The continued story of NNLOCAL:

Integrating the subtraction terms

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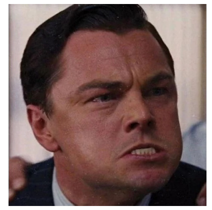
Quick recap

- ✦ **Precision** is important.
- ✦ Hurdles at higher order corrections are the **singularities**.
- ✦ Handling the singularities is well understood and extensively **implemented at NLO** to high precision.
- ✦ But it's real **challenge** to do the same at **NNLO**.
- ✦ The challenges are overcome and automated in **CoLoRFulNNLO subtraction scheme**.

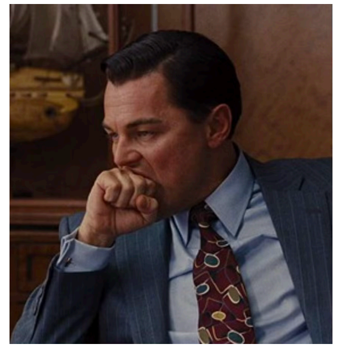
Generic procedure in a nutshell

Step 6: NNLOCAL

Step 5: explicit integration

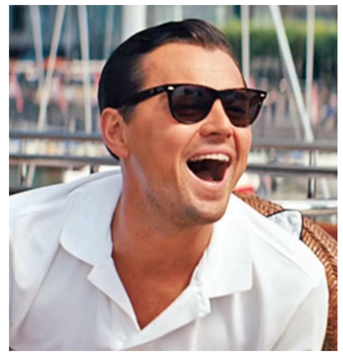


Step 4: well-defined counterterms



Step 3: iterate the first two steps until all singularities are gone

Step 2: integrate the subtraction over unresolved emission and add back



Step 1: subtract IR singularities for k- fold emission

Formula at NNLO

✦ The **arrangement of counterterms** to handle the singularities is done in the following way :

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} J_m \right]$$

$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1}} + \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right] J_{m+1} - \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} + \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_2}} + \int_2 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} \right] + \int_1 \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right\} J_m$$

- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$.
- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR}}$.
- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_{12}}$ accounts for the overlap of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_1}$ and $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_2}$.
- $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},A_1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma_{m+1}^{\mathrm{RV}}$.
- $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_1} \right)^{A_1}$ regularizes the singly-unresolved limits of $\displaystyle\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_1}$.

This talk ...

✦ In this talk : generic overview of the steps needed in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_2}$ **integration** procedure.

✦ $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},A_2}$ involves counterterms when **two** partons become **unresolved** :

- **Triple collinear**: three momenta become parallel, $p_i \parallel p_r \parallel p_s$
- **Double collinear**: two momentum pairs become parallel, $p_i \parallel p_r,\ p_j \parallel p_s$
- **Soft collinear**: one momentum pair becomes parallel, $p_i \parallel p_r$, and a third becomes soft $p_s \to 0$
- **Double Soft**: two momenta become soft $p_s \to 0,\ p_r \to 0$

✦ Mainly discuss about the **double collinear subtraction term** denoted by the operator $C_{ir;js}$

Preface to the computation

✦ **Reverse unitarity**: a very well known concept. For a generic massless external momentum q, we can write the on-shell condition as the difference of two propagators with opposite prescription for their imaginary part, thereby:

[Anastasiou, Melnikov ,'02]

$$\delta_+(q^2) \to \left(\frac{1}{q^2}\right)_c$$

c = cut propagators

✦ **Topology**: is a family of Feynman integrals characterized by the same set of propagators. A generic Feynman integral of a given topology can be expressed as:

$$I(n_1, \cdots, n_N) = \int d^d k_1 \cdots d^d k_l f(k_1, \cdots, k_l, p_1, \cdots, p_g)$$

$$f(k_1, \cdots, k_l, p_1, \cdots, p_g) = \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

Canonical form:

✦ **IBP reduction** : to express "complicated" Feynman integrals and their derivatives as a linear combination of "simple" ones called master integrals (MIs) . The relation stems from the equation.

[Chetyrkin, Tkachov ,'81]

$$0 = \int d^D k_1 \cdots d^D k_L \frac{\partial}{\partial k_i^\mu} \frac{N^\mu}{D_1 \cdots D_N}$$

✦ **ϵ-form of the differential equation** : the MIs obtained after reduction are put into a differential form using differential operators which reads as

$$d\vec{f}(\epsilon, \{x_j\}) = \left(\sum_{i=1}^{N} A_i(\epsilon, \{x_j\})\ dx_i \right) \vec{f}(\epsilon, \{x_j\})$$

where A_i are $N_{\text{master}} \times N_{\text{master}}$ -matrices of rational functions in $\{x_j\}$ and ϵ.

Canonical form:

- **Transforming** the basis of MIs with an invertible transformation T,

$$\vec{f} = T(\epsilon, \{x_j\})\vec{f}'$$

sets up the differential equation in **canonical form**: **[Henn ,'13]**

$$d\vec{f}'(\epsilon, \{x_j\}) = \epsilon \left(\sum_{i=1}^{N} A_i'(\{x_j\})\ dx_i \right) \vec{f}'(\epsilon, \{x_j\})$$

- the only singularities of the differential are **simple poles** and the only dependence on ϵ is given by the **explicit prefactor.**

- The algorithm for finding a **rational transformation** of a differential equation into ϵ-form is automated in several packages.

- We use the **Canonica** package in *Mathematica* for this purpose.

[Meyer,´17]

Iterated integrals :

- And the solution to the set of such differential equation is given as :

$$\vec{f}' = \mathbb{P} \exp[\epsilon \int A'(\vec{x})] \vec{f}'_0$$

- where, $\mathbb{P} \exp[\epsilon \int A'(\vec{x})]$ gives the **general solutions** and $\vec{f}'_0$ is the boundary vector determined from **initial conditions**.

- The canonical form allows the above path ordered exponential to be written in terms of **iterated integrals**.

- This enables a systematic study of the **function space** of the differential equation's general solution.

Function space of the result

✦ Even if a result is **simple** , it might be that our approach to the problem leads to a **complicated answer** :

✦ Hence the final goal is to obtain an expression of the general solution in terms of :

- **Transcendental numbers**: $\zeta_2, \ \log 2, \cdots$
- **Transcendental functions** : a whole zoo was discovered
  - Classical polylogarithms
  - Harmonic polylogarithms
  - 2d harmonic polylogarithms
  - Cyclotomic harmonic polylogarithms
  - All these are just special classes of multiple polylogarithms
  - Elliptic polylogarithms

✦ In this talk : will concentrate exclusively on **multiple polylogarithms** .

Polylogarithms

✦ Recursive definition of **multiple polylogarithms (MPLs)**,

$$G(a_1, \cdots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \cdots, a_n; t), \qquad \Bigg| \qquad \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t)$$

✦ All the other polylogarithms are just **special cases of MPLs**,

- ◉ Classical polylogarithms : $\mathrm{Li}_n(z) = -G(0,\cdots,0,1,z)$
- ◉ Harmonic polylogarithms : $a_i \in \{-1,0,1\}$
- ◉ 2d harmonic polylogarithms : e.g., $a_i \in \{0,1,a\}$
- ◉ Cyclotomic harmonic polylogarithms : roots of unity

✦ Natural "invariants" attached to MPLs: **weight** = **number of integrations**

✦ The polylogarithms satisfy various complicated **functional equations** and **the simplicity of the answer** might be hidden behind it.

Hopf algebra of MPLs

✦ **Algebra** : vector space with an operation that allows one to "fuse" two elements into one (**multiplication**)

a linear map $\mu : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$

or in other words, a linear map , multiplication : $(a, b) \to ab$

✦ **Coalgebra** : Vector space with an operation that allows one to "break into" elements apart (**comultiplication**)

a linear map $\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$

$\Delta(a) = a \otimes 1 + 1 \otimes a,\ a \in \mathscr{A},$ its coproduct $\Delta(a) \in \mathscr{A} \otimes \mathscr{A}$

✦ **Hopf algebra** : Vector space with **both multiplication and comultiplication**, i.e., one can "fuse" and "break into" in a consistent manner.

Preface to the computation

✦ Multiple polylogarithms form a **Hopf algebra**: **[Goncharov,´05]**

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1$$

$$\Delta(\mathrm{Li}_2(z)) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log(z)$$

✦ **Symbols** : maximal iteration of the coproduct (modulo $i\pi$),

$$S(F) \equiv \Delta_{1,\ldots,1(F)} \quad \mod i\pi$$

$$S(\mathrm{Li}_2(z)) = -\log(1-z) \otimes \log(z)$$

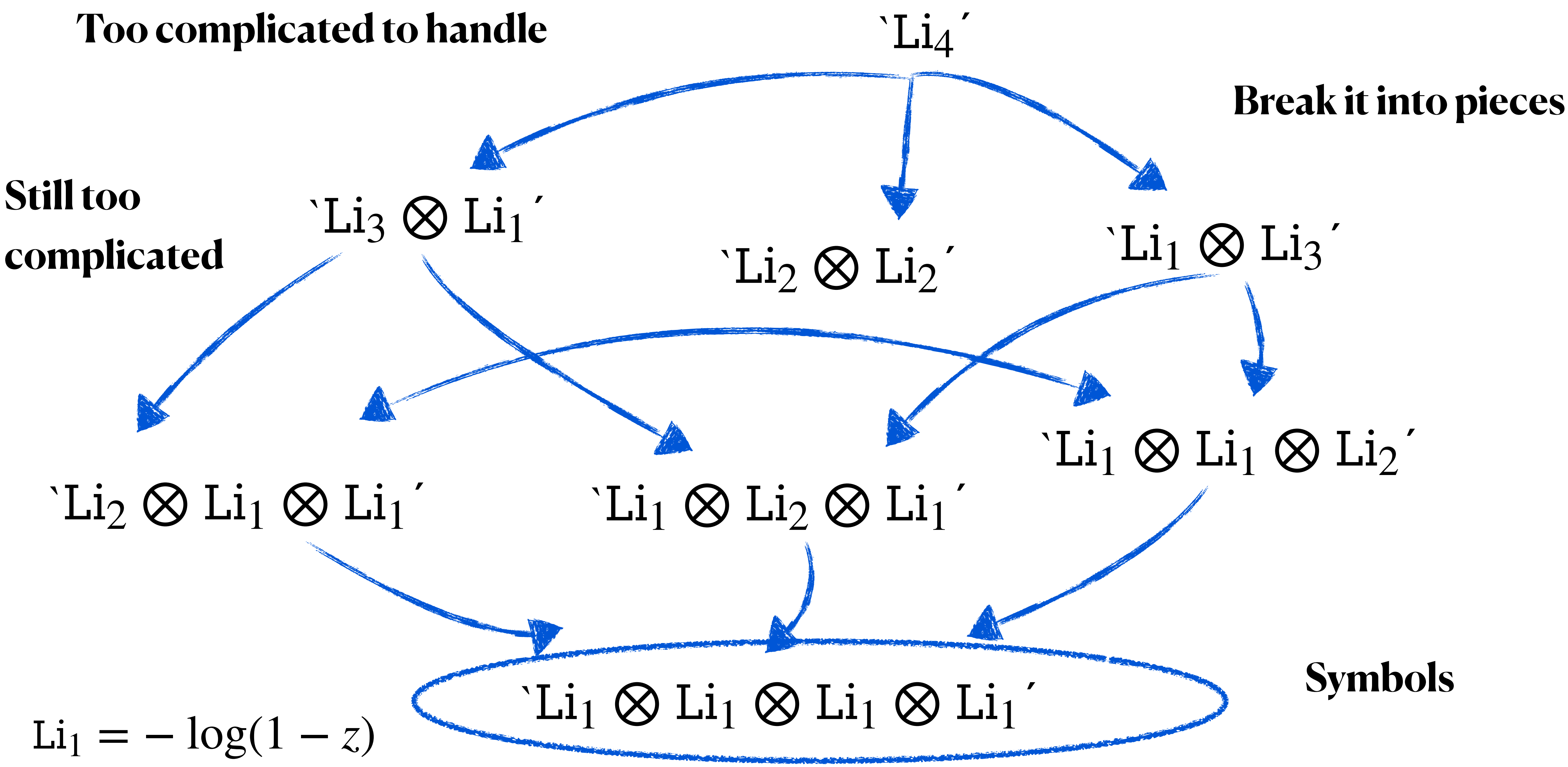
✦ Since symbols are only in terms of **logarithms** conventionally the log-signs are dropped and written as :

$$S(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1}$$

Preface to the computation

✦ How are all these important?

✦ Imagine a two-loop multi-scale integral that evaluates to 1000´s of Li_4´s



$\mathrm{Li}_1 = -\log(1-z)$

Preface to the computation

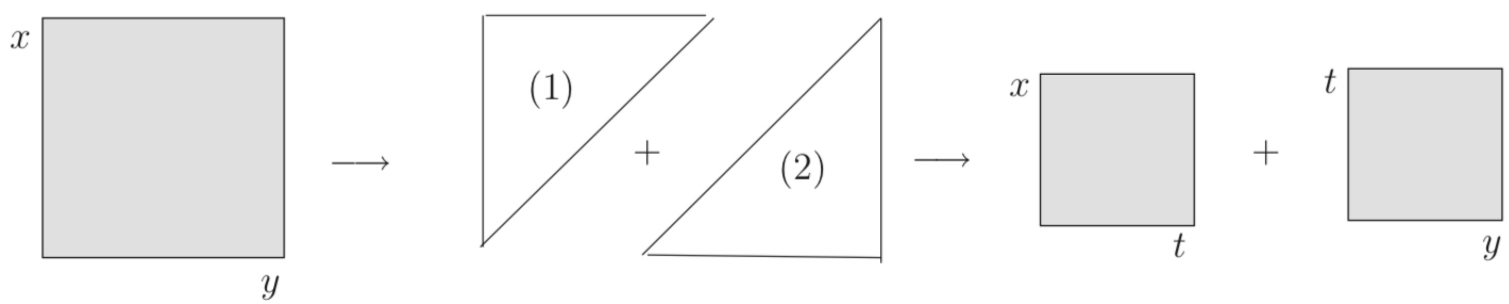
- At the end of this procedure , we have broken everything into little pieces (symbol), for which **all identities are known**.
- We then need to **reassemble the pieces** to find the simplified expression.
- Recall that the **weight** = **number of integrations**
- Examples:
  - $\log x \to$ **weight 1**
  - $\log x \,.\, \log y \to$ **weight 2**
  - $\mathrm{Li}_n(x), G(a_1, \ldots, a_n; x) \to$ **weight** n
  - $i\pi = \log(-1) \to$ **weight 1**, $\dfrac{\pi^2}{6} = \mathrm{Li}_2(1) \to$ **weight 2**, $\zeta_n = \mathrm{Li}_n(1) \to$ **weight** n,
  - Rational numbers → **weight 0**.

Sector decomposition

✦ **Sector decomposition** : corresponds to a resolution of singularities by a sequence of blow-ups. For instance, **[Heinrich ,'08]**

$$I = \int_0^1 dx \int_0^1 dy \; x^{-1-\epsilon} \; y^{-\epsilon} \left(x + y\right)^{-1}$$

✦ The integral has **overlapping singularity** for $x \to 0$ and $y \to 0$. **But if we divide the integration region in two sectors then we get,**



$$I = \int_0^1 dx \int_0^1 dt \; x^{-1-2\epsilon} \; t^{-1+\epsilon}\left(1 + t\right)^{-1} + \int_0^1 dy \int_0^1 dt \; y^{-1-2\epsilon} \; t^{\epsilon}\left(1 + t\right)^{-1}$$

✦ In practice, a single step of SD is not sufficient. We need to **iterate** this procedure.

Preface to the computation

✦ At the end of the iteration : All **singularities factorized** .

✦ **subtractions** of the poles in ϵ : expand the singular factors into **distribution**

$$\int_0^1 dx x^{-1-\epsilon} f(x) \to \int_0^1 dx x^{-1-\epsilon} \Big[f(x) - f(0) \Big] + \int_0^1 dx x^{-1-\epsilon} f(0) \,.$$

✦ This is equivalent to applying **plus prescriptions** .

✦ After this : Integral can be evaluated **numerically** for checks for each coefficient of ϵ

Integration of double collinear subtraction term

✦ From previous talk we know that there is a **factorization of the singularities** in the collinear/ soft limit :

$$\mathbf{U}_j |M_{ab,m+j}(\{p\}_{m+j})|^2_{l\text{-loop}} \propto \sum_{i=0}^{l} \text{Sing}_j^{(i)} \times \underbrace{|M_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(l-i)\text{-loop}}}_{j \text{ partons removed}}$$

✦ Here the **collinear factorization** for final partons r and s collinear to initial partons a and b

$$C_{ir;js} |M^{(0)}_{m+2}(p_i, p_r, p_j, p_s, \ldots, p_a, p_b)|^2 \propto \langle M^{(0)}_m(p_{ir}, p_{js}, \ldots, p_a, p_b) | \hat{P}^{(0)}_{f_i f_r}(z_i, z_r, k_\perp, \epsilon) \hat{P}^{(0)}_{f_j f_s}(z_j, z_s, k_\perp, \epsilon) | M^{(0)}_m(p_{ir}, p_{js}, \ldots, p_a, p_b) \rangle$$

- $\hat{P}^{(0)}_{f_i f_j}$ is the d- dimensional Altarelli–Parisi splitting function
- $M^{(0)}_m(p_{ir}, p_{js}, \cdots, p_a, p_b)$ is the reduced matrix element.

Integration of double collinear subtraction term

✦ The **momentum mapping** for well defined $\text{Sing}_j^{(i)}$ and reduced matrix elements

$$\hat{p}_a^\mu = \xi_{a,rs} p_a^\mu, \qquad \hat{p}_m^\mu = \Lambda(P,\hat{P})^\mu_\nu p_m^\mu \quad, m \neq r, s$$

$$\hat{p}_b^\mu = \xi_{b,rs} p_b^\mu,$$

- $\Lambda(P,\hat{P})$ is a **proper Lorentz transformation** that takes the massive momentum P into a momentum of the same mass $\hat{P}$.
- the fact that $P^2 = \hat{P}^2$ fixes the product of $\xi_{a,rs}$ and $\xi_{b,rs}$.
- and the values of $\xi_{a,rs}$ and $\xi_{b,rs}$ are chosen as: **[Gehrmann et.al. ;'07]**

$$\xi_{a,rs} = \sqrt{\frac{s_{ab} - s_{b(rs)}}{s_{ab} - s_{a(rs)}} \frac{s_{ab} - s_{(rs)(ab)} + s_{rs}}{s_{ab}}} \qquad s_{j(ab)} = 2p_j\,.\,(p_a + p_b)$$

$$\xi_{b,rs} = \sqrt{\frac{s_{ab} - s_{a(rs)}}{s_{ab} - s_{b(rs)}} \frac{s_{ab} - s_{(rs)(ab)} + s_{rs}}{s_{ab}}}. \qquad s_{(ij)(ab)} = 2(p_i + p_j)\,.\,(p_a + p_b)$$

Integration of double collinear subtraction term

✦ Now after all the re-definition we want to compute the following integral for **double collinear** counterterm :

$$\int_2 \mathrm{d}\phi_{rs}(p_r, p_s, x_1, x_2) \frac{x_1 x_2}{x_{a,r} s_{ar} x_{b,s} s_{bs}} \frac{1}{\mathbf{T}_{ar}^2 \mathbf{T}_{bs}^2} \hat{P}^{(0)}_{\hat{a}\hat{f}_r}(x_{a,r}, k_{\perp r}, \epsilon) \hat{P}^{(0)}_{\hat{b}\hat{f}_s}(x_{b,s}, k_{\perp s}, \epsilon) F_{ifif}(x_{a,r}, x_{b,s}, \xi_{a,rs}, \xi_{b,rs}) \,.$$

◉ $x_{a,r} = 1 - \frac{s_{r(ab)}}{s_{ab}}, \; x_{b,s} = 1 - \frac{s_{s(ab)}}{s_{ab}},$

◉ $F_{ifif}(x_{a,r}, x_{b,s}, \xi_{a,rs}, \xi_{b,rs}) = \frac{(s_{ab} - s_{r(ab)})(s_{ab} - s_{s(ab)})}{s_{ab}(s_{ab} - s_{(rs)(ab)} + s_{rs})}$

✦ Making use of **reverse unitarity** we define the **cut propagators** :

$$D_1 = p_r^2, D_2 = p_s^2, D_3 = (p_a + p_b - p_r - p_s)^2 - x_1 x_2 s_{ab}, D_4 = x_2(s_{ab} - s_{bs} - s_{br}) - x_1(s_{ab} - s_{as} - s_{ar})$$

✦ Then the integral measure becomes :

$$\mathrm{d}\phi_{rs} = d^d p_r d^d p_s \frac{1}{D_1 D_2 D_3 D_4}$$

Finding the topologies

✦ After some simplification this is what the final form looks like :

$$\int_2 \mathrm{d}\phi_{rs} \left(\frac{1}{x_{a,r}} + \frac{1}{1 + x_{a,r}} \right) \left(\frac{1}{x_{b,s}} + \frac{1}{1 + x_{b,s}} \right) \frac{s_{ab}\big(x_1\big(s_{ab} - s_{ar} - s_{as}\big) + x_2\big(s_{ab} - s_{br} - s_{bs}\big)\big)}{x_{a,r} x_{b,s} s_{ar} s_{bs}}$$

$$\times \frac{\big(s_{ab} - s_{ar} - s_{as}\big)\big(s_{ab} - s_{br} - s_{bs}\big)}{x_1 x_2 s_{ab}^2}$$

✦ Now we want to define some **topologies** that include the above integrals and **reduce** them to MIs.

✦ A topology with two external momenta (p_a, p_b) and two loop momenta (p_r, p_s) has to contain **7 independent propagators**.

Finding the master integrals

- Using **partial fraction decomposition**, all integrals were mapped into specific topologies, ensuring each topology includes D_1, D_2, D_3 and D_4 as cut propagators, while the remaining three propagators were defined to maintain independence.

- The Double Soft counterterm was decomposed into **3 topologies**, Collinear Soft into **3 topologies**, Double Collinear into **6 topologies**, and Triple Collinear into **17 topologies**.

- These topologies were then taken for IBP reductions in **Kira**, yielding **15** MIs for Double Soft, **14** for Collinear Soft, **24** for Double Collinear, and **28** for Triple Collinear.

[Maierhoefer et. al.,´17]

Finding the canonical form

✦ MIs were evaluated via **Differential Equations** in ϵ-**factorized form**.

✦ For Double Soft and Collinear Soft, the **Canonica package** successfully obtained the ϵ-factorized form.

✦ For Triple Collinear and Double Collinear, canonization had to be performed **block by block** .

✦ Sub-sectors not requiring **non-rational transformations** were directly canonized using **Canonica**.

✦ For the remaining sub-sectors, a two-step basis transformation was applied:

- First, transformed to a **new basis** ensuring only ϵ and ϵ^0 dependency.
- Then, another change of basis was performed to **integrate out** ϵ^0 dependency.

Canonization of G_1 topology

- Let's take one of the topologies (G_1) of double collinear and break down the method of **canonization** procedure.

$$\text{Topology } G_1 : D_5 = s_{ar}, D_6 = s_{ab} - s_{as} - s_{bs}, D_7 = s_{bs} \,.$$

- The system of differential equations w.r.t the variables x_1 and x_2 is:

$$\partial_{x_1} \vec{f}(x_1, x_2, \epsilon) = A^{(1)}(x_1, x_2, \epsilon) \vec{f}(x_1, x_2, \epsilon)$$

$$\partial_{x_1} \vec{f}(x_1, x_2, \epsilon) = A^{(2)}(x_1, x_2, \epsilon) \vec{f}(x_1, x_2, \epsilon)$$

  - where $A^{(1)}$ and $A^{(2)}$ are 11×11 **matrices** whose coefficients depend in rational way on x_1, x_2 and ϵ.
  - the first 7×7 sub-block does not require **non- rational transformation**.
  - hence this part was done using **Canonica.**

Canonization of G_1 topology

✦ Then the change of basis leads to the following new matrices which look like this :

$$A_{x_1} = \begin{pmatrix} e & e & e & 0 & 0 & e & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & e & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & e & * & 0 \\ * & * & * & * & * & * & * & * & e & 0 & * \end{pmatrix} \qquad A_{x_2} = \begin{pmatrix} e & e & e & 0 & 0 & e & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & e & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & * \end{pmatrix}$$

- e denotes a non-zero element **proportional to** ϵ.
- $*$ denotes a non-zero element **proportional to polynomial in** ϵ.

Canonization of G_1 topology

- At first we proceed with the **all the under diagonal terms** for rows from 8 to 11.

- The ϵ dependence of these coefficients are in the form :

$$\epsilon\left(-4+32\epsilon-95\epsilon^2+130\epsilon^3-81\epsilon^4+18\epsilon^5\right)\equiv\epsilon\alpha(\epsilon)$$

- Now to get rid of this polynomial we make a **change of basis** :

$$\vec{f}' \to T_{\alpha(\epsilon)}\vec{f}'$$

$$\tilde{A}_{x_i} \to T^{-1}_{\alpha(\epsilon)}A_{x_i}T_{\alpha(\epsilon)}$$

with,

$$T_{\alpha(\epsilon)}=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&\alpha(\epsilon)&0&0&0\\
0&0&0&0&0&0&0&0&\alpha(\epsilon)&0&0\\
0&0&0&0&0&0&0&0&0&\alpha(\epsilon)&0\\
1&0&0&0&0&0&0&0&0&0&\alpha(\epsilon)
\end{pmatrix}.$$

- with this transformation all the **diagonal sub-blocks are proportional to** ϵ **and** ϵ^0**.**

Canonization of G_1 topology

✦ **Non-rational transformation** of the basis are needed for rows from 8 to 11.

✦ The ϵ dependency for the sub-blocks 8 to 10 reads as :

$$\partial_{x_1}\vec{f}'(x_1,x_2,\epsilon) = \tilde{A}_{x_1}(x_1,x_2,\epsilon)\vec{f}'(x_1,x_2,\epsilon)$$

$$\partial_{x_2}\vec{f}'(x_1,x_2,\epsilon) = \tilde{A}_{x_2}(x_1,x_2,\epsilon)\vec{f}'(x_1,x_2,\epsilon)$$

$$\tilde{A}_{x_1} = \begin{pmatrix} e & e & e & 0 & 0 & e & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & e & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & e & e & e & e & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & e & * & 0 \\ e & e & e & e & e & e & e & * & e & 0 & * \end{pmatrix} \qquad \tilde{A}_{x_2} = \begin{pmatrix} e & e & e & 0 & 0 & e & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & 0 & e & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & e & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ e & e & e & e & e & e & e & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & e & e & 0 & 0 & 0 & 0 & * & 0 \\ e & e & e & e & e & e & e & * & 0 & 0 & * \end{pmatrix}$$

◉ e denotes a non-zero element proportional to ϵ.

◉ $*$ denotes a non-zero element proportional to ϵ and ϵ^0.

Canonization of G_1 topology

✦ At first we begin with **sub- block from 8 to 10** where the differential equation reads as : e.g. for 8th master integral :

$$\partial_{x_i} f_8 = \epsilon \left(\sum_{j=1}^{8} \tilde{A}_{x_i,8,j} \; f_j \right) + B_{x_i,8,8} \; f_8, \quad i \in \{1,2\}$$

✦ So in order to get the above equation proportional to ϵ we have to make a change of basis : $f_8 = t_8(x_1, x_2) g_8$ where

$$\partial_{x_i} t_8(x_1, x_2) = B_{x_i,8,8} \; t_8(x_1, x_2), \quad i \in \{1,2\}$$

✦ Now for the **11 th sub-block** , the differential equation reads as:

$$\partial_{x_i} f_{11} = \epsilon \left(\sum_{j=1}^{10} \tilde{A}_{x_i,11,j} \; f_j \right) + B_{x_i,11,11} \; f_{11} + B_{x_i,11,8} \; f_8, \quad i \in \{1,2\}$$

✦ As before we can **integrate out** $B_{x_i,11,11}$ term via a change of basis : $f_{11} = t_{11}(x_1, x_2) g_{11}$

Canonization of G_1 topology

✦ And to remove $B_{x_i,11,8}$ we make a **shift transformation**:

$$f_{11}(x_1, x_2, \epsilon) \to f_{11}(x_1, x_2, \epsilon) + G_8(x_1, x_2) f_8(x_1, x_2, \epsilon)\,.$$

✦ then the differential equation reads as:

$$\frac{\partial}{\partial x_i}(f_{11} - G_8 f_8) = \epsilon \tilde{A}^{(i)}_{x_i,11,11}(f11 - G_8 f_8) + \epsilon \sum_{j=1}^{10} \tilde{A}^{(i)}_{x_i,11,j} f_j + B^{(i)}_{x_i,11,8} f_8$$

✦ Implying the **constraint**:

$$\frac{\partial G_8(x_1, x_2)}{\partial x_i} + B^{(i)}_{11,8}(x_1, x_2) = 0$$

✦ With this the **canonical form** is reached :

$$d\vec{f} = \epsilon \left(\sum_{i=1}^{11} C_i \, \mathrm{d}\log L_i \right) \vec{f}$$

Finding the boundary constants

✦ C_i is a 11×11 -matrix, whose entries are **algebraic numbers**.

✦ the general solution was obtained in terms of MPLs up to **weight 3**

✦ For boundary constant the **phase space integrals** are then solved as :

- choose explicit **parametrization** of phase space.
- write the **parametric integral** representation in chosen variables.
- resolve the ϵ poles by **sector decomposition**.
- pole coefficients are **finite** parametric integrals.
- evaluate the parametric integrals in terms of **MPLs**.

✦ Details on direct integration method : **Sam Van Thurenhout's talk in the future** .

Simplifying the solutions

✦ **Simplification** is carried out in two steps :

Step 1: the MPLs were reduced to a function which is a linear combination of (products of) **classical polylogarithms** :

$$G(a,b;c) = \sum_i c_i \, \mathrm{Li}_2(f_i(a,b,c)) + \sum_{j,k} c_{jk} \log(g_j(a,b,c)) \; \log(h_k(a,b,c))$$

- ✦ c_i, c_{jk} are **rational numbers** and f_i, g_j, h_k are **rational functions**.
- ✦ The arguments of these functions lie within the **unit circle** such that no branch cuts singularities are crossed.

Step 2: find an **independent set of Li_2** by exploiting the relations among classical polylogarithms of different arguments.

An example of simplification :

✦ Let's simplify $G(-1,1;x)$ in terms of **dilogarithms and logs** :

✦ The symbol of $G(-1,1;x)$:

$$S(G(-1,1;x)) = (1+x)\otimes 2 + (1-x)\otimes(1-x) - (1-x)\otimes 2$$

✦ Make an **ansatz** for $G(-1,1;x)$:

$$G(-1,1;x) = \sum_i c_i \,\mathrm{Li}_2(f_i(x)) + \sum_{j,k} c_{jk} \log(g_j(x)) \;\log(h_k(x))$$

✦ Now the coefficients can be determined by **projecting the symmetric and anti-symmetric part** of the symbol.

An example of simplification :

✦ Notations for **decomposition** of symmetric and anti-symmetric part :.

$$a \odot b \equiv a \otimes b + b \otimes a, \qquad a \wedge b \equiv a \otimes b - b \otimes a,$$

such that, $$a \otimes b = \frac{1}{2} a \odot b + \frac{1}{2} a \wedge b\,.$$

✦ Then the **anti-symmetric** part of the $S(G(-1,1;x))$ simplifies to :

$$\left(1 - \frac{1+x}{2}\right) \wedge \frac{1+x}{2} = -\,S\left(\mathrm{Li}_2\left(\frac{1+x}{2}\right)\right).$$

✦ and since the product of **logarithms is totally symmetric** , the coefficient of the product of logarithms can be determined as :

$$S\left(G(-1,1;x) + \mathrm{Li}_2\left(\frac{1+x}{2}\right)\right) = 2 \odot (1+x) - \frac{1}{2}(2 \odot 2)$$

$$= S\left(\log 2\ \log(1+x) - \frac{1}{2}\log^2 2\right)$$

An example of simplification :

✦ We have to also account for weight 2 **constants independent** of x. This can be done by taking some limit.

$$\left\{ G(-1,1;x) - \left[-\mathrm{Li}_2\left(\frac{1+x}{2}\right) + \log 2\ \log(1+x) - \frac{1}{2}\log^2 2 \right] \right\}\Big|_{x=0} = \frac{\pi^2}{12}$$

✦ Thus we obtain,

$$G(-1,1;x) = -\mathrm{Li}_2\left(\frac{1+x}{2}\right) + \log 2\ \log(1+x) - \frac{1}{2}\log^2 2 + \frac{\pi^2}{12}\,.$$

✦ Computation of symbols and projector operators is implemented in **PolyLogTools package**.

[Duhr,´19]

Conclusion

- Following all these methods and tools , **fully analytical** integrated counterterms are now available.

- With this we have **extended** the CoLoRFulNNLO scheme to **hadron- hadron collisions**.

- All the results are implemented in local subtraction code : **NNLOCAL**

  https://github.com/nnlocal/nnlocal

- So far **only the gluon channel** in gluon fusion Higgs production in HEFT ($M_t \to \infty$) with no light quarks ($n_f \to 0$) is implemented.

- The inclusion of all quark channels and optimisation of the code is **currently underway.**

Thank you !