An Axiomatic Road to General Relativity

Gergely Székely

MTA Alfréd Rényi Institute of Mathematics [1]

2014.02.21. MTA Wigner Research Centre for Physics

[1] joint work with H.Andréka, J.X.Madarász, I.Németi

General goal:

„Investigate/understand the logical structure of relativity theories."

In more detail:

- Explore the tacit assumptions and make them explicit.
- Axiomatize relativity theories (in the sense of math. logic).
- Derive the predictions from a few natural basic assumptions.
- Analyze the relations between assumptions and consequences.

Terminology

- **Axioms:** Starting/basic assumptions.
  (Things that we don't prove from other assumptions.)
  They are NOT final/basic truths.
- **Theory:** A list of axioms.
- **Model:** A (mathematical) structure, from which we can decide whether it satisfies the axioms or not.
- **Model of the axioms:** A model satisfying the axioms.

Axiomatization in general:

Axioms:

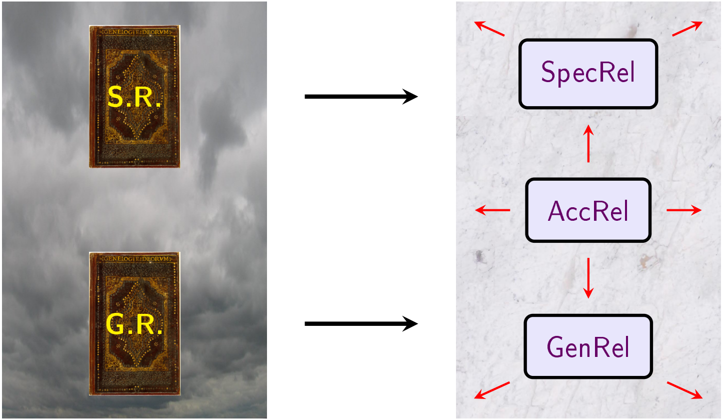
Ax.1.
Ax.2.
Ax.3.
Etc.

Theorems:

Thm.1.
Thm.2.
Thm.3.
Etc.

Economical
Streamlined
Transparent

Rich
Complex


S.R.
G.R.
SpecRel
AccRel
GenRel

Relativity theory is axiomatic (in its spirit) since its birth.

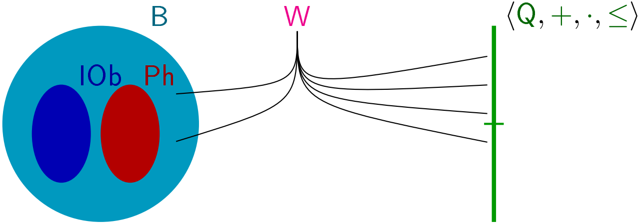
Two informal postulates of Einstein (1905):

- **Principle of relativity:** „The laws of nature are the same for every inertial observer."
- **Light postulate:** „Any ray of light moves in the 'stationary' system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body."

Corollary: „Any ray of light moves in all the inertial systems of co-ordinates with the same velocity."

SpecRel

Logic Language: $\{\, B, IOb, Ph, Q, +, \cdot, \leq, W \,\}$



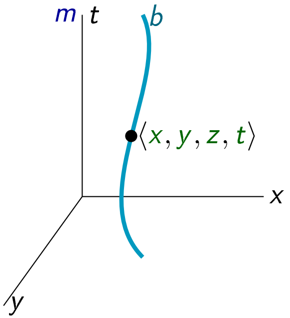
B $\rightsquigarrow$ Bodies (things that move)

IOb $\rightsquigarrow$ Inertial Observers   Ph $\rightsquigarrow$ Photons (light signals)

Q $\rightsquigarrow$ Quantities $+$, $\cdot$ and $\leq$ $\rightsquigarrow$ field operations and ordering

W $\rightsquigarrow$ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, x, y, z, t) \leftrightsquigarrow$ „observer m coordinatizes body b at spacetime location $\langle x, y, z, t\rangle$."

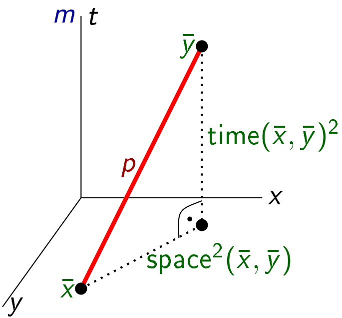


Worldline of body b according to observer m

$$wline_m(b) = \{\langle x, y, z, t\rangle \in \mathsf{Q}^4 \ : \ \mathsf{W}(m, b, x, y, z, t)\}$$

AxPh :

For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.



$$\forall m\Big(\mathsf{IOb}(m) \to \exists c\Big[c > 0 \land \forall \bar{x}\bar{y}\ \Big(\exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow \mathrm{space}^2(\bar{x},\bar{y}) = c^2 \cdot \mathrm{time}(\bar{x},\bar{y})^2\Big)\Big]\Big)$$

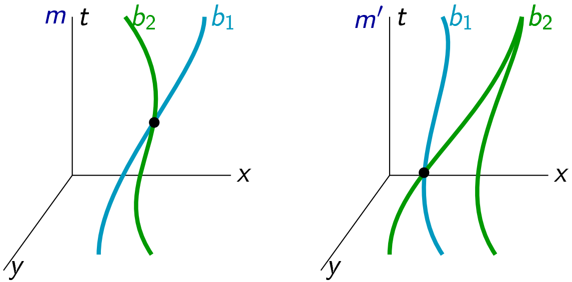
AxOField :

The **structure of quantities** $\langle Q, +, \cdot, \leq \rangle$ *is an ordered field,*

- Rational numbers: $\mathbb{Q}$,
- $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\pi)$, ...
- Computable numbers,
- Constructable numbers,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- **Real numbers:** $\mathbb{R}$,
- Hyperrational numbers: $\mathbb{Q}^*$,
- Hyperreal numbers: $\mathbb{R}^*$,
- Etc.

AxEv :

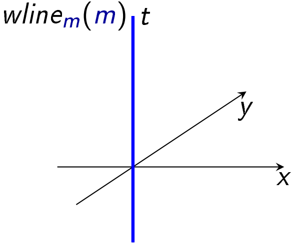
Inertial observers coordinatize the same events (meetings of bodies).



$$\forall m\, m' \bar{x} \ \ \mathsf{IOb}(m) \land \mathsf{IOb}(m') \to \left[\exists \bar{x}' \ \ \forall b \ \ \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(m', b, \bar{x}')\right].$$

AxSelf :

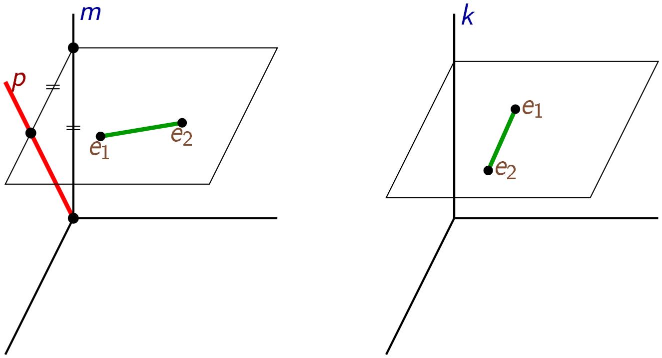
Every Inertial observer is stationary according to himself.



$$\forall mxyzt\ \left(IOb(m) \rightarrow \left[W(m,m,x,y,z,t) \leftrightarrow x = y = z = 0\right]\right).$$

AxSym :

Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them. Furthermore, the speed of light is 1.



What follows from SpecRel?

SpecRel:		Theorems:
AxPh		?
AxEv		??
AxOField	$\longrightarrow$	???
AxSelf		Etc.
AxSym		

Theorems of SpecRel

SpecRel = AxOField + **AxPh** + AxEv + AxSelf + AxSym

Theorem:

SpecRel $\Rightarrow$ *„Worldlines of inertial observers are straight lines."*

Theorem:

SpecRel − AxSym $\Rightarrow$ *„No inertial observer can move FTL."*

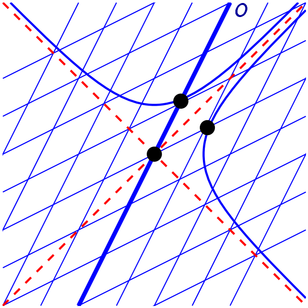
Theorem:

SpecRel $\Rightarrow$ {
„Relatively moving clocks slow down.",
„Relatively moving spaceships shrink."
etc.

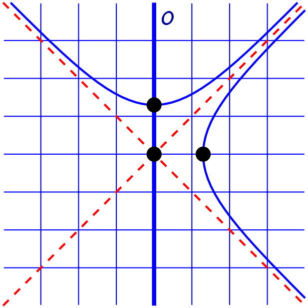
Theorems of SpecRel

Theorem:

SpecRel $\Rightarrow$ „*The worldview transformations between inertial observers are Poincaré transformations.*”



wordview of o'



wordview of o

Theorems about SpecRel

Theorem: (Consistency)

SpecRel *is consistent.*

Theorem: (Independence)

No axiom of SpecRel *is provable from the rest.*

Theorem: (Completeness)

SpecRel *is complete with respect to the „standard model of SR”, i.e., the Minkowski spacetimes over ordered fields.*

SpecRel:

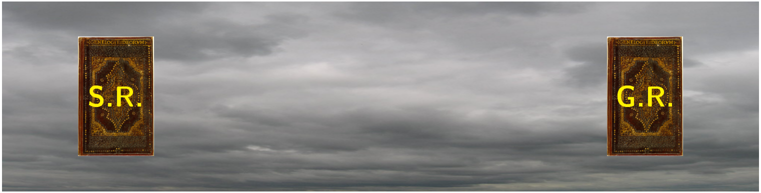
AxPh
AxEv
AxOField.
AxSelf
AxSym

$\longrightarrow$

Theorems:

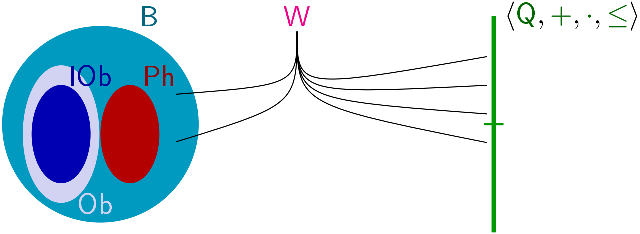
Slowing down of clocks, etc.
$\nexists$ FTL *observers*
Poincaré transformations
etc.

AccRel



SpecRel → AccRel → GenRel

The language is the same.



B $\leftrightsquigarrow$ Bodies (things that move)

IOb $\leftrightsquigarrow$ Inertial Observers Ph $\leftrightsquigarrow$ Photons (light signals)

Q $\leftrightsquigarrow$ Quantities

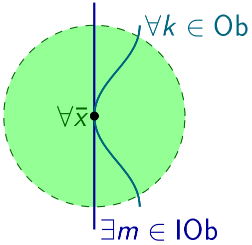
$+$, $\cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

W $\leftrightsquigarrow$ Worldview (a 6-ary relation of type BBQQQQ)

Observers: $\mathsf{Ob}(k) \overset{def}{\Longleftrightarrow} \exists xyzt\ b\ \mathsf{W}(k, b, x, y, z, t)$

AxCmv :

At each moment of its life, every observer coordinatizes the nearby world for a short while in the same way as an inertial observer does.



$$\forall k \in \mathsf{Ob}\ \forall \bar{x} \in wline_k(k)\ \exists m \in \mathsf{IOb} \quad d_{\bar{x}}\mathsf{w}_{mk} = Id, \text{ where}$$

$$d_{\bar{x}}\mathsf{w}_{mk} = L \overset{def}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall \bar{y}\ \ |\bar{y} - \bar{x}| \le \delta$$
$$\to |\mathsf{w}_{mk}(\bar{y}) - L(\bar{y})| \le \varepsilon |\bar{y} - \bar{x}|.$$

AxEv$^-$:

Any observer encounters the events in which he was observed.

AxSelf$^-$:

The worldline of an observer is an open interval of the time-axis, in his own worldview.

AxDiff :

The worldview transformations have linear approximations at each point of their domain (i.e., they are differentiable).

CONT :

Every definable, bounded and nonempty subset of Q has a supremum.

- ~~Rational numbers: $\mathbb{Q}$~~,
- ~~$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\pi)$, . . .~~
- Computable numbers,
- ~~Constructable numbers~~,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- **Real numbers:** $\mathbb{R}$,
- ~~Hyperrational numbers: $\mathbb{Q}^*$~~,
- Hyperreal numbers: $\mathbb{R}^*$,
- etc.

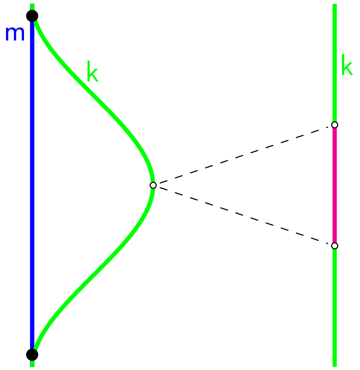
AccRel:		Theorems:
SpecRel		
AxCmv		?
AxEv^-		??
AxSelf^-	$\longrightarrow$	???
AxDiff		etc.
CONT		

Twin paradox $\rightsquigarrow$ TwP

Theorem:

$$\mathsf{AccRel} - \mathsf{AxDiff} \Rightarrow \mathsf{TwP}$$

$$Th(\mathbb{R}) + \mathsf{AccRel} - \mathsf{CONT} \not\Rightarrow \mathsf{TwP}$$



<u>AccRel:</u>

SpecRel

AxCmv

AxEv$^-$

AxSelf$^-$

AxDiff

CONT

$\longrightarrow$

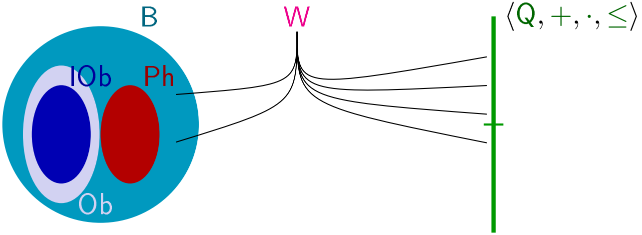
<u>Theorems:</u>

Twin paradox

etc.

GenRel

The language is the same.



B $\leftrightsquigarrow$ Bodies (things that move)

IOb $\leftrightsquigarrow$ Inertial Observers Ph $\leftrightsquigarrow$ Photons (light signals)

Q $\leftrightsquigarrow$ Quantities

$+$, $\cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

W $\leftrightsquigarrow$ Worldview (a 6-ary relation of type BBQQQQ)

Observers: $\mathsf{Ob}(k) \stackrel{def}{\Longleftrightarrow} \exists xyzt\, b\ \mathsf{W}(k, b, x, y, z, t)$

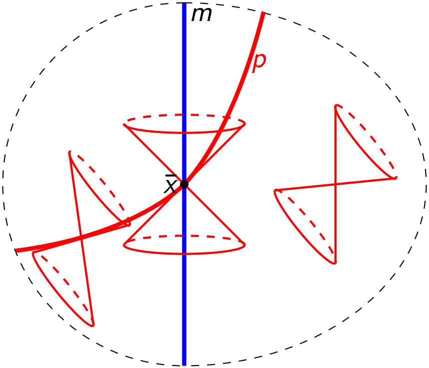
„Let all observers be equal at the level of axioms." (Einstein)

$$\left.\begin{array}{c}\mathsf{AxPh}\\ \mathsf{AxEv}\\ \mathsf{AxSelf}\\ \mathsf{AxSym}\end{array}\right\} \xrightarrow[\mathsf{AxDiff}]{\mathsf{AxCmv}} \left\{\begin{array}{c}\mathsf{AxPh}^-\\ \mathsf{AxEv}^-\\ \mathsf{AxSelf}^-\\ \mathsf{AxSym}^-\end{array}\right.$$

For example: $\mathsf{AxPh}, \mathsf{AxCmv} \Rightarrow \mathsf{AxPh}^-$.

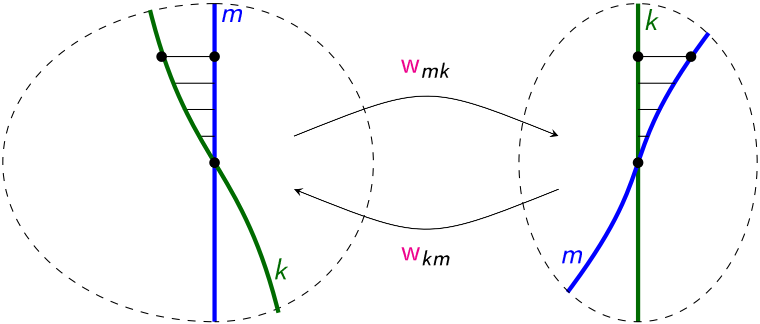
AxPh^{-} :

The instantaneous velocity of light signals is 1 in the moment when they are sent out according to the observer who sent them out, and any observer can send out a light signal in any direction with this instantaneous velocity.



AxSym$^-$:

Any two observers meeting see each others' clocks behaving in the same way at the event of meeting.



GenRel:

AxPh^-

AxEv^-

AxOField

AxSelf^-

AxSym^-

AxDiff

CONT

$\longrightarrow$

Theorems:

?

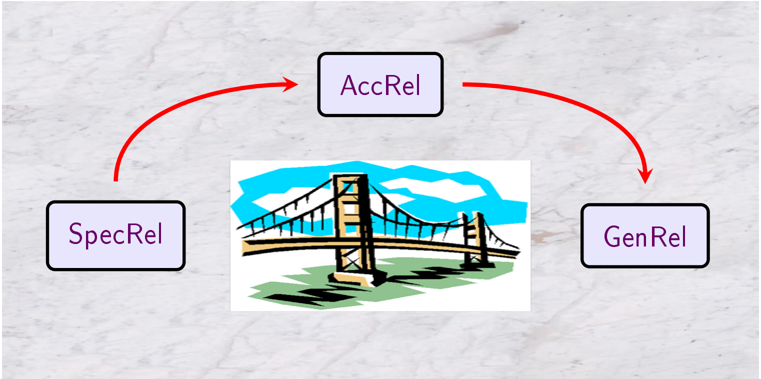
??

???

etc.

Theorem:

$$\mathsf{SpecRel} \Dashv \mathsf{AccRel} \models \mathsf{GenRel}$$

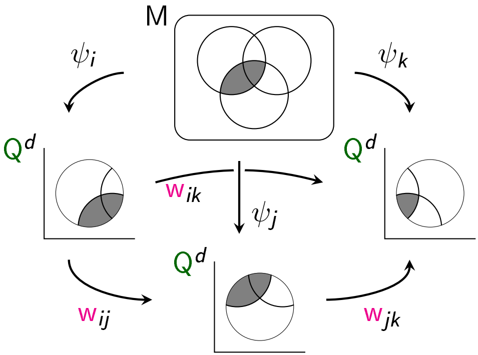


Theorem:

GenRel $\Rightarrow \forall m,k \in Ob\ \forall \bar{x} \in wline_m(k) \cap wline_m(m) \rightarrow$ "w_{mk} *is differentiable at* $\bar{x}$ *and* $d_{\bar{x}} w_{mk}$ *is a Lorentz transformation.*"

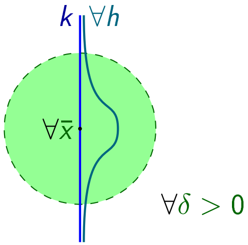
Theorem: (Completeness)

GenRel *is complete with respect to the „standard models of GR", i.e., Lorentzian manifolds over real closed fields.*



Def. (Geodesic):

The worldline of an observer is called timelike geodesic if it „locally maximizes measured time."



COMPR :

For any parametrically definable timelike curve in any observers worldview, there is another observer whose worldline is the range of this curve.

$$\mathrm{GenRel}^{+} = \mathrm{GenRel} + \mathrm{COMPR}$$

In GenRel$^+$ the notion of geodesics coincides with its standard notion. Via geodesics, we can define the other notions of general relativity, such as Riemann curvature tensor.

Einstein's field equations: $\boxed{R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}}$.

Definition or axiom? No real difference.

GenRel$^+$:

AxPh$^-$

AxEv$^-$

AxOField

AxSelf$^-$

AxSym$^-$

AxDiff$^-$

CONT

COMPR

$\longrightarrow$

Theorems:

Loc. Lorenz transf.

Completeness

Geodetics

Thank you for your attention!

Background materials:

www.renyi.hu/~turms