, 2022], we were then able to derive the alien couplings to leading order in g_s but for all values of N [Falcioni et al., 2024b]

Summary & outlook

- Hadronic structure is characterized by non-perturbative parton distributions
- The scale dependence of such distributions can be computed perturbatively as the **anomalous dimensions** of the operators that define them
- Such perturbative calculations are complicated due to several sources of operator mixing
- Uniform approach: Consistency relations based on **conjugations**
- Several **extensions** in principle possible but still to be looked at (e.g. properly taking into account aliens for the non-forward flavour-singlet anomalous dimensions)

End [1]

Thank you for your attention!

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Appendices and references

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned}\sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a).\end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

Suppose

$$\frac{g(N)}{g(N-1)}$$

is a rational function in N. The algorithm consists of three main steps. Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that
$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$
and
$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$
2. Solve the so-called Gosper equation,
$$f(x) = g(x)y(x+1) - h(x)y(x),$$
for the polynomial $y(x)$.
3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by
$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Creative telescoping

Classical telescoping works when dealing with sequences that depend on one variable only. When we want to determine a closed form for a summation of a sequence depending on two variables, we can use the creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{b} f(N,k) \equiv S(N).$$

The way to go about this is by attempting to find d functions $c_0(N), \ldots, c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N) f(N,k) + \ldots + c_d(N) f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,b+1) - g(N,a) = c_0(N) \sum_{k=a}^{b} f(N,k) + \ldots + c_d(N) \sum_{k=a}^{b} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N+d).$$

Typically, one starts this procedure at $d = 0$, which is equivalent to classical telescoping. The value of d is then increased stepwise until a solution is found. The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N+1)S_1(N) - (2N+3)S_1(N+1) + (N+2)S_1(N+2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovŝek et al., 1996].

4-step algorithm for constructing the non-forward ADM

In practical computations we use a different representation of the consistency relations

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}.$$

- ✓ Order-independent consistency check
- ✓ Can be used to construct the full ADM from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ + boundary condition to ensure uniqueness of the solution ($\gamma^{\mathcal{D}}_{N,0}$, from Feynman diagrams)

4-step algorithm for constructing the non-forward ADM

1. Calculate

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,\,j+k}$$

   and construct an Ansatz for the off-diagonal piece

2. Calculate

$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,j}$$

3. Substitute into the consistency relation $\Rightarrow$ System of equations, solution not necessarily unique $\Rightarrow$ Need boundary condition!
4. Determine all-N expression for $\gamma^{\mathcal{D}}_{N,0}$ from Feynman diagrams

Getting actual predictions for hard exclusive processes

To obtain predictions for physical observables in hard exclusive processes, like cross-sections and spin/charge asymmetries, one needs to combine the coefficient functions (state of the art: NNLO for DVCS [Braun et al., 2022]) with a GPD model. The GPD evolution kernels (operator anomalous dimensions) are needed to evolve the GPDs from some reference scale to the scale of interest.

→ Several numeric codes for this purpose exist, e.g.

- PARTONS (numeric code for GPD phenomenology) [Berthou et al., 2018]
  → `https://partons.cea.fr/partons/doc/html/index.html`
- Vinnikov code (LO GPD evolution) [Vinnikov, 2006]
- GPD evolution for DVCS @ NLO [Freund and McDermott, 2002]
- Gepard [Kumericki et al., 2008]
  → `https://gepard.phy.hr/index.html`
- Twist-2 GPD evolution in momentum space [Bertone et al., 2022, Bertone et al., 2024]
  → available through APFEL++ [Bertone et al., 2014, Bertone, 2018] and PARTONS

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