NNLOCAL: completely local subtractions for color-singlet production in hadron collisions

Gábor Somogyi
HUN-REN Wigner Research Centre for Physics

work in collaboration with V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, F. Tramontano and S. Van Thurenhout, arXiv:2412.21028 [hep-ph]

HUN-REN Wigner RCP theory seminar, February 7, 2025

Outline

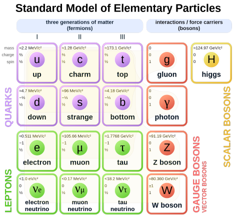
1. Introduction
2. Higher-order corrections
3. The CoLoRFulNNLO scheme
4. The NNLOCAL code
5. Conclusions and outlook

Introduction

What is the question if the answer is particle physics?

The question: what is the world made of and what holds it together, i.e., what are the laws of Nature governing the smallest length scales?

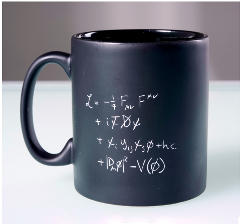
- What are the basic building blocks of matter?
- What is the nature of interactions between them?



[Source: Wikipedia]

Our best answer so far: the **Standard Model** of elementary particle physics

- 12 elementary fermions (6 leptons + 6 quarks)
- 3 basic gauge interacions (electromagnetic, weak, strong) + Higgs mechanism



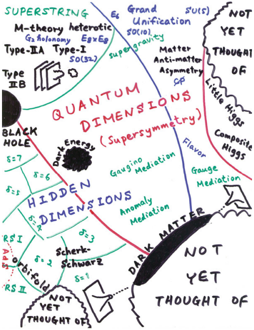
The challenge

The **Standard Model** is very successful at describing a wide array of particle physics phenomena.

Yet, it cannot be the final description of Nature, since it leaves unanswered some **great questions**:

- How does gravity work at microscopic scales?
- What is dark matter and dark energy?
- Why are we made of matter and not antimatter (baryogenesis)?
- Why are neutrinos so light?
- What is the origin of flavor and CP violation?
- Is our vacuum stable on cosmological time scales?
- ...

What lies beyond the Standard Model?



Where do we stand?

Large Hadron Collider (LHC): spectacular confirmation of the SM!

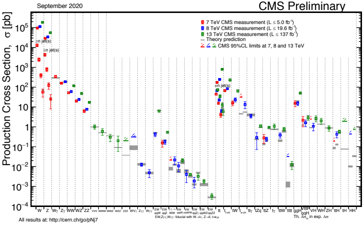
- Basic processes (W, Z production) measured to sub-percent accuracy

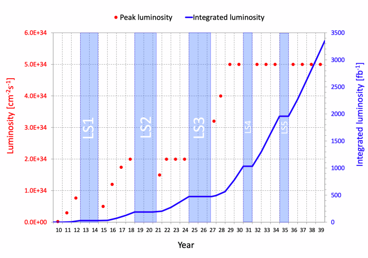
So far no **direct** signs of beyond Standard Model (BSM) physics

- Typically huge backgrounds

However, we have only seen a small amount of the total expected data

- **HL-LHC**: increase integrated luminosity by a factor of 10 beyond LHC design value
- We have gathered only $\sim 10\%$ of all data foreseen for the complete LHC program

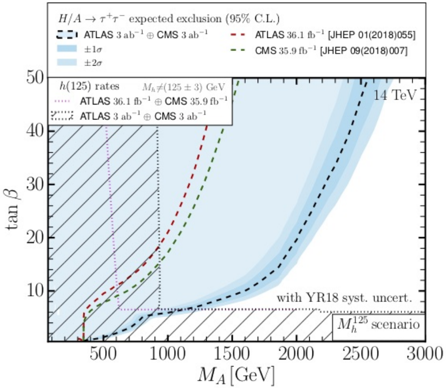




Direct detection vs. indirect signals

How do we best exploit the physics potential of LHC?

- Two Higgs Doublet Models (2HDM) are a well-motivated class of BSM models
- They generally predict new elementary scalar particles $\Rightarrow$ direct searches
- Higgs couplings can deviate from their SM values $\Rightarrow$ precision measurements
- At the LHC the **two search strategies** are complementary: they are **sensitive in different regions** of parameter space.



[ATLAS coll., ATL-PHYS-PUB-2022-018]

New Physics can show up directly or in precision measurements: precision is key!

Early measurements of the $pp \to WW$ cross section at LHC

- ATLAS @ 8 TeV [ATLAS-CONF-2014-033]

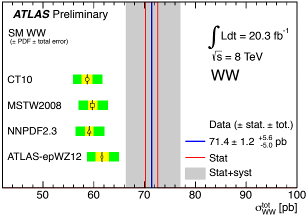
$$\sigma(pp \to WW) = 71.4^{+1.2}_{-1.2}(\text{stat})^{+5.0}_{-4.4}(\text{syst})^{+2.2}_{-2.1}(\text{lumi})\ \text{pb}$$

- Standard Model predictions at NLO

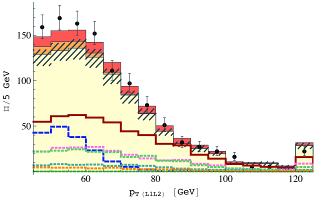
$$\sigma_{\text{NLO}}(pp \to WW) = 58.7^{+3.0}_{-2.7}\ \text{pb}$$

- Similar **discrepancy** at CMS and at 7 TeV

$$\boxed{\Delta(\text{data}/\text{NLO, ATLAS+CMS}) \sim 3\sigma}$$



Great! Signal of a **new particle** (supersymmetric chargino) being produced.



[Phys. Rev. D 87 (2013) 3, 031701]

- contribution from chargino pair production
- $m(\chi^{\pm}) = 110$ GeV

BUT! Is the theoretical modelling precise enough?

A recent lesson

Theoretical developments

- Including some new contributions previously neglected in the computation (NNLO QCD corrections) increases the prediction by $\sim +10\%$.

[Phys. Rev. Lett. 113 (2014) 21, 212001]

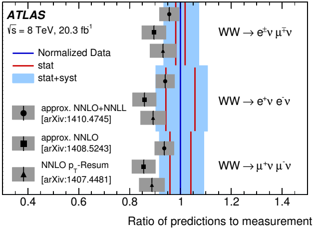
	$\sigma_{\text{inclusive}}$ [fb]		$\sigma/\sigma_{\text{NLO}} - 1$	
$\sqrt{s}$	8 TeV	13 TeV	8 TeV	13 TeV
LO	$425.41(4)\,^{+2.8\%}_{-3.6\%}$	$778.99\ (8)\,^{+5.7\%}_{-6.7\%}$	-31.8%	-35.4%
NLO	$623.47(6)\,^{+3.6\%}_{-2.9\%}$	$1205.11(12)\,^{+3.9\%}_{-3.1\%}$	0	0
NLO′	$635.95(6)\,^{+3.6\%}_{-2.8\%}$	$1235.82(13)\,^{+3.9\%}_{-3.1\%}$	$+\ 2.0\%$	$+\ 2.5\%$
NLO′+gg	$655.83(8)\,^{+4.3\%}_{-3.3\%}$	$1286.81(13)\,^{+4.8\%}_{-3.7\%}$	$+\ 5.2\%$	$+\ 6.8\%$
NNLO	$690.4(5)\,^{+2.2\%}_{-1.9\%}$	$1370.9(11)\,^{+2.6\%}_{-2.3\%}$	$+10.7\%$	$+13.8\%$

[JHEP 08 (2016) 140]

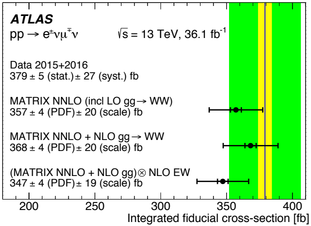
- These previously neglected terms also play an important role in the proper modelling of other aspects of the analysis (e.g., the jet veto used to suppress backgrounds).

A recent lesson

The final comparison at 8 TeV and 13 TeV



[JHEP 09 (2016) 029]



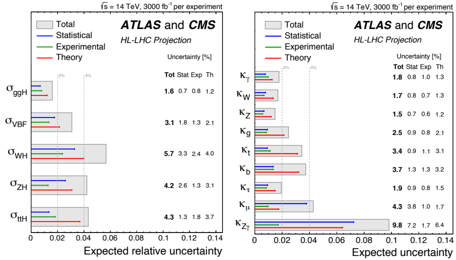
[Eur. Phys. J. C 79 (2019) 10]

We **did not find a new particle**, but we did learn important **lessons**:

- **Precise** and **properly controlled** theoretical predictions are of essential importance.
- Terms that are **neglected** at the usual theoretical accuracy of NLO (i.e., higher-order perturbative corrections) can play a pivotal role.

The High Luminosity LHC program (3 ab^{-1} integrated luminosity per experiment) is expected to measure Higgs boson production cross sections and couplings to an accuracy of $\sim$ 2–4%.

- This is precise enough to constrain the parameter spaces of many BSM models.
- The forecasts in the presented figures **assume a substantial decrease of current theoretical uncertainties**.
- The role of theoretical uncertainties is especially noteworthy for the couplings to t- and b-quarks and gluons.

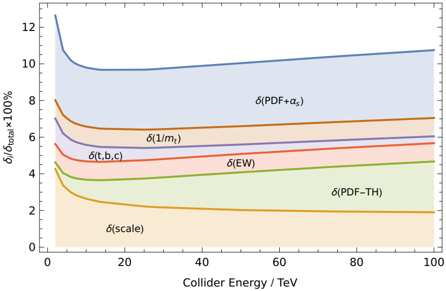


[ATLAS coll., ATL-PHYS-PUB-2022-018]

The theory challenge

The uncertainty budget for Higgs production in gluon fusion as a function of energy

- $\delta(\mathrm{PDF} + \alpha_S)$ and $\delta(\mathrm{PDF} - \mathrm{TH})$: uncertainties related to our knowledge of proton structure
- $\delta(\mathrm{scale})$ and $\delta(\mathrm{EW})$: uncertainties coming from uncalculated higher-order corrections in theory predictions
- $\delta(1/m_t)$ and $\delta(t, b, c)$: uncertainties associated with neglecting quark masses in the theory predictions

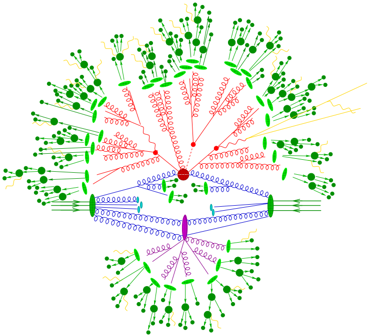


[CERN Yellow Rep. Monogr. 7 (2019) 221]

How to decrease the theoretical uncertainty?

In order to fully exploit the physics potential of LHC, QCD (and EW) must be understood and modelled as best as possible: many aspects of precision

- Incoming protons are beams of partons: parton distribution functions (PDFs)
- Primary hard scattering: **fixed-order perturbation theory**, resummation
- Partonic evolution: parton showers
- Hadronization and heavy hadron decay
- Multiple parton interactions and underlying event



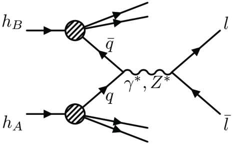
[A Sherpa artist]

Diverse market of uncertainties, but higher orders are very important!

Higher-order corrections

QCD cross sections at the LHC

Hadronic cross sections can be computed as convolutions of parton distribution functions (PDFs) with hard scattering partonic cross sections: **collinear factorization** theorem



$$d\sigma_{AB} = \sum_{a,b} \int dx_a \int dx_b \underbrace{f_{a/A}(x_a, \mu_F) f_{b/B}(x_b, \mu_F)}_{\text{PDFs}} \times \underbrace{d\sigma_{ab}(x_a, x_b, \mu_F)}_{\text{partonic x-sect}} \left[1 + \mathcal{O}\left(\frac{\Lambda^n_{\text{QCD}}}{Q^n} \right) \right]$$

- $f_{a/A}(x_a, \mu_F)$, $f_{b/B}(x_b, \mu_F)$: non-perturbative PDFs, to be measured in experiment or computed on the lattice
- $d\sigma_{ab}(x_a, x_b, \mu_F)$: hard partonic cross section, can be computed in perturbative QCD
- Collinear factorization is valid at high Q (recall $\Lambda_{\text{QCD}} \sim 300$ MeV), value of n is not predicted (believed to be $n = 2$ for many cases)
- Factorization scale dependence (μ_F): separation of long and short distance physics

The partonic cross section in perturbation theory

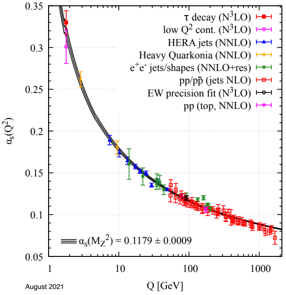
High-energy parton scattering can be described in QCD via **perturbation theory**.

- The hard partonic cross section computed as a power series of the strong coupling α_S

$$\mathrm{d}\sigma_{ab}(\mu_F) = \alpha_S^p(\mu_R)\Big(\mathrm{d}\sigma_{ab}^{LO} + \underbrace{\alpha_S(\mu_R)\mathrm{d}\sigma_{ab}^{NLO}(\mu_R,\mu_F)}_{\mathcal{O}(10\%)} + \underbrace{\alpha_S^2(\mu_R)\mathrm{d}\sigma_{ab}^{NNLO}(\mu_R,\mu_F)}_{\mathcal{O}(1\%)} + \dots\Big)$$

The **strong coupling**

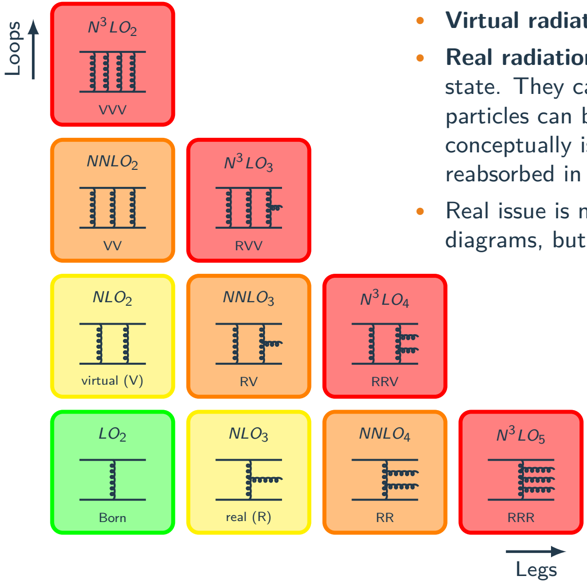
- The basis for the applicability of perturbation theory in QCD is **asymptotic freedom**: the strength of the interaction decreases with increasing energy.
- $\alpha_S(M_Z) \approx 0.118 \Rightarrow$ roughly NLO $\sim \mathcal{O}(10\%)$, NNLO $\sim \mathcal{O}(1\%)$, etc., but naive counting can be **off by an order of magnitude**.
- Scale dependence (μ_R): due to truncation of the perturbative series $\Rightarrow$ indicates size of missing higher orders.
- Note: $\alpha/\alpha_S \approx 0.1$ but beware $\ln\left(\frac{M_Z^2}{Q^2}\right)$!



[PDG Review of Particle Physics (2022)]

Ingredients of a fixed-order calculation

Beyond leading order, we must account for **extra radiation**, both virtual as well as real.



- **Virtual radiation**: loops
- **Real radiation**: extra particles in the final state. They can appear because the extra particles can be **unresolved**, which conceptually is the same as being reabsorbed in the process, like in loops.
- Real issue is not the growing number of diagrams, but the **presence of singularities**.

Unresolved partons

We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n+p$-parton event, then the $n+p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Unresolved partons

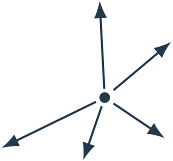
We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n + p$-parton event, then the $n + p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Examples:

- 5 resolved partons, 0 unresolved partons
- all momenta "well-separated" and "hard"



Unresolved partons

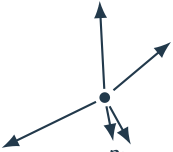
We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n+p$-parton event, then the $n+p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Examples:

- 4 resolved partons, 1 unresolved parton
- one pair of momenta is collinear, $p_i || p_r$



Unresolved partons

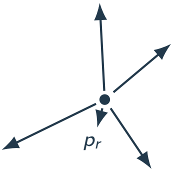
We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n+p$-parton event, then the $n+p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Examples:

- 4 resolved partons, 1 unresolved parton
- one momentum is soft, $p_r \to 0$



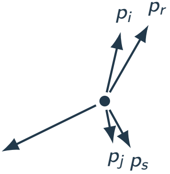
We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n+p$-parton event, then the $n+p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Examples:

- 3 resolved partons, 2 unresolved partons
- two pairs of momenta are collinear, $p_i||p_r$ and $p_j||p_s$



Unresolved partons

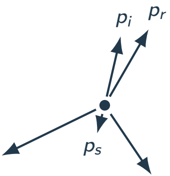
We call a parton **unresolved** if its energy is much smaller than the typical energies of the rest of the partons (**soft** limit) or if its momentum is nearly collinear with the momentum of another parton (**collinear** limit).

Figuratively, if p partons are unresolved in an $n+p$-parton event, then the $n+p$-parton momentum configuration is indistinguishable from the momentum configuration of an n-parton event. In this case we speak of a p-fold unresolved configuration.

At LO, all partons are resolved (by definition), at NLO at most one parton can become unresolved, at NNLO single and double unresolved configurations can occur.

Examples:

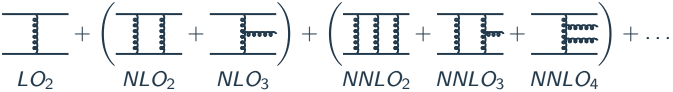
- 3 resolved partons, 2 unresolved partons
- one pair of momenta is collinear, a third momentum is soft $p_i || p_r$ and $p_s \to 0$



Perturbation theory in practice

The **application of perturbation theory** in particle collisions raises **two basic issues**.

1. We must evaluate the mathematical expressions corresponding to the diagrams describing the process ⇒ **multi-loop integrals**



2. The various contributions must be summed. Although the sum of contributions is finite at each order, the separate contributions are naively infinite due to the presence of IR singularities ⇒ **IR pole treatment**

$$LO_2 = \text{finite}$$

$$NLO_2 + NLO_3 = \text{finite} \quad \textbf{BUT} \quad NLO_{2,3} = \infty$$

$$NNLO_2 + NNLO_3 + NNLO_4 = \text{finite} \quad \textbf{BUT} \quad NNLO_{2,3,4} = \infty$$

Note: both issues understood at NLO for general processes ⇒ automation of solutions lead to the NLO revolution of the 2010s.

Where do the infinities come from?

Recall that in order to obtain physical quantities, i.e., cross sections, we must perform some momentum integrals:

$$\sigma = \int |\mathcal{M}|^2 d\phi$$

- over loop momenta to obtain $|\mathcal{M}|^2$ (loop integrals)
- over real momenta to obtain σ (phase space integrals)

It turns out that integrations over momenta of **unobserved** particles, either in loops or in real radiation lead to **divergent integrals**!

Two distinct sources:

1. Divergences "at infinity", i.e., $p \to \infty$: **UV divergences**, removed by suitable redefinitions of the parameters of the theory (renormalization); only in loops
2. Divergences "at zero", i.e., $p \to 0$ (zero energy) or $p_1 || p_2$ (zero angle): **IR divergences**, cancel in properly defined physical quantities; in real radiation and loops

Technically, UV divergences are (much) easier to handle: renormalization can be performed once and for all at a given perturbative order. On the other hand, IR divergences manifest in each new calculation even after UV renormalization has been implemented.

Sources of IR singularities

The NNLO correction to a generic m-jet observable is the sum of three terms

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} + \int_{m+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+1} + \int_{m} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2}\right) J_m$$

Double real (RR)

- Tree level squared MEs with $m+2$-parton kinematics
- MEs diverge as one or two partons become unresolved
- **phase space integral divergent** (up to $\mathrm{O}(\varepsilon^{-4})$ poles from PS integration in dim. reg.)
- no loops, so no explicit ε poles in dim. reg.

Real-virtual (RV)

- One-loop squared MEs with $m+1$-parton kinematics
- MEs diverge as one parton becomes unresolved
- **phase space integral divergent** (up to $\mathrm{O}(\varepsilon^{-2})$ poles from PS integration in dim. reg.)
- one loop, **explicit** ε **poles** up to $\mathrm{O}(\varepsilon^{-2})$ from loop integration in dim. reg.

Double virtual (VV)

- Two-loop squared MEs with m-parton kinematics
- jet function screens divergences in MEs as partons become unresolved
- phase space integral is finite
- two loops, **explicit** ε **poles** up to $\mathrm{O}(\varepsilon^{-4})$ from loop integration in dim. reg.

Perturbation theory in practice – the problem

Clearly, the infinities that appear must be treated in a **consistent way** before any explicit calculation can be performed.

How to perform computations in practice beyond NLO?

The basic idea of the subtraction method

One general approach to dealing with the infrared divergences that appear in higher-order perturbative computations is the so-called **subtraction method**.

Idea

Using appropriately chosen **subtraction terms**, reshuffle divergences between each contribution of the full higher-order correction in such a way that each contribution is finite after the reshuffling!

The subtraction method in practice - a caricature

Suppose we want to evaluate (at $\varepsilon \to 0$)

$$\sigma = \int_0^1 \sigma_3^{NLO}(x,\varepsilon)dx + \sigma_2^{NLO}(\varepsilon) \qquad \text{where} \qquad \begin{aligned} \sigma_3^{NLO}(x,\varepsilon) &= x^{-1-\varepsilon} R(x)\,, \\ \sigma_2^{NLO}(\varepsilon) &= \frac{V_{-1}}{\varepsilon} + V_0 + V_1\,\varepsilon + \ldots\,, \end{aligned}$$

with $R(0) = R_0 < \infty$, and $V_{-1} = R_0$ (such that σ is finite).

- Define the **counterterm**

$$\sigma_3^{NLO,A}(x,\varepsilon) \equiv x^{-1-\varepsilon} R_0$$

- Use it to **reshuffle singularities** between the contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\sigma_3^{NLO}(x,\varepsilon) - \sigma_3^{NLO,A}(x,\varepsilon)\right] dx \bigg|_{\varepsilon=0} + \left[\sigma_2^{NLO}(\varepsilon) + \int_0^1 \sigma_3^{NLO,A}(x,\varepsilon)dx\right]_{\varepsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0}{x^{1+\varepsilon}}\right] dx \bigg|_{\varepsilon=0} + \left[\frac{V_{-1}}{\varepsilon} + V_0 + V_1\,\varepsilon + \ldots - \frac{R_0}{\varepsilon}\right]_{\varepsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0}{x} dx + V_0\,, \qquad\qquad \text{recall } V_{-1} = R_0! \end{aligned}$$

The two terms on the last line are both **finite**!

The CoLoRFulNNLO scheme

CoLoRFulNNLO: a local, analytic subtraction scheme for NNLO

- Exact perturbative result without slicing parameters
  (reduced source of numerical uncertainty)
- Point-by-point subtraction in phase space including spin and color correlations
  (no integrals that are finite in d dims., but undefined in 4 dims.)
- Analytic computation of integrated subtraction terms
  (rigorously show cancellation of virtual poles)
- Explicit and general expressions
  (coding, automation)

General subtraction procedure

Consider the N^{k}LO partonic cross section: total of k extra emissions

$$\sigma_{ab}^{\mathrm{N}^k\mathrm{LO}} = \sum_{l=0}^{k} \int_{m+k-l} \left(\mathrm{d}\sigma_{ab}^{\mathrm{R}_{k-l}\mathrm{V}_l} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_l} \right) J_{m+k-l} = \sum_{l=0}^{k} R_{k-l} V_l$$

- $\mathrm{d}\sigma_{ab}^{\mathrm{R}_{k-l}\mathrm{V}_l}$: the l-loop contribution with $(k-l)$ extra real emissions
- $\mathrm{d}\sigma_{ab}^{\mathrm{C}_l}$: the l-loop collinear remnant from PDF renormalization. E.g., ($P^{(0)}$ is the standard space-like splitting function)

$$\mathrm{d}\sigma_{ab}^{\mathrm{C}_1} = \frac{\alpha_{\mathrm{S}}}{2\pi\varepsilon} \left(P_{ac}^{(0)} \otimes \mathrm{d}\sigma_{cb}^{\mathrm{B}} + \mathrm{d}\sigma_{ac}^{\mathrm{B}} \otimes P_{cb}^{(0)} \right)$$

At NNLO, we have

$$\sigma_{ab}^{\mathrm{NNLO}} = \underbrace{\int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}}}_{\equiv R_2 V_0} + \underbrace{\int_{m+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1} \right)}_{\equiv R_1 V_1} + \underbrace{\int_{m} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2} \right)}_{\equiv R_0 V_2}$$

The **CoLoRFul** scheme to regularize IR singularities proceeds in **3 steps**

General subtraction procedure

Step 1: subtract all IR singularities from k-fold real emission

- Construct an approximation to $R_k V_0$ valid when all k real emissions are unresolved. Denote this as $\mathbf{A}_k[R_k V_0]$. Then the difference

  $$(\mathbf{1} - \mathbf{A}_k)[R_k V_0]$$

  is free of singularities as k partons become unresolved.
- However, it is still singular as $(k-1)$ partons become unresolved, hence construct $\mathbf{A}_{k-1}(\mathbf{1} - \mathbf{A}_k)[R_k V_0]$. Now the expression

  $$(\mathbf{1} - \mathbf{A}_{k-1})(\mathbf{1} - \mathbf{A}_k)[R_k V_0]$$

  is free of singularities as k or $(k-1)$ partons become unresolved.
- Iterate the above procedure until all singularities are removed. The expression

  $$(\mathbf{1} - \mathbf{A}_1)(\mathbf{1} - \mathbf{A}_2)(\mathbf{1} - \mathbf{A}_3)\cdots(\mathbf{1} - \mathbf{A}_k)[R_k V_0] = \left[\prod_{i=1}^{k}(\mathbf{1} - \mathbf{A}_i)\right][R_k V_0]$$

  is free of all IR singularities.

Step 2: integrate the subtractions over unresolved emissions and add back

- Specifically, all terms that involve taking a j-fold limit will involve integrations over j unresolved emissions. These terms are precisely

$$(\mathbf{1}-\mathbf{A}_1)(\mathbf{1}-\mathbf{A}_2)(\mathbf{1}-\mathbf{A}_3)\cdots(-\mathbf{A}_j)[R_k V_0] = \left[\prod_{i=1}^{j-1}(\mathbf{1}-\mathbf{A}_i)\right](-\mathbf{A}_j)[R_k V_0]$$

- After integration, they must be grouped with $R_{k-j}V_j$. After this rearrangement we have

$$\sigma_{ab}^{\mathrm{N}^k\mathrm{LO}} = \left[\prod_{i=1}^{k}(\mathbf{1}-\mathbf{A}_i)\right][R_k V_0] + \sum_{j=1}^{k}\left\{R_{k-j}V_j - \int_j \left[\prod_{i=1}^{j-1}(\mathbf{1}-\mathbf{A}_i)\right](-\mathbf{A}_j)[R_k V_0]\right\}$$

- The first term is free of IR singularities, but the j-th term of the sum in the second term still has singularities as up to $j-k$ partons become unresolved.

General subtraction procedure

Step 3: iterate the first two steps until all IR singularities are removed

- The final result of this iteration can be written in a compact form

$$\sigma_{ab}^{\mathrm{N}^k\mathrm{LO}} = \sum_{l=0}^{k} \left[\prod_{i=1}^{k-l} (\mathbf{1} - \mathbf{A}_i) \right] \sum_{j=0}^{l} \Bigg\{ \delta_{lj}\, \mathbf{1} + \sum_{\pi \in \mathcal{P}(l-j)} \prod_{m=1}^{|\pi|} \int_{\pi(m)} \left[\prod_{n_m=1}^{\pi(m)-1} (\mathbf{1} - \mathbf{A}_{n_m}) \right] \mathbf{A}_{\pi(m)} \Bigg\} [R_{k-j} V_j]$$

- Here $\mathcal{P}(l-j)$ denotes the set of *ordered partitions* of the integer $(l-j)$ and $\pi = \{\pi(1), \pi(2), \ldots, \pi(|\pi|)\}$ is a particular element of length $|\pi|$ of this set.

General subtraction procedure

LO ($k = 0$):

$$\sigma_{ab}^{\mathrm{LO}} = R_0 V_0 = \int_m \mathrm{d}\sigma_{ab}^{\mathrm{B}} J_m$$

NLO ($k = 1$):

$$\begin{aligned}\sigma_{ab}^{\mathrm{NLO}} &= (\mathbf{1} - \mathbf{A}_1)[R_1 V_0] + R_0 V_1 + \int_1 \mathbf{A}_1[R_1 V_0] \\ &= \int_{m+1} \left[\mathrm{d}\sigma_{ab}^{\mathrm{R}} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{R},\mathrm{A}_1} J_m\right] + \int_m \left[\mathrm{d}\sigma_{ab}^{\mathrm{V}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1} + \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{R},\mathrm{A}_1}\right] J_m\,,\end{aligned}$$

- On the second line we have introduced the notation

$$\mathbf{A}_1[R_1 V_0] = \mathbf{A}_1\left[\mathrm{d}\sigma_{ab}^{\mathrm{R}} J_{m+1}\right] = \mathrm{d}\sigma_{ab}^{\mathrm{R},\mathrm{A}_1} J_m$$

- Notice that $\mathbf{A}_1$ also acts on the measurement function J.
- Infrared and collinear safety of the observable ensures that $\mathbf{A}_1[J_{m+1}] = J_m$ so the integrated subtraction term can be combined with the virtual contribution.

NNLO ($k=2$):

$$\begin{aligned}\sigma_{ab}^{\mathrm{NNLO}} &= (\mathbf{1}-\mathbf{A}_1)(\mathbf{1}-\mathbf{A}_2)[R_2 V_0] \\ &+ (\mathbf{1}-\mathbf{A}_1)[R_1 V_1] + (\mathbf{1}-\mathbf{A}_1)\int_1 \mathbf{A}_1 [R_2 V_0] \\ &+ R_0 V_2 + \left[\int_2 (\mathbf{1}-\mathbf{A}_1)\mathbf{A}_2 + \int_1 \mathbf{A}_1 \int_1 \mathbf{A}_1\right][R_2 V_0] + \int_1 \mathbf{A}_1 [R_1 V_1]\end{aligned}$$

- Following the notation introduced above

$$\begin{aligned}\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2}\Big[&\mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} J_m\Big] \\ + \int_{m+1}\Bigg\{&\left[\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1} + \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right] J_{m+1} - \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV},\mathrm{A}_1} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1,\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}\right] J_m\Bigg\} \\ + \int_m \Bigg\{&\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2} + \int_2\left[\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}}\right] + \int_1\left[\mathrm{d}\sigma_{ab}^{\mathrm{RV},\mathrm{A}_1} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1,\mathrm{A}_1}\right] + \int_1\left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}\Bigg\} J_m\end{aligned}$$

Role of individual terms

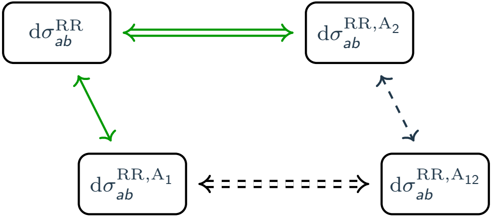
$$\left\{\begin{array}{c} d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \\ d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2} \\ d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} \\ d\sigma_{ab}^{\mathrm{RV},\mathrm{A}_1} \\ d\sigma_{ab}^{\mathrm{C}_1,\mathrm{A}_1} \\ \left(\int_1 d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1} \end{array}\right\} \text{ regularizes } \left\{\begin{array}{c} d\sigma_{ab}^{\mathrm{RR}} \\ d\sigma_{ab}^{\mathrm{RR}} \\ d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2}\,,\, d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \\ d\sigma_{ab}^{\mathrm{RV}} \\ d\sigma_{ab}^{\mathrm{C}_1} \\ \int_1 d\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \end{array}\right\} \text{ as } \left\{\begin{array}{c} 1 \\ 2 \\ 1\,,2 \\ 1 \\ 1 \\ 1 \end{array}\right\} \text{ partons are unresolved}$$

The dual role of A_{12}

To define $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$, we can set

- $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \equiv \mathbf{A}_1 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}}\right]$
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \equiv \mathbf{A}_2 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}\right]$

Whichever option one chooses, the other becomes a **constraint** on the construction

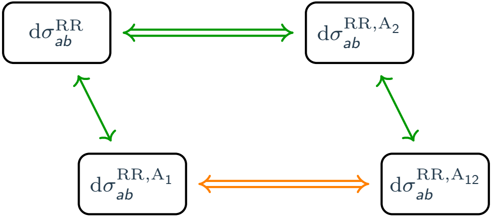


The dual role of A_{12}

To define $\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}}$, we can set

- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} \equiv \mathbf{A}_1\left[\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2}\right]$
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} \equiv \mathbf{A}_2\left[\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right]$

Whichever option one chooses, the other becomes a **constraint** on the construction

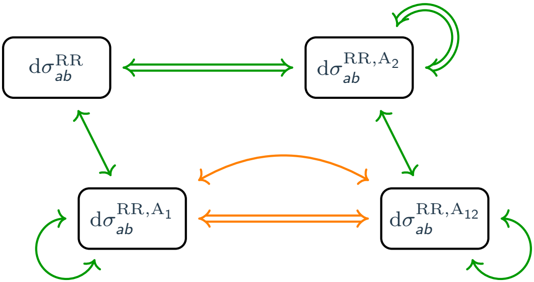


The dual role of A_{12}

To define $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$, we can set

- $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \equiv \mathbf{A}_1 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}}\right]$
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \equiv \mathbf{A}_2 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}\right]$

Whichever option one chooses, the other becomes a **constraint** on the construction



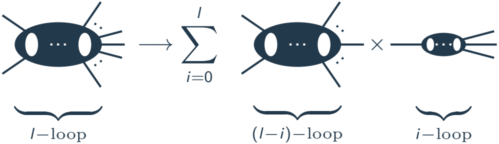
Fine print:

- Internal cancellations within each piece to avoid multiple subtraction (see below).
- Spurious single unresolved singularities in $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$ regularized by $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$.

Constructing the subtraction terms

In the CoLoRFul scheme, subtraction terms are built from **IR factorization formulae**

$$\boldsymbol{U}_j|\mathcal{M}_{ab,m+j}(\{p\}_{m+j})|^2_{l-\text{loop}} = \left(\frac{\alpha_S}{2\pi}\right)^j \sum_{i=0}^{l} \text{Sing}_j^{(i)} \times \underbrace{|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(l-i)-\text{loop}}}_{j \text{ partons removed}}$$



- $\boldsymbol{U}_j$: formal operator that takes some j-fold unresolved limit
- $\text{Sing}_j^{(i)}$: universal (independent of $\mathcal{M}$) IR singular structure for i-loop, j-fold unresolved emission
- $|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(l-i)-\text{loop}}$: $(l-i)$-loop reduced matrix element with j partons removed

Known ingredients

Explicit forms of all $\mathrm{Sing}_j^{(i)}$ relevant at NNLO are known.

- Tree-level three-parton splitting functions and double soft gg and $q\bar{q}$ currents

[Campbell, Glover 1997; Catani, Grazzini 1998; Del Duca, Frizzo, Maltoni 1999; Kosower 2002]

- One-loop two-parton splitting functions and soft gluon current

[Bern, Dixon, Dunbar, Kosower 1994; Bern, Del Duca, Kilgore, Schmidt 1998-9; Kosower, Uwer 1999; Catani, Grazzini 2000; Kosower 2003]

Are we done? Just set $\mathrm{A}_j = \boldsymbol{U}_j$!

Unfortunately NO!

From limits to subtraction terms

Issues

1. Unresolved regions in phase space overlap $\Rightarrow$ care must be taken to **avoid multiple subtraction** in overlapping regions
2. IR limit formulae are only well-defined in the strict limits $\Rightarrow$ **definitions must be carefully extended** over the full phase space away from the limits
3. Subtraction terms must be **integrated** over the momenta of unresolved radiation $\Rightarrow$ see Pooja Mukherjee's talk next week

1. Overlapping singularities can be addressed by the **inclusion-exclusion principle**: subtract each limit once, add back pairwise overlaps, subtract triple overlaps and so on. E.g., at NLO

$$\mathbf{A}_1 = \sum_{ir} \mathbf{C}_{ir} + \sum_{r} \mathbf{S}_r - \sum_{\{ir,r\}} \mathbf{C}_{ir} \cap \mathbf{S}_r$$

2. In order to obtain true subtraction terms, two additional steps are needed

   - **Make** $|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(l-i)-\mathrm{loop}}$ **well-defined** $\Rightarrow$ specify precisely the momenta entering the reduced matrix elements via momentum mappings that implement momentum conservation and mass-shell conditions

$$\{p\}_{m+j} \to \{\tilde{p}\}_m\,, \qquad j = 1, 2$$

   - **Make** $\mathrm{Sing}_j^{(i)}$ **well-defined** $\Rightarrow$ the various quantities entering the singular structures such as momentum fractions, transverse momenta and eikonal factors need to be precisely specified.

The adopted definitions must be such that they respect the structure of cancellations in all overlapping limits, which is a constraint for the entire construction.

From limits to subtraction terms

After these definitions are fixed, the **IR limit formula can be promoted** to a (sum of) **true subtraction term(s)** that is unambiguously defined at any point in phase space,

$$\boldsymbol{U}_j |\mathcal{M}_{ab,m+j}(\{p\}_{m+j})|^2_{l-\text{loop}} \to \sum_{i=0}^{l} \mathcal{U}_j^{(i,l-i)}$$

where

$$\mathcal{U}_j^{(i,l-i)} = \left(\frac{\alpha_S}{2\pi}\right)^j \widetilde{\text{Sing}}_j^{(i)} \times |\mathcal{M}_{\tilde{a}\tilde{b},m}(\{\tilde{p}\}_m)|^2_{(l-i)-\text{loop}}$$

- $\widetilde{\text{Sing}}_j^{(i)}$ represents the expression of the corresponding singular structure incorporating the precise definitions of momentum fractions, eikonal factors and so on.
- The matrix element is evaluated over the set of mapped momenta, $\{\tilde{p}\}_m$.

CoLoRFulNNLO for color-singlet production: A_1

For color-singlet production in hadron-hadron collisions, $h_A + h_B \to X$, we have

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_1^{(0)}$$

where

$$\mathcal{A}_1^{(0)} = \sum_{r\in F}\left[\mathcal{S}_r^{(0,0)} + \sum_{\substack{i\in F\\ i\neq r}}\left(\frac{1}{2}\mathcal{C}_{ir}^{FF(0,0)} - \mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}\right) + \sum_{c\in I}\left(\mathcal{C}_{cr}^{IF(0,0)} - \mathcal{C}_{cr}^{IF}\mathcal{S}_r^{(0,0)}\right)\right]$$

- I and F denote the sets of initial-state and final-state partons
- The various subtraction terms correspond to the limit implied by the notation
- The $(0,0)$ superscript signals that these terms originate from IR limit formulae that involve tree-level singular structures multiplying tree-level reduced matrix elements
- Each term is explicitly defined and can be evaluated in any point of the double real emission phase space

CoLoRFulNNLO for color-singlet production: A_2

For color-singlet production in hadron-hadron collisions, $h_A + h_B \to X$, we have

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_2^{(0)}$$

where

$$\mathcal{A}_2^{(0)} = \frac{1}{2}\sum_{r\in F}\sum_{\substack{s\in F\\ s\neq r}}\left\{\mathcal{S}_{rs}^{(0,0)} + \sum_{c\in I}\left[\mathcal{C}_{crs}^{IFF(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)} + \sum_{\substack{d\in I\\ d\neq c}}\left(\mathcal{C}_{cr,ds}^{IF,IF(0,0)} + \mathcal{C}_{cr,ds}^{IF,IF}\mathcal{S}_{rs}^{(0,0)}\right)\right]\right\}$$

- Again, various counterterms correspond to the limit implied by the notation
- Notice no triple collinear-double collinear overlap: general feature at NNLO
- Notice no soft-collinear terms: these cancel for color-singlet production due to the precise definitions we adopt
- Each term is explicitly defined and can be evaluated in any point of the double real emission phase space

CoLoRFulNNLO for color-singlet production: A_{12}

For color-singlet production, we have $\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_{12}^{(0)}$ where

$$\mathcal{A}_{12}^{(0)} = \sum_{s\in F}\left[\mathcal{A}_2^{(0)}\,\mathcal{S}_s + \sum_{\substack{r\in F\\ r\neq s}}\left(\frac{1}{2}\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF} - \mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s\right) + \sum_{c\in I}\left(\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF} - \mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s\right)\right]$$

with

$$\mathcal{A}_2^{(0)}\,\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left[\mathcal{S}_{rs}^{(0,0)}\,\mathcal{S}_s + \sum_{c\in I}\left(\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{S}_s - \mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)}\,\mathcal{S}_s\right)\right]$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF} = \mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{rs}^{FF} + \sum_{c\in I}\left(\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{C}_{rs}^{FF} - \mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{rs}^{FF}\right)$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s = \sum_{c\in I}\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF} = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{C}_{csr}^{IFF(0,0)}\,\mathcal{C}_{cs}^{IF} + \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}_{cs,dr}^{IF,IF(0,0)}\,\mathcal{C}_{cs}^{IF}\right)$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s + \mathcal{C}_{csr}^{IFF(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s - \mathcal{C}_{csr}^{IFF}\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s\right)$$

Full set of double real subtraction terms

Representative example: $gg \to Hgg$ as double real for $gg \to H$

$$\int_{m+2} \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} J_{m} + \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} J_{m} \right]$$

	Types	Terms	Counter events	Master integrals
$\mathcal{A}_1^{(0)}$	5	13	7	11
$\mathcal{A}_2^{(0)}$	5(11)	9(29)	5	42
$\mathcal{A}_{12}^{(0)}$	13(23)	39(81)	17	104
Total	**23(39)**	**61(123)**	**29**	**157**

- Numbers shown with(without) accounting for cancellation of soft-collinear type terms

Full set of real-virtual subtraction terms

Representative example: $gg \to Hg$ as real-virtual for $gg \to H$

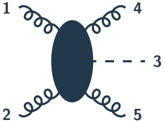
$$\int_{m+1}\left\{\left[\mathrm{d}\sigma_{ab}^{\mathrm{RV}}+\mathrm{d}\sigma_{ab}^{\mathrm{C}_1}+\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right]J_{m+1}-\left[\mathrm{d}\sigma_{ab}^{\mathrm{RV},\mathrm{A}_1}+\mathrm{d}\sigma_{ab}^{\mathrm{C}_1,\mathrm{A}_1}+\left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}\right]J_m\right\}$$

	Types	Terms	Counter events	Master integrals
$\mathcal{A}_1^{(1)}$	2	3	2	24
$\mathcal{A}_1^{\Gamma}$	3	6	2	10
$\mathcal{A}_1^{I}$	5	13	2	65
Total	**10**	**22**	**2**	**99**

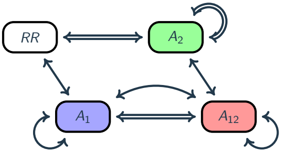
- In the real-virtual contribution only single unresolved real radiation $\Rightarrow$ construction of subtraction terms is only "NLO complexity" (integrals are NNLO complexity though)

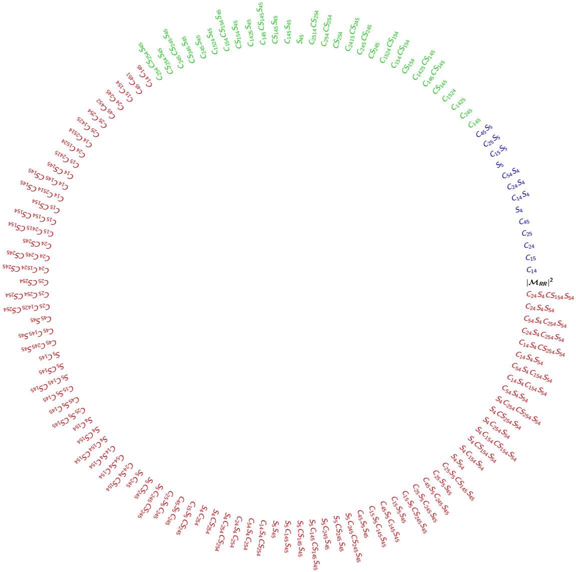
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



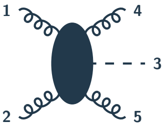
- Interplay of terms



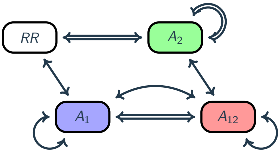


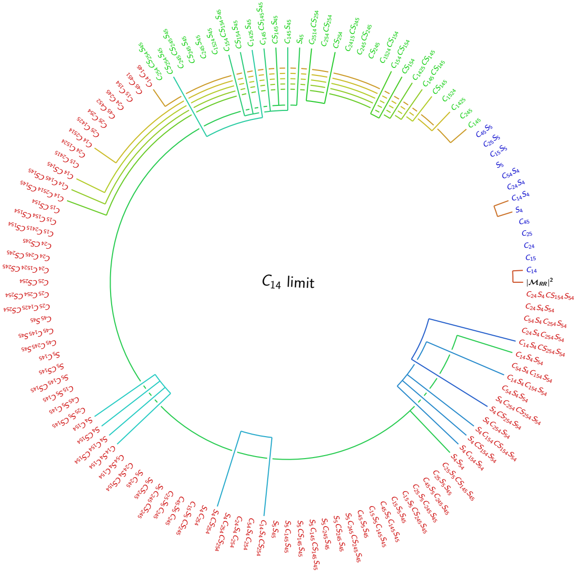
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



- Interplay of terms

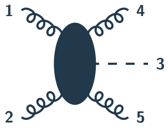




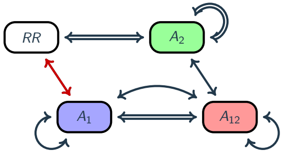
Explicit structure of cancellations in RR

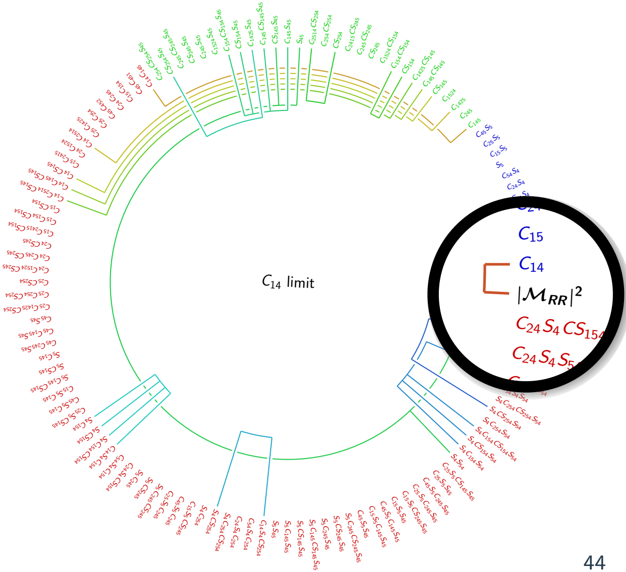
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



- Interplay of terms

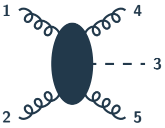




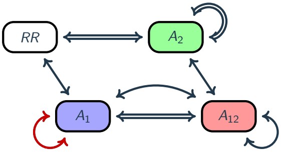
Explicit structure of cancellations in RR

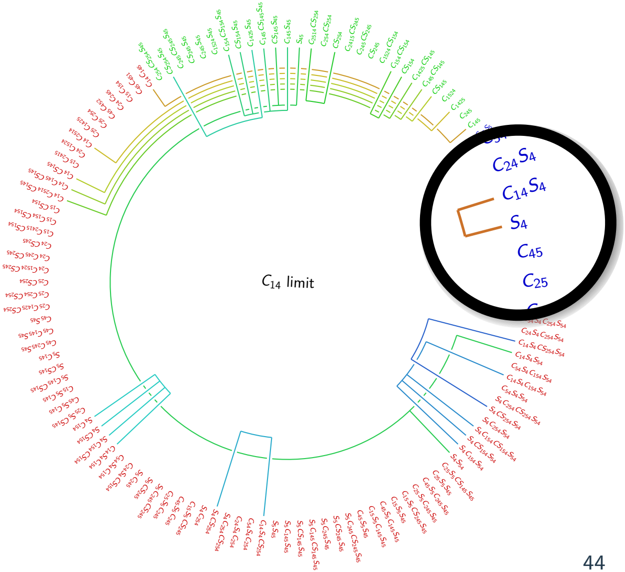
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



- Interplay of terms

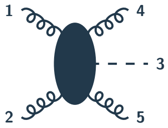




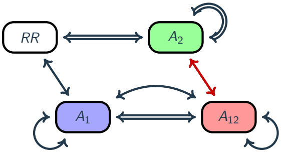
Explicit structure of cancellations in RR

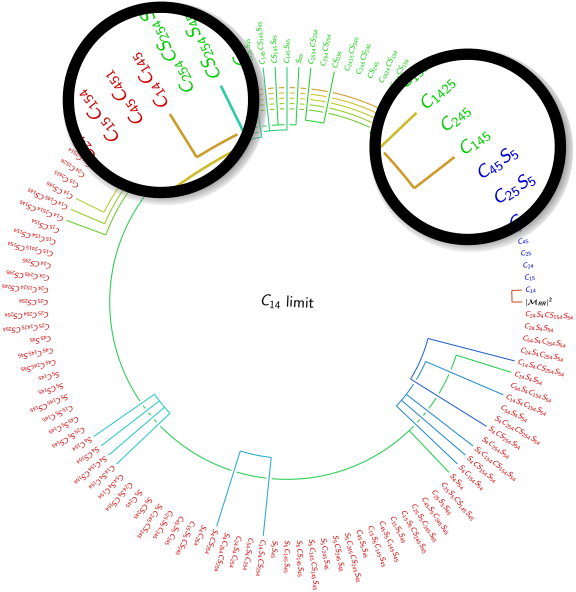
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



- Interplay of terms

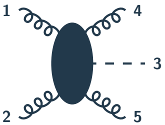




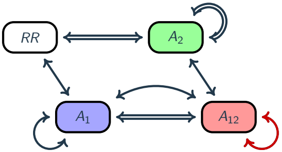
Explicit structure of cancellations in RR

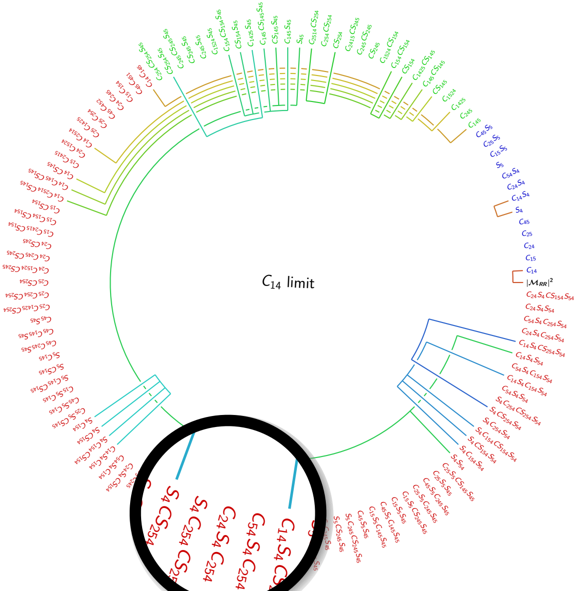
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



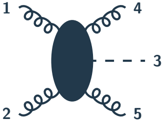
- Interplay of terms



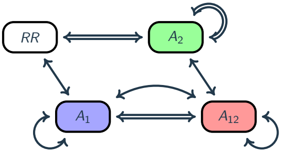


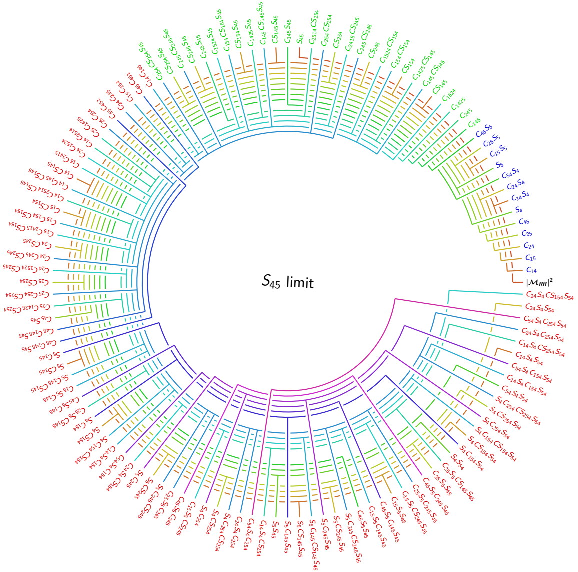
Delicate structure of cancellations between matrix element and subtraction terms

- $gg \to Hgg$



- Interplay of terms

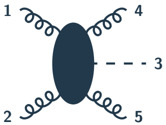




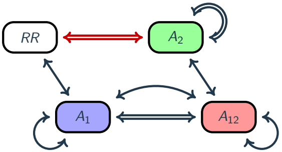
Explicit structure of cancellations in RR

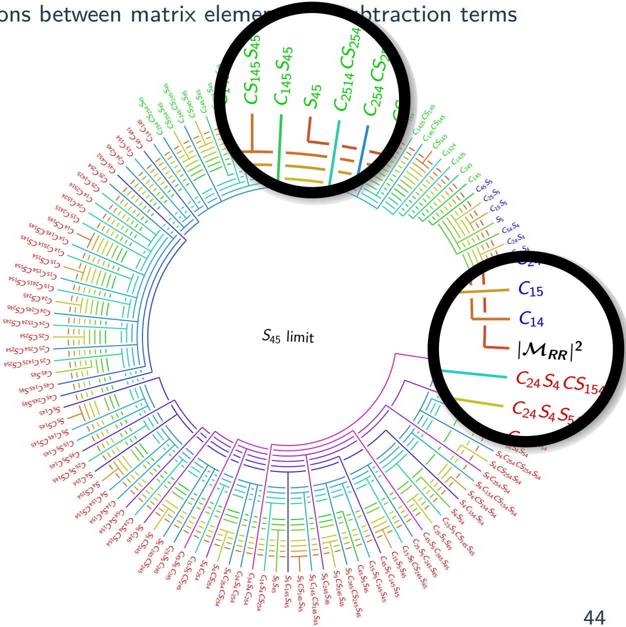
Delicate structure of cancellations between matrix elemen… …btraction terms

- $gg \to Hgg$



- Interplay of terms





The `NNLOCAL` code

```
~/research/colorsinglet/runs_for_nnlocalgg/nnlocal/bin/testrun-H
› ../nnlocal
************************** version  beta  **************************
*                                                                  *
* ##    ## ##    ## ##        #######   ######     ###    ##       *
* ###   ## ###   ## ##       ##     ## ##    ##   ## ##   ##       *
* ####  ## ####  ## ##       ##     ## ##        ##   ##  ##       *
* ## ## ## ## ## ## ##       ##     ## ##       ##     ## ##       *
* ##  #### ##  #### ##       ##     ## ##       ######### ##       *
* ##   ### ##   ### ##       ##     ## ##    ## ##     ## ##       *
* ##    ## ##    ## ########  #######   ######  ##     ## ######## *
*                                                                  *
*********************** December 20th, 2024 ************************
*                                                                  *
* Authors:                                                         *
* Vittorio Del Duca <Vittorio.Del.Duca@cern.ch>                    *
* Claude Duhr <cduhr@uni-bonn.de>                                  *
* Levente Fekeshazy <levente.fekeshazy@desy.de>                    *
* Flavio Guadagni <guadagni.flavio@gmail.com>                      *
* Pooja Mukherjee <pooja.mukherjee@desy.de>                        *
* Gabor Somogyi <somogyi.gabor@wigner.hun-ren.hu>                  *
* Sam Van Thurenhout <sam.van.thurenhout@wigner.hun-ren.hu>        *
* Francesco Tramontano <francesco.tramontano@unina.it>             *
*                                                                  *
* https://github.com/nnlocal/nnlocal.git                           *
*                                                                  *
********************************************************************
* Using input file named input.DAT
```

The NNLOCAL code

NNLOCAL: a proof-of-concept **Monte Carlo program** implementing the described scheme

- Architecture based on MCFM-4.0 (note current version is 10.3) written in Fortran77. (Have ensured that this is tolerated by the authors!)
- Publicly available, only external dependency is LHAPDF.

https://github.com/nnlocal/nnlocal.git

Proof-of-concept code:

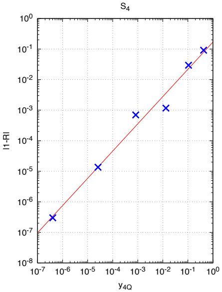
- Gluon fusion Higgs production in HEFT ($m_t \to \infty$) with no light quarks ($n_f = 0$): not a restriction on the structure of the subtraction scheme, on the contrary, all possible IR singularities for color-singlet production are present.
- Minimal optimisation only (especially in VV part): some issues with numerical cancellations in integrated subtraction terms handled by dynamical switching to quadruple precision.
- Many features to support checking/validation.

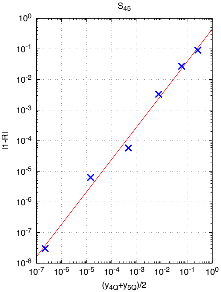
The NNLOCAL code

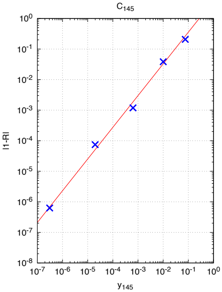
Main features

- Publicly available local analytic subtraction code at NNLO.
- Computation of any infrared and collinear-safe observables via user defined analysis routine.
- Support for parallel running if desired (managed by shell scripts).
- Support for efficient building and monitoring of Monte Carlo integration grids (via parallel running and scripts for visualisation).
- Support for checking cancellation of kinematical singularities through dedicated phase space routines.
- Support for checking cancellation of ε-poles order by order.

Validate **cancellation** of kinematical singularities in double real emission ($R = \sum \mathcal{A}/|\mathcal{M}|^2$)







Check **cancellation** of ε-poles

```
na = 0.755605220795
nb = 0.458650112152
                                           1/eps^4                 1/eps^3                 1/eps^2                   1/eps                   eps^0
--------------------------------------------------------------------------------------------------------------------------------------------------
ICT  f(xa/na)*f(xb/nb) =            0.000000000000          0.000000000000         -0.000000000001         -0.000000001763       5835.169801514889
ICT  f(xa/na)*f(xb)    =            0.000000000000         -0.000000000000          0.000000000002         -0.000000000002      -3532.639553984512
ICT  f(xa)*f(xb/nb)    =            0.000000000000          0.000000000000          0.000000000000         -0.000000000001      -4741.126731759623
ICT  f(xa)*f(xb)       =          -18.000000000000        -90.750000000000        162.250675918787        440.936804453876       3724.953259726909
VV/B f(xa)*f(xb)       =           18.000000000000         90.750000000000       -162.250675918791       -440.936804453905       1358.713586569218
```

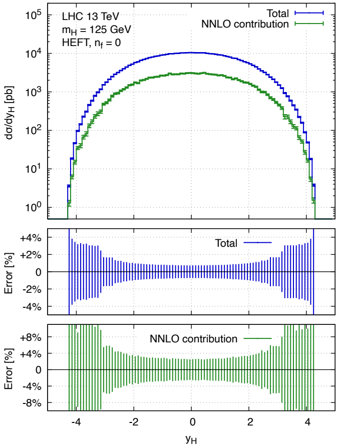
Code validation

Tuned comparison of **inclusive cross section** to `n3loxs` [Baglio, Duhr, Mistlberger, Szafron 2022] (modified to exclude quark channels and use same α_S running as `NNLOCAL`)

m_H [GeV]	`n3loxs` (gg)	`NNLOCAL` (gg)
100	65.72 pb	65.74(4) pb
125	42.94 pb	42.94(2) pb
250	9.730 pb	9.733(5) pb
500	1.626 pb	1.626(1) pb
1000	173.7 fb	173.7(1) fb
2000	8.794 fb	8.790(5) fb

- Several choices of μ_R, μ_F checked, shown values are for $\mu_R = \mu_F = m_H$.
- Runtime per mass value: $\sim$ 20 mins. on a MacBook Pro M2 with 8 CPUs.

Rapidity distribution of a Higgs boson of mass $m_H = 125$ GeV at the 13 TeV LHC



- Error shown is estimated Monte Carlo integration uncertainty.
- Runtime $\sim$ 1 hr 15 mins. on a MacBook Pro M2 with 8 CPUs.

Conclusions and outlook

Conclusions and outlook

Precision is key

- Understand subtle features of the Standard Model.
- First signs of New Physics might very well be indirect.
- Exact higher-order calculations required beyond NLO.

CoLoRFulNNLO at work for cancelling initial state singularities

- Completely local subtraction terms.
- Fully analytic integrated subtraction terms.

Implemented in `NNLOCAL`

- Gluon fusion Higgs production in HEFT has very simple matrix elements but otherwise all essential features.
- Quark channels, full mass dependence will be available in the near future.

Conclusions and outlook

Outlook

- NNLOCAL: from proof-of-concept to useful tool.
  (all channels, more color-singlet processes, optimisation)
- The inclusion of final-state jets and heavy quarks in hadron collisions appears feasible in our methodology.
- Extension to next order (N^3LO) for color-singlet appears conceivable.

Outlook

- `NNLOCAL`: from proof-of-concept to useful tool.
  (all channels, more color-singlet processes, optimisation)
- The inclusion of final-state jets and heavy quarks in hadron collisions appears feasible in our methodology.
- Extension to next order (N^3LO) for color-singlet appears conceivable.

Thank you for your attention!

Acknowledgments

This work was supported by grant K 143451 of the National Research, Development and Innovation Fund in Hungary and by the Bolyai Fellowship programme of the Hungarian Academy of Sciences.