Schwarzschild spacetime with extra compact dimensions

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Einstein gravity

Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi\kappa T_{\mu\nu}$$

without cosmological constant, $\Lambda = 0$, and in vacuum $T_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$\Downarrow$$

$$\boxed{R_{\mu\nu} = 0}$$

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\rho\nu}$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right)$$

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Schwarzschild solution

Minkowski spacetime

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2 \underbrace{\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)}_{d\Omega^2_{(2)}} \end{aligned}$$

Schwarzschild spacetime

$$ds^2 = -\left(1 + \frac{a}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{a}{r}} + r^2 d\Omega^2_{(2)}$$

$$a = -2\kappa M$$

K. **Schwarzschild**: *On the gravitational field of a mass point according to Einstein's theory*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189-196 (1916).

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Schwarzschild spacetime

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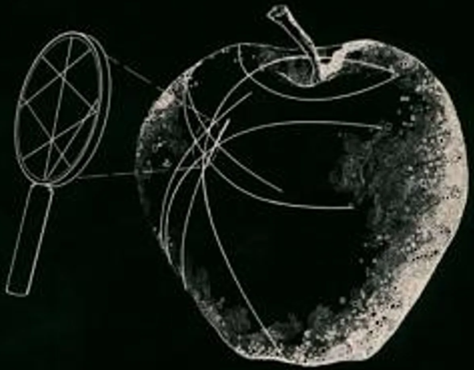
K. Schwarzschild: *On the gravitational field of a mass point according to Einstein's theory*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189-196 (1916).

GRAVITATION

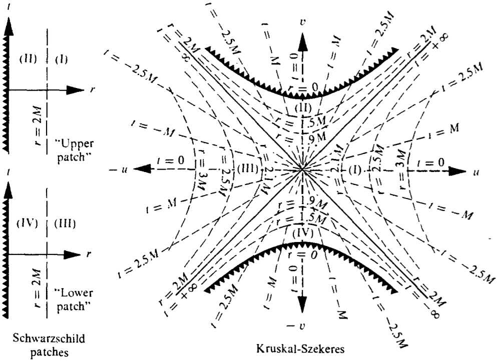
Charles W. MISNER Kip S. THORNE John Archibald WHEELER



Schwarzschild solution

D. Finkelstein: *Past-Future Asymmetry of the Gravitational Field of a Point Particle*, Phys. Rev. **110**, 965-967 (1958).

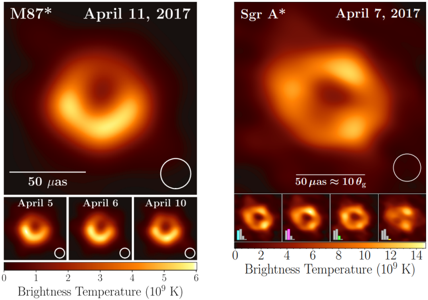
M. D. Kruskal: *Maximal extension of Schwarzschild metric*, Phys. Rev. **119**, 1743-1745 (1960).



Black holes

Event Horizon Telescope Collaboration: *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. Lett. **875**, 17 (2019).

Event Horizon Telescope Collaboration: *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, Astrophys. J. Lett. **930**, 21 (2022).



Extra dimensions

Kaluza–Klein theory

Th. Kaluza: *On the Unification Problem in Physics*, Int. J. Mod. Phys. D **27**, No. 14 (2018) 1870001 (translation); Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966-972 (original).

L. Randall, R. Sundrum: *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83**, 3370-3373 (1999).

string theory

E. Witten: *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. B **471**, 135-158 (1996).

Nuclear Physics B360 (1991) 197–209
North-Holland

BLACK STRINGS AND p-BRANES

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Received 4 March 1991

It is shown that low-energy string theory admits a variety of solutions with the structure of an extended object surrounded by an event horizon. In particular there is a family of black string solutions, labelled by the mass and axion charge per unit length, corresponding to a string in ten dimensions surrounded by an event horizon. The extremal member of this family is the known supersymmetric singular solution corresponding to a macroscopic fundamental string. A similar family of solutions is found describing a fivebrane surrounded by an event horizon, whose extremal member is a previously discovered non-singular supersymmetric fivebrane. Additional charged, extended black hole solutions are presented for each of the antisymmetric tensors that arise in heterotic and type II string theories.

In this section we find extrema of the action

$$S=\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left[R+4(\nabla\phi)^2\right]-\frac{2e^{2\alpha\phi}}{(D-2)!}F^2\right], \tag{1}$$

where F is a $(D-2)$-form satisfying $dF=0$. We will assume $D\geqslant 4$. For certain values of α and D this is part of the low-energy action from string theory. The

Finally, using eqs. (2), (3) and (11), one obtains black $(10-D)$-brane solutions of (1)

$$\begin{aligned}
&F=Q\epsilon_{D-2},\\
&ds^2=-\left[1-(r_+/r)^{D-3}\right]\left[1-(r_-/r)^{D-3}\right]^{\gamma_x-1}dt^2\\
&\qquad+\left[1-(r_+/r)^{D-3}\right]^{-1}\left[1-(r_-/r)^{D-3}\right]^{\gamma_r}dr^2\\
&\qquad+r^2\left[1-(r_-/r)^{D-3}\right]^{\gamma_r+1}d\Omega_{D-2}^2+\left[1-(r_-/r)^{D-3}\right]^{\gamma_x}dx^i\,dx_i,\\
&e^{-2\phi}=\left[1-(r_-/r)^{D-3}\right]^{\gamma_\phi},
\end{aligned} \tag{15}$$

re the exponents are given by

Vacuum solution

ansatz with n extra dimensions $\zeta^A = \zeta^1, ..., \zeta^n$:

$$ds^2 = -f(r)^\alpha dt^2 + f(r)^\beta dr^2 + r^2 d\Omega^2_{(2)} + f(r)^\gamma \delta_{AB} \underbrace{d\zeta^A d\zeta^B}_{\text{extra}}$$

Ricci tensor $R_{\mu\nu}$:

$$R_{00} = \alpha f^{\alpha-\beta} F_1 \quad R_{rr} = F_2 \quad R_{\vartheta\vartheta} = F_3$$
$$R_{\varphi\varphi} = F_3 \sin^2\vartheta \quad R_{AB} = -\gamma f^{\gamma-\beta} F_1 \delta_{AB}$$

$$F_1 = \frac{1}{r}\frac{f'}{f} + \frac{1}{4}(\alpha - \beta + n\gamma - 2)\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\frac{f''}{f}$$

$$F_2 = \beta\frac{1}{r}\frac{f'}{f} + \frac{1}{4}\left[\alpha(-\alpha+\beta+2) + n\gamma(\beta-\gamma+2)\right]\left(\frac{f'}{f}\right)^2 - \frac{1}{2}(\alpha + n\gamma)\frac{f''}{f}$$

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}(\alpha-\beta+n\gamma)\, r\frac{f'}{f}\right]$$

Vacuum solution

ansatz with n extra dimensions $\zeta^A = \zeta^1, ..., \zeta^n$:

$$ds^2 = -f(r)^\alpha dt^2 + f(r)^\beta dr^2 + r^2 d\Omega_{(2)}^2 + f(r)^\gamma \delta_{AB} \underbrace{d\zeta^A d\zeta^B}_{\text{extra}}$$

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$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}(\alpha - \beta + n\gamma)\, r\frac{f'}{f}\right]$$

Vacuum solution

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}\left(\alpha - \beta + n\gamma\right) r \frac{f'}{f}\right] = 0$$

$$\Downarrow$$

$$\boxed{f = \left(1 + ar^q\right)^{-1/\beta}} \qquad q = \frac{2\beta}{\alpha - \beta + n\gamma}$$

exception: $\alpha - \beta + n\gamma = 0 \Rightarrow f = 1 \Rightarrow$ Minkowski spacetime

the rest is then

$$F_1 = C_1 \Phi \quad F_2 = \left(C_2 + C_3 a r^q\right)\Phi \quad \text{where:}$$

$$\Phi = \frac{1}{\left(\alpha - \beta + n\gamma\right)^2} \frac{ar^{q-2}}{\left(1 + ar^q\right)^2}$$

$$C_1 = -\alpha - \beta - n\gamma$$

$$C_2 = 2\beta^2 + \beta\left(\alpha + n\gamma\right) - \left(\alpha + n\gamma\right)^2$$

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Vacuum solution(s)

algebraic equations

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\end{aligned}
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can be solved by $\alpha = -\beta - n\gamma \Rightarrow C_1 = 0, C_2 = 0 \Rightarrow$

$$C_3 = -n\gamma\left[2\beta + \left(n+1\right)\gamma\right]$$

possible solutions are:

• $\gamma = 0 \rightarrow \alpha = -\beta$:
$f^\alpha = 1 + \frac{a}{r} \quad f^\beta = \left(1 + \frac{a}{r}\right)^{-1} \quad f^\gamma = 1$
trivial extension of the Schwarzschild spacetime

• $2\beta + \left(n+1\right)\gamma = 0 \rightarrow \alpha/\beta = (n-1)/(n+1), \gamma/\beta = -2/(n+1)$:
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nontrivial extension

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nontrivial extension

Vacuum solutions

trivial extension

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ds^2 = & - \left(1+\frac{a}{r}\right) dt^2 + \left(1+\frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \\
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nontrivial extension

$$
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existence of horizon for $a < 0$ at $r = |a|$!!!

Vacuum solutions

trivial extension

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nontrivial extension

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existence of horizon for $a < 0$ at $r = |a|$!!!

Size of extra dimensions

trivial extension

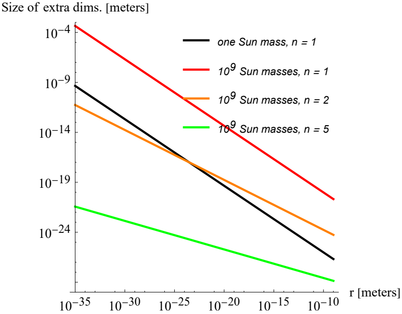
$$g_{AB} = \delta_{AB}$$

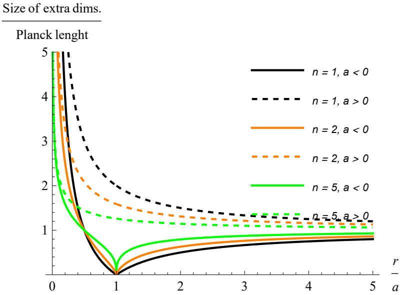
nontrivial extension

$$\boxed{g_{AB} = \left(1 + \frac{a}{r}\right)^{\frac{2}{n+1}} \delta_{AB}}$$

Size of extra dimensions

$$\left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}$$





Newtonian limit

$$ds^2 \approx -(1+2\phi)dt^2 + (1-2\psi)\delta_{ij}dx^i dx^j$$

$$g_{00} \approx -(1+2\phi) \quad \boxed{\phi = -\frac{\kappa M}{r}}$$

trivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right) \Rightarrow \phi = \frac{1}{2}\frac{a}{r} \Rightarrow \boxed{M = -\frac{a}{2\kappa}}$$

nontrivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} \Rightarrow \phi = -\frac{n-1}{n+1}\frac{a}{2}\frac{1}{r} \Rightarrow \boxed{M = \frac{n-1}{n+1}\frac{a}{2\kappa}}$$

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nontrivial extension

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$$g_{00} = -\left(1+\frac{a}{r}\right) \Rightarrow \phi = \frac{1}{2}\frac{a}{r} \Rightarrow \boxed{M = -\frac{a}{2\kappa}}$$

nontrivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} \Rightarrow \phi = -\frac{n-1}{n+1}\frac{a}{2}\frac{1}{r} \Rightarrow \boxed{M = \frac{n-1}{n+1}\frac{a}{2\kappa}}$$

Horizon

Schwarzschild spacetime

$$ds^2 = -\left(1 + \frac{a}{r}\right) dt^2 + \frac{dr^2}{1 + \dfrac{a}{r}} + r^2 d\Omega^2_{(2)}$$

$$a = -2\kappa M$$

has only the central singularity

Kretschmann scalar proves it

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12a^2}{r^6}$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\mu\rho,\nu\sigma} - g_{\nu\sigma,\mu\rho}\right) + g_{\alpha\beta}\left(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\beta}_{\nu\sigma}\right)$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right)$$

Horizon

Schwarzschild spacetime

$$ds^2 = -\left(1 + \frac{a}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{a}{r}} + r^2 d\Omega^2_{(2)}$$

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Kretschmann scalar

trivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \delta_{AB}d\zeta^A d\zeta^B$$

$$K = \frac{12a^2}{r^6} \quad \text{the same as original Schwarzschild}$$

nontrivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}}dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}\delta_{AB}d\zeta^A d\zeta^B$$

$$K = \frac{12a^2}{r^6}\left[1 - \frac{1}{3}\frac{n(n-1)}{(n+1)^3}\left(1+\frac{r}{a}\right)^{-2}\left(\frac{3}{4}n + 1 + (n+1)\frac{r}{a}\right)\right]$$

Kretschmann scalar

trivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \delta_{AB}d\zeta^A d\zeta^B$$

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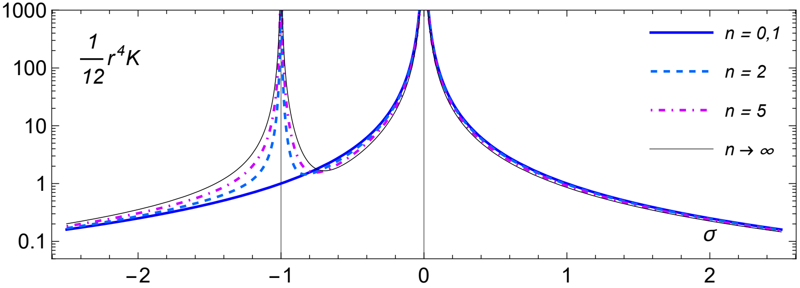
$$ds^2 = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}}dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}\delta_{AB}d\zeta^A d\zeta^B$$

$$K = \frac{12a^2}{r^6}\left[1 - \frac{1}{3}\frac{n(n-1)}{(n+1)^3}\left(1+\frac{r}{a}\right)^{-2}\left(\frac{3}{4}n + 1 + (n+1)\frac{r}{a}\right)\right]$$

Kretschmann scalar

dimensionless radial coordinate $\sigma = \dfrac{r}{a}$ for both $a > 0$ and $a < 0$

$$\frac{1}{12} r^4 K = \frac{1}{\sigma^2} \left[1 - \frac{1}{3} \frac{n(n-1)}{(n+1)^3} \frac{1}{(1+\sigma)^2} \left(\frac{3}{4} n + 1 + (n+1)\sigma \right) \right]$$

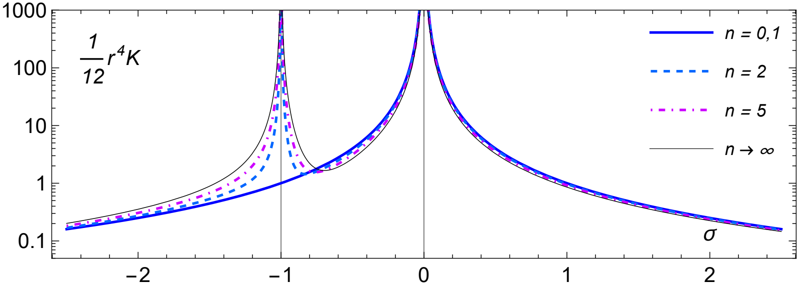


for $a < 0$ there is **horizon singularity** at $r = |a|$!!!

Kretschmann scalar

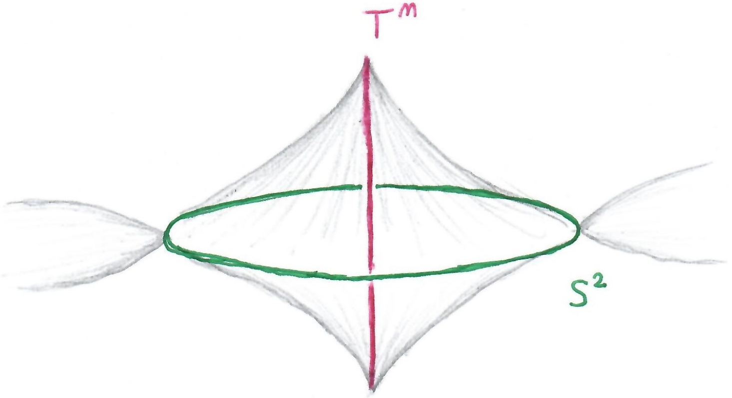
dimensionless radial coordinate $\sigma = \dfrac{r}{a}$ for both $a > 0$ and $a < 0$

$$\frac{1}{12} r^4 K = \frac{1}{\sigma^2} \left[1 - \frac{1}{3} \frac{n(n-1)}{(n+1)^3} \frac{1}{(1+\sigma)^2} \left(\frac{3}{4} n + 1 + (n+1)\sigma \right) \right]$$



for $a < 0$ there is **horizon singularity** at $r = |a|$!!!

Kaluza–Klein bubble - "bubble of nothing"



Kaluza–Klein bubble - "bubble of nothing"

Weyl formalism:

H. Weyl: *"Zur gravitationstheorie"*, Ann. Phys. **54**, 117 (1917), [*The theory of gravitation*, DOI: 10.1007/s10714-011-1310-7].

R. Emparan, H. S. Reall: *Generalized Weyl Solutions*, Phys. Rev. D **65**, 084025 (2002), [arXiv:hep-th/0110258].

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Weyl solutions

Weyl metrics

1 language

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In general relativity, the **Weyl metrics** (named after the German-American mathematician Hermann Weyl)[1] are a class of *static* and *axisymmetric* solutions to Einstein's field equation. Three members in the renowned Kerr–Newman family solutions, namely the Schwarzschild, nonextremal Reissner–Nordström and extremal Reissner–Nordström metrics, can be identified as Weyl-type metrics.

Standard Weyl metrics [edit]

The Weyl class of solutions has the generic form[2][3]

$$ds^2 = -e^{2\psi(\rho,z)}\,dt^2 + e^{2\gamma(\rho,z)-2\psi(\rho,z)}(d\rho^2 + dz^2) + e^{-2\psi(\rho,z)}\rho^2\,d\phi^2\,, \qquad (1)$$

where $\psi(\rho,z)$ and $\gamma(\rho,z)$ are two metric potentials dependent on *Weyl's canonical coordinates* $\{\rho\,,z\}$. The coordinate system $\{t,\rho,z,\phi\}$ serves best for symmetries of Weyl's spacetime (with two Killing vector fields being $\xi^t = \partial_t$ and $\xi^\phi = \partial_\phi$) and often acts like cylindrical coordinates,[2] but is *incomplete* when describing a black hole as $\{\rho\,,z\}$ only cover the horizon and its exteriors.

Schwarzschild solution [edit]

The Weyl potentials generating Schwarzschild's metric as solutions to the vacuum equations Eq(**8**) are given by[2][3][4]

$$\psi_{SS} = \frac{1}{2}\ln\frac{L-M}{L+M}\,,\quad \gamma_{SS} = \frac{1}{2}\ln\frac{L^2-M^2}{l_+ l_-}\,, \qquad (12)$$

where

$$L = \frac{1}{2}(l_+ + l_-)\,,\quad l_+ = \sqrt{\rho^2+(z+M)^2}\,,\quad l_- = \sqrt{\rho^2+(z-M)^2}\,. \qquad (13)$$

From the perspective of Newtonian analogue, ψ_{SS} equals the gravitational potential produced by a rod of mass M and length $2M$ placed symmetrically on the z-axis; that is, by a line mass of uniform density $\sigma = 1/2$ embedded the interval $z \in [-M, M]$. (Note: Based on this analogue, important extensions of the Schwarzschild metric have been developed, as discussed in ref.[2])

Given ψ_{SS} and γ_{SS}, Weyl's metric Eq(**1**) becomes

$$ds^2 = -\frac{L-M}{L+M}dt^2 + \frac{(L+M)^2}{l_+ l_-}(d\rho^2+dz^2) + \frac{L+M}{L-M}\rho^2 d\phi^2\,, \qquad (14)$$

and after substituting the following mutually consistent relations

$$L+M = r\,,\quad l_+ - l_- = 2M\cos\theta\,,\quad z = (r-M)\cos\theta\,,$$
$$\rho = \sqrt{r^2-2Mr}\,\sin\theta\,,\quad l_+ l_- = (r-M)^2 - M^2\cos^2\theta\,, \qquad (15)$$

one can obtain the common form of Schwarzschild metric in the usual $\{t,r,\theta,\phi\}$ coordinates,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\,. \qquad (16)$$

The metric Eq(**14**) cannot be directly transformed into Eq(**16**) by performing the standard cylindrical-spherical transformation $(t,\rho,z,\phi) = (t, r\sin\theta, r\cos\theta, \phi)$, because $\{t,r,\theta,\phi\}$ is complete while (t,ρ,z,ϕ) is incomplete. This is why we call $\{t,\rho,z,\phi\}$ in Eq(**1**) as Weyl's canonical coordinates rather than cylindrical coordinates, although they have a lot in common; for example, the Laplacian $\nabla^2 := \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}$ in Eq(**7**) is exactly the two-dimensional geometric Laplacian in cylindrical coordinates.

Generalized Weyl solutions

ansatz

$$ds^2 = -e^{2U_1(r,z)}dt^2 + \sum_{a=2}^{D-2} e^{2U_a(r,z)}d\phi_a^2 + e^{2\nu(r,z)}(dr^2 + dz^2),$$

vacuum solutions satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) U_a = 0, \qquad \sum_{a=1}^{D-2} U_a = \ln r,$$

$\Rightarrow$ **Laplace's equation with rod source $\longrightarrow$ "rod diagrams"**

$$\frac{\partial \nu}{\partial r} = -\frac{1}{2r} + \frac{r}{2}\sum_{a=1}^{D-2}\left[\left(\frac{\partial U_a}{\partial r}\right)^2 - \left(\frac{\partial U_a}{\partial z}\right)^2\right], \quad \frac{\partial \nu}{\partial z} = r\sum_{a=1}^{D-2}\frac{\partial U_a}{\partial r}\frac{\partial U_a}{\partial z}.$$

Generalized Weyl solutions

ansatz

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Generalized Weyl solutions

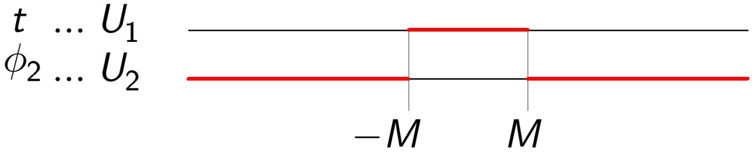
$D = 4$ Minkowski spacetime

t ... U_1

ϕ_2 ... U_2

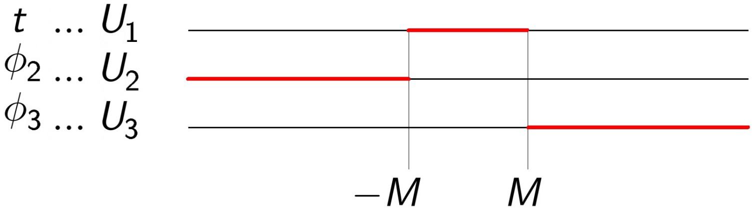
Generalized Weyl solutions

$D = 4$ Schwarzschild spacetime



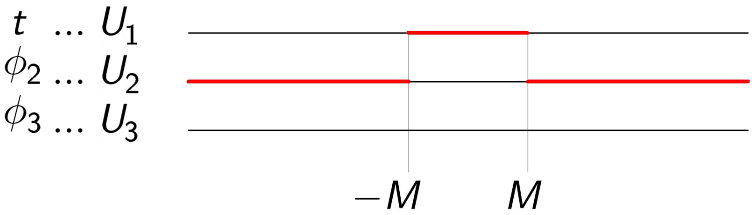
Generalized Weyl solutions

$D = 5$ Schwarzschild–Tangherlini spacetime



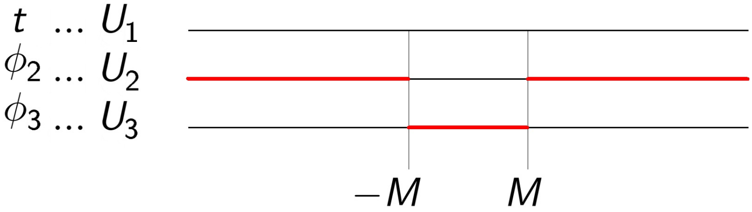
Generalized Weyl solutions

$D = 5$ black string



Generalized Weyl solutions

$D = 5$ Kaluza–Klein bubble



Weyl form of the nontrivial extension

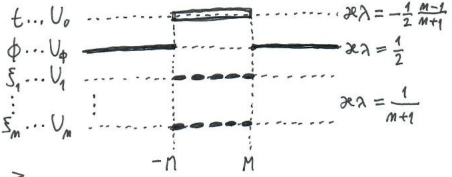
$$ds^2 = -e^{2U_0}dt^2 + e^{2U_\phi}d\phi^2 + \sum_{A=1}^{m} e^{2U_A}d\xi_A^2 + e^{2\nu}(d\rho^2 + dz^2)$$

$$U_0 = \frac{1}{2}\,\frac{1-m}{1+m}\ln\frac{L-M}{L+M} \qquad L = \frac{1}{2}(R_+ + R_-) \qquad R_\pm = \sqrt{(z\pm M)^2 + \rho^2}$$

$$U_\phi = \ln\rho - \frac{1}{2}\ln\frac{L-M}{L+M}$$

$$U_A = \frac{1}{m+1}\ln\frac{L-M}{L+M}$$

$$\nu = \frac{1}{2}\ln\frac{(L+M)^2}{R_+R_-}$$



$$z = r\left(1-\frac{M}{r}\right)\cos\vartheta$$

$$\rho = r\sqrt{1-\frac{2M}{r}}\,\sin\vartheta$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)^{-\frac{m-1}{m+1}}dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2d\vartheta^2 + r^2\sin^2\vartheta\, d\phi^2 + \left(1-\frac{2M}{r}\right)^{\frac{2}{m+1}}\sum_{A=1}^{m} d\xi_A^2$$

Conserved energy

Landau–Lifshitz stress-energy pseudotensor

conserved D-momentum in $D-1$ dimensional space region Ω

$$P^{\mu} = \oint_{\partial\Omega} h^{\mu 0 \nu} d\Sigma_{\nu}$$

where $\quad h^{\mu\nu\lambda} = \frac{1}{16\pi\kappa}\left[(-g)\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}\right)\right]_{,\sigma}$

choice of $\partial\Omega$ such that $d\Sigma_i = dS_i$ and $d\Sigma_A = 0$

only $P^0 = \mathcal{E}$ is nonzero

$$\mathcal{E} = \oint_{\partial\Omega} h^{00i} dS_i$$

Conserved energy

Landau–Lifshitz stress-energy pseudotensor

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Conserved energy

def.: $\mathcal{E} = \oint_{\partial\Omega} h^{00i} dS_i \quad h^{00i} = \frac{1}{16\pi\kappa}\left[(-g)\left(g^{00}g^{i\sigma} - g^{0i}g^{0\sigma}\right)\right]_{,\sigma}$

trivial extension

$$\mathcal{E} = -\frac{a}{2\kappa}$$

nontrivial extension

$$\mathcal{E} = -\frac{a}{2(n+1)\kappa}$$

Conserved energy

def.: $\mathcal{E} = \oint_{\partial\Omega} h^{00i} dS_i \quad h^{00i} = \frac{1}{16\pi\kappa}\left[(-g)\left(g^{00}g^{i\sigma} - g^{0i}g^{0\sigma}\right)\right]_{,\sigma}$

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trivial extension

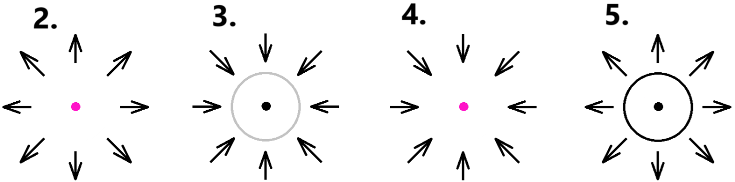
$$\mathcal{E} = -\frac{a}{2\kappa}$$

nontrivial extension

$$\mathcal{E} = -\frac{a}{2(n+1)\kappa}$$

Physical properties

	horizons	singularities	Newt. lim. mass	cons. energy
1. Mink. ($a=0$)	none	none	$M=0$	$\mathcal{E}=0$
2. trivial ($a>0$)	none	$r=0$	$M=\frac{-a}{2\kappa}<0$	$\mathcal{E}=M<0$
3. trivial ($a<0$)	$r=-a$	$r=0$	$M=\frac{-a}{2\kappa}>0$	$\mathcal{E}=M>0$
4. nontriv. ($a>0$)	none	$r=0$	$M=\frac{n-1}{n+1}\frac{a}{2\kappa}>0$	$\mathcal{E}=\frac{-a}{2(n+1)\kappa}<0$
5. nontriv. ($a<0$)	$r=-a$	$r=\{0,-a\}$	$M=\frac{n-1}{n+1}\frac{a}{2\kappa}<0$	$\mathcal{E}=\frac{-a}{2(n+1)\kappa}>0$



Instability

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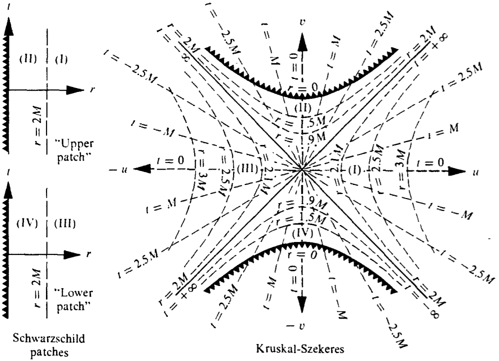


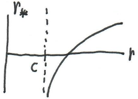
Maximal extension

$$ds^2 \supset -\left(1-\frac{c}{r}\right)dt^2+\left(1-\frac{c}{r}\right)^{-1}dr^2 = -\left(1-\frac{c}{r}\right)\left(dt^2-dr_*^2\right)$$

$$r_* = \int_0^r \left(1-\frac{c}{r'}\right)^{-1} dr'$$

$r \in (c, \infty)$

$r_* \in (-\infty, \infty)$



EDDINGTON–FINKELSTEIN

$u = t - r_*(r)$

$v = t + r_*(r)$

KRUSKAL–SZEKERES

$U = -e^{-\frac{u}{2c}}$

$V = e^{+\frac{v}{2c}}$

$\tilde{t} = \frac{V+U}{2}$

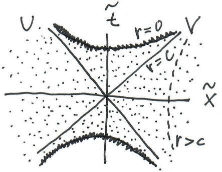
$\tilde{x} = \frac{V-U}{2}$

$U = \tilde{t} - \tilde{x}$

$V = \tilde{t} + \tilde{x}$

$$\tilde{t}^2 - \tilde{x}^2 = UV = -e^{-\frac{u-v}{2c}} = -e^{+\frac{r_*(r)}{c}}$$

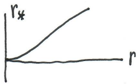
$$\tilde{t}^2 + e^{+\frac{r_*(r)}{c}} = \tilde{x}^2$$



Maximal extension

$$ds^2 \supset -\left(1+\frac{c}{r}\right) dt^2 + \left(1+\frac{c}{r}\right)^{-1} dr^2 = -\left(1+\frac{c}{r}\right)\left(dt^2 - dr_*^2\right)$$

$$r_* = \int_0^r \left(1+\frac{c}{r'}\right)^{-1} dr' \qquad r \in (0,\infty) \qquad r_* \in (0,\infty)$$



EDDINGTON – FINKELSTEIN

$$u = t - r_*(r)$$

$$v = t + r_*(r)$$

KRUSKAL – SZEKERES

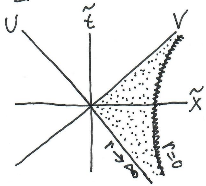
$$U = -e^{+\frac{u}{2c}} \qquad V = e^{-\frac{v}{2c}}$$

$$\tilde{t} = \frac{V+U}{2} \qquad \tilde{x} = \frac{V-U}{2}$$

$$U = \tilde{t} - \tilde{x} \qquad V = \tilde{t} + \tilde{x}$$

$$\tilde{t}^2 - \tilde{x}^2 = UV = -e^{+\frac{u-v}{2c}} = -e^{-\frac{r_*(r)}{c}}$$

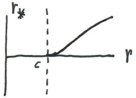
$$\tilde{t}^2 + e^{-\frac{r_*(r)}{c}} = \tilde{x}^2$$



Maximal extension

$$ds^2 \supset -\left(1-\frac{c}{r}\right)^{-\frac{m-1}{m+1}} dt^2 + \left(1-\frac{c}{r}\right)^{-1} dr^2 = -\left(1-\frac{c}{r}\right)^{-\frac{m-1}{m+1}} \left(dt^2 - dr_*^2\right)$$

$$r_* = \int_c^r \left(1-\frac{c}{r'}\right)^{-\frac{1}{m+1}} dr' \qquad r \in (c, \infty) \qquad r_* \in (0, \infty)$$



EDDINGTON – FINKELSTEIN

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$$v = t + r_*(r)$$

KRUSKAL–SZEKERES

$$U = -e^{-\frac{u}{2c}}$$

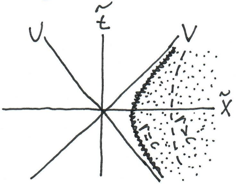
$$V = e^{+\frac{v}{2c}}$$

$$\tilde{t} = \frac{V+U}{2} \qquad U = \tilde{t} - \tilde{x}$$

$$\tilde{x} = \frac{V-U}{2} \qquad V = \tilde{t} + \tilde{x}$$

$$\tilde{t}^2 - \tilde{x}^2 = UV = -e^{-\frac{u-v}{2c}} = -e^{+\frac{r_*(r)}{c}}$$

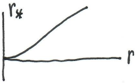
$$\tilde{t}^2 + e^{+\frac{r_*(r)}{c}} = \tilde{x}^2$$



Maximal extension

$$ds^2 \supset -\left(1+\frac{c}{r}\right)^{-\frac{m-1}{m+1}} dt^2 + \left(1+\frac{c}{r}\right)^{-1} dr^2 = -\left(1+\frac{c}{r}\right)^{-\frac{m-1}{m+1}} \left(dt^2 - dr_*^2\right)$$

$$r_* = \int_0^r \left(1+\frac{c}{r'}\right)^{-\frac{1}{m+1}} dr' \qquad r \in (0,\infty) \qquad r_* \in (0,\infty)$$



EDDINGTON-FINKELSTEIN

$$u = t - r_*(r)$$

$$v = t + r_*(r)$$

KRUSKAL-SZEKERES

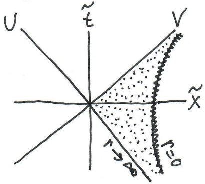
$$U = -e^{+\frac{u}{2c}} \qquad V = e^{-\frac{v}{2c}}$$

$$\tilde{t} = \frac{V+U}{2} \qquad \tilde{x} = \frac{V-U}{2}$$

$$U = \tilde{t} - \tilde{x} \qquad V = \tilde{t} + \tilde{x}$$

$$\tilde{t}^2 - \tilde{x}^2 = UV = -e^{+\frac{u-v}{2c}} = -e^{-\frac{r_*(r)}{c}}$$

$$\tilde{t}^2 + e^{-\frac{r_*(r)}{c}} = \tilde{x}^2$$



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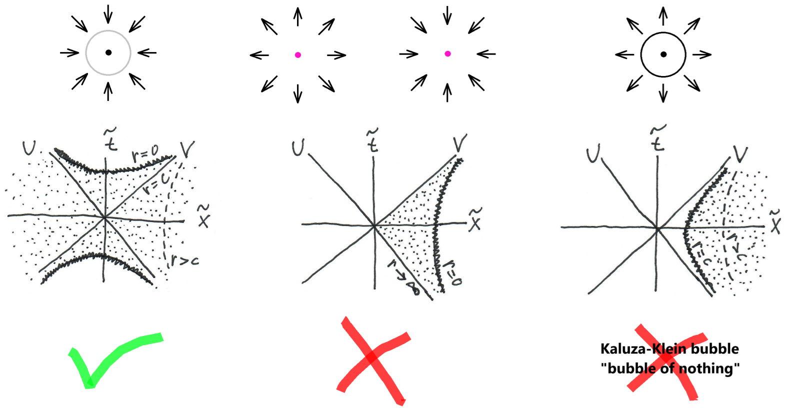

Work in progress

- more than $1+3$ standard dimensions (straightforward)

$$ds^2 = -\left(1+\frac{a}{r^{q-1}}\right)^{-\frac{n-1}{n+1}} dt^2 + \left(1+\frac{a}{r^{q-1}}\right)^{-1} dr^2 +$$
$$+ r^2 {d\Omega_{(q)}}^2 + \left(1+\frac{a}{r^{q-1}}\right)^{\frac{2}{n+1}} \delta_{AB} d\zeta^A d\zeta^B$$

- nonzero cosmological constant (complicated)

$$R_{\mu\nu} = \frac{2\Lambda}{n+q} g_{\mu\nu}$$

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- Schwarzschild: $ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega^2_{(2)}$
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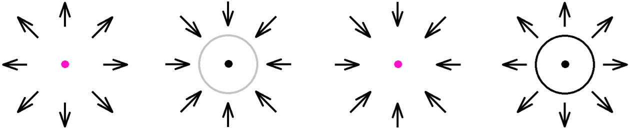

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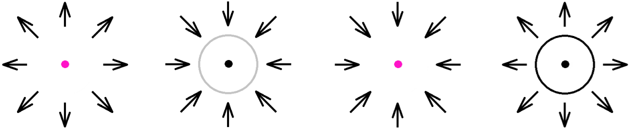

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Thank You for listening!