Wetterich's equation for regularised QFTs and general probability measures

[Ziebell, 2023]

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Outline

Path Integrals

Scalar Quantum Field Theories as Measures (Glimm & Jaffe)

- μ a Borel probability measure on a distribution space, typically $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{D}'(\mathbb{R}^d)$.
- + a couple of other axioms
- Free fields modelled as Gaußian measures

Path Integrals

Free Fields are boring!

- (Wick-rotated) Feynman path integrals suggests that interactions should be modelled as densities, i.e.

$$\nu = (-\exp \circ J) \cdot \mu \tag{1}$$

  where J would typically be a point-local polynomial in the field, e.g. $J(\phi) = g \int_{\mathbb{R}^d} \phi(x)^4 \mathrm{d}^d x$.
- However, μ is not supported on a function space $\implies$ J is defined on a zero-measure subset
- How to extend J, resp. can it be extended?

Path Integrals

Regularisation & Renormalisation

- Regularise in some way (e.g. UV/IR cutoffs) $\implies$ μ_{reg} lives on a function space. Then

$$\nu = (-\exp \circ J) \cdot \mu_{\text{reg}} \tag{2}$$

is fine.
- Take limit regularisation $\to 0$.

Advertisement: Wilsonian RG

'Family (ν_η) of measures satisfying certain compatibility conditions always has a UV limit' – [arXiv:2502.16319: Laszlo, Tarcsay, Ziebell]

What is a Density anyway?

The Onsager-Machlup Function

'Let μ be a measure on a metric space X and let $K(x, \epsilon)$ be the closed ball of radius ϵ centred at x. In diverse applications one has to study the existence of the following limit:

$$\lim_{\epsilon \to 0} \frac{\mu\left(K\left(a, \epsilon\right)\right)}{\mu\left(K\left(b, \epsilon\right)\right)} = I\left(a, b\right) \tag{3}$$

' – [Bogachev: Gaussian Measures]

What was the Density?

Example

- $\mu = (\exp \circ -f) \cdot \lambda$ on $\mathbb{R}$ with f continuous, λ the Lebesgue measure.
- $\implies I(a, b) = \exp[f(b) - f(a)]$
- $\implies J(a) = f(0) - \ln I(a, 0)$.

Density $\sim$ OM Function

- Beware: Even equivalent metrics may produce different OM functions! [Ayanbayev, Klebanov, Lie, Sullivan: 2021]

Wetterich RG I

The (regularised) Setup

- Partition Function:

$$Z_k(\phi) = \frac{\int_X \exp\left[\phi(x) - J(x) - \frac{1}{2}B_k(x,x)\right] \mathrm{d}\mu(x)}{\int_X \exp\left[-J(x) - \frac{1}{2}B_k(x,x)\right] \mathrm{d}\mu(x)} =: \frac{\dots}{N_k} \tag{4}$$

- $X = \mathcal{S}(\mathbb{R}^d)$, μ a Gaußian measure, $\phi \in X^*$, J 'sufficiently nice'
- B_k measurable bilinear functionals, with $B_k = 0$, $\lim_{k\to\infty} B_k(x,x) = \infty$ 'sufficiently fast'

Wetterich RG II

The Effective Average Action

- $W_k = \ln \circ Z_k$
- For every $h \in H(\mu)$ (Cameron-Martin space of μ),

$$\Gamma_k(h) = \sup_{\phi \in X^*} \left[\phi(h) - W_k(\phi)\right] - \frac{1}{2} B_k(h,h) \tag{5}$$

$$\implies \lim_{k \to \infty} \Gamma_k(h) = \frac{1}{2} \langle h, h \rangle_{H(\mu)} + J(x) - J(0) \; . \tag{6}$$

Wetterich RG II

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$$\Gamma_k(h) = \sup_{\phi \in X^*} \left[\phi(h) - W_k(\phi)\right] - \frac{1}{2} B_k(h,h) \tag{7}$$

$$\Longrightarrow \lim_{k \to \infty} \Gamma_k(h) = \boxed{\tfrac{1}{2}\langle h, h\rangle_{H(\mu)} + J(x)} - J(0) \ . \tag{8}$$

- a.k.a. the (regularised) classical action
- a.k.a the Onsager-Machlup function $-\ln I(h,0)$ of $(\exp \circ - J)\mu$

Wetterich RG III

Some Abbreviations

- $(X^k, \mathcal{A}^k, m_k)$ a σ-finite measure space
- $U^k : X^k \to (H(\mu)^*)_{\mathbb{C}}, x \mapsto U^k_x$

$$\forall h \in H(\mu) : \qquad \frac{\mathrm{d}}{\mathrm{d}k} B_k(h, h) = \int_{X^k} \left| U^k_x(h) \right|^2 \mathrm{d}m_k(x) \tag{9}$$

The RG Equation

$$\frac{\mathrm{d}}{\mathrm{d}k} \Gamma_k(h) = \frac{1}{2} \int_{X^k} \overline{U^k_x} \left[\left(D^2_h \Gamma_k + B_k \right)^{-1} \left(U^k_x \right) \right] \mathrm{d}m_k(x) + \frac{\mathrm{d}}{\mathrm{d}k} \ln N_k \tag{10}$$

Features

Really Cool Things

- An infinite-dimensional PDE
- Except for N_k-part, the form of the equation depends only on B_k
- N_k-part is independent of h
- Well-defined boundary condition

Features

The Reverse Programme

- An infinite-dimensional PDE with boundary condition
- Which boundary conditions admit solutions down to $k = 0$?
- Uniqueness of solutions?

ϕ^4 theory

The Massive Free Scalar Field

Let μ be defined by

$$\int_{\mathcal{S}(\mathbb{R}^d)'} \exp\left[iT(\phi)\right] \mathrm{d}\mu\left(T\right) = \exp\left[-\frac{1}{2}\int_{\mathbb{R}^d} \frac{\tilde{\phi}\left(-p\right)\tilde{\phi}\left(p\right)}{p^2+m^2} \mathrm{d}^d p\right] \quad (11)$$

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ (Bochner-Minlos).

The Interaction

Define $J(\phi) = \lambda \int_{\mathbb{R}^d} \phi^4$.

How to Regularise?

Regularisation

- Define regulator as continuous linear map $\mathcal{R} : \mathcal{S}'_\beta(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$
- Consider the pushforward measure $\nu = \mathcal{R}_* \mu$ on $\mathcal{S}(\mathbb{R}^d)$

An Example

$$\mathcal{R}T = \chi \cdot (\xi * T) \tag{12}$$

with

$$\begin{aligned} \chi(x) &= \exp\left[-\frac{\Lambda_{\mathrm{IR}}^2}{2} x^2\right] \\ \xi(x) &= \left(\frac{\Lambda_{\mathrm{UV}}^2}{2\pi}\right)^{d/2} \exp\left[-\frac{\Lambda_{\mathrm{UV}}^2}{2} x^2\right] \end{aligned} \tag{13}$$

Note, $w - \lim_{\Lambda_{\mathrm{UV}} \to \infty, \Lambda_{\mathrm{IR}} \to 0} \mathcal{R}T = T$ for all $T \in \mathcal{S}'(\mathbb{R}^d)$.

Convex Conjugation

The Method of Proof

- Let $W_k(T) = \ln Z_k(T)$

  $\Rightarrow$ W_k is convex ('trivial' from measure theory)

  $\Rightarrow$ $\mathrm{D}W_k$ is injective (almost 'trivial' - Gaußian measures)

  $\Rightarrow$ W_k extends to closure of $\mathcal{S}'_\beta$ in $L^2(\nu)$ ('trivial' - Gaußian measures)

  $\Rightarrow$ range of $\mathrm{D}W_k$ is all of $H(\nu)$ - the 'Cameron-Martin space' of ν (tricky)

The Effective Average Action

Straightforward Formula

$$\Gamma_k(\phi) = (DW_k)^{-1}(\phi)(\phi) - W_k\left((DW_k)^{-1}(\phi)\right) - \frac{1}{2}B_k(\phi)(\phi) \quad (14)$$

FRGE

The Flow Equation

Usually, $B_k \to R_k$ with R_k function in Fourier space. Then,

$$\partial_k \Gamma_k(\phi) = \frac{1}{2} \mathrm{Tr}_{L^2(\mathbb{R}^d)} \left\{ (\partial_k R_k) \left[\mathcal{F} \circ \mathrm{D}^2_\phi \Gamma_k \circ \mathcal{F}^{-1} + R_k \right]^{-1} \right\} + \partial_k \ln N_k \tag{15}$$

and

$$\lim_{k \to \infty} \Gamma_k(\phi) = \frac{1}{2} \langle \phi, \phi \rangle_{H(\nu)} + J(\phi) \ . \tag{16}$$

Wait, Where are the Counterterms?

In four Dimensions

- Counterterm $-\Delta m^2 \int_{\mathbb{R}^4} \phi^2$ (among others) should be added to J
- $-J$ blows up since $\lambda \int_{\mathbb{R}^4} \phi^4$ cannot control $\Delta m^2 \int_{\mathbb{R}^4} \phi^2$
- Still fine, since

$$\int_{\mathcal{S}} \exp\left[-\lambda \int_{\mathbb{R}^4} \phi^4 + \Delta m^2 \int_{\mathbb{R}^4} \phi^2\right] \mathrm{d}\nu(\phi) < \infty$$

for all $\Delta m^2 > 0$. [Janssen, Hinrichs, Ziebell: 2023].

Do We Really Need to Regularise?

Taking the Limit regularisation $\to 0$

- Corresponding measures converge if corresponding family Γ_0^{reg} converges in 'complicated sense from paper'.
- Does the equation itself also converge?
- What about the boundary condition? (Renormalisation suggests infinities)

Do We Really Need to Regularise? II

Can we Derive the Equation for Other Measures μ? (WIP)

- Problem of domain of Γ_k
  - Candidates: kernel of μ? Lusin affine kernel of μ, etc.
- Differentiability properties of Γ_k
  - Correspondence to minimisation properties of W_k
  - W_k on a larger domain than X^* (A closure in some topology?)
- Compatibility with repsect to k-differentiation
  - Sufficiently uniform differentiability of Γ_k

Do We Really Need to Regularise? III

Can we Derive the Equation for Other Measures μ? (WIP)

- What happens to the boundary condition?
  - OM-functions do not generally exist
  - Even when they do, their domains are not necessarily vector spaces
- The 'non-regularised classical action' should not exist either!
  - Are there divergences e.g. like

$$\lim_{k\to\infty} \frac{D^2\Gamma_k(h,h)}{k^2} \in \mathbb{R}\setminus\{0\}\,? \tag{17}$$

- Connection to behaviour of boundary condition in limit regularisation $\to 0$.

Outlook

Further Directions

- Application to nonrelativistic problems (in progress)
    - UV limit well understood $\Rightarrow$ study convergence of RG equation and boundary conditions
- Fermions? (Non-commutative Probability?)
- Coupled systems?