The Need for Beyond-Second-Order Dissipative Fluid Dynamics for Heavy Ion Collisions and Astrophysics

Azwinndini Muronga

Nelson Mandela University

HUN-REN Wigner Research Centre for Physics Seminar
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1121 Budapest, Konkoly-Thege Miklós út 29-33, Hungary

Nelson Mandela University, Summerstrand Campus, Gqeberha

Touring Budapest in May 2014 with Thendo and Rotondwa

Some background of my involvement in the topic

The journey

ECT*/HICforFAIR/CATHIE/Nikhef workshop at ECT* Trento

Flow and dissipation in ultrarelativistic Heavy Ion Collisions

Monday September 14 - Friday September 18, 2009

Organisers:
Marcus Bleicher, Frankfurt
Carsten Greiner, Frankfurt
Pasi Huovinen, Frankfurt
Peter Petreczky, BNL
Raimond Snellings, NIKHEF

Trento Workshop

My first public presentation of the topic with the Title: *Beyond second order theories of relativistic dissipative fluids* was in 2009.

Some background on my involvement

SQM2009

Figure 1. Participants at the 14th International Conference on Strangeness in Quark Matter (SQM2009) held at Búzios, Rio de Janeiro, Brazil from 27 September to 2 October 2009.

J. Phys. G: Nucl. Part. Phys. **35** (2008) 090301

Preface

SQM2009

IOP PUBLISHING JOURNAL OF PHYSICS G: NUCLEAR AND PARTICLE PHYSICS

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New developments in relativistic dissipative fluid dynamics

Azwinndini Muronga

Centre for Theoretical Physics and Astrophysics, Department of Physics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa
and
UCT-CERN Research Centre, Department of Physics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa

E-mail: Azwinndini.Muronga@uct.ac.za

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Abstract
The recent notion of the perfect fluid created at the relativistic heavy ion collider (RHIC) has been embraced by many experimentalists and theorists alike. However, much of the evidence to this notion has been based on the success of describing some experimental observables by non-viscous hydrodynamics or by small shear viscosity to entropy density ratio. Developments on viscous hydrodynamics evolved from (0+1) dimensions (Bjorken scaling solution) over

Some background on my involvement

Towards EIC

Third order relativistic dissipative fluid dynamics

Azwinndini Muronga[1,2]

[1] *UJ Theoretical and Astro-particle Physics, Department of Physics, University of Johannesburg, Auckland Park, South Africa*

[2] *UJ Soweto Science Centre, Faculty of Science, University of Johannesburg, Soweto, South Africa*

"Exploring QCD frontiers: from RHIC and LHC to EIC"
30 January - 3 February, 2012
Stellenbosch — Western Cape — South Africa

SQM2009

PHYSICAL REVIEW C **102**, 034902 (2020)

Third order viscous hydrodynamics from the entropy four current

Mohammed Younus *
Department of Physics, Nelson Mandela University, Port Elizabeth 6031, South Africa

Azwinndini Muronga †
Faculty of Science, Nelson Mandela University, Port Elizabeth 6031, South Africa

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Nonequilibrium dynamics for relativistic fluid or quark gluon plasma (QGP) have already been calculated earlier up to third order using both kinetic and thermodynamic approaches. Calculations presented in this article are based on thermodynamics principles. The expressions for third order dissipative fluxes have been derived from equation for entropy four-current developed earlier by Muronga. The relaxation equations in the present work have been developed in a simple Bjorken $(1+1)$ dimensional scenario and Eckart frame. The relaxation equations are found to have slightly different values for the coupling coefficients as compared to calculations from earlier models. The solutions to the differential equations have been found to be sensitive to values of these coefficients. The shear relaxation equations derived in third order theory are discussed term by term. Effects of third order theory on shear relaxation time have been discussed. Thermodynamic quantities related to hot and dense matter have been calculated as functions of proper time. Moreover, various initial conditions for the relaxation equations have been assumed to study their effects on above mentioned observables. A CERN Large Hadron Collider QGP formation time of $\tau_0 = 0.4\,\mathrm{fm}/c$ and temperature of $T_0 = 500\,\mathrm{MeV}$ have been assumed.

DOI: 10.1103/PhysRevC.102.034902

I. INTRODUCTION

High energy heavy ion collisions offer the opportunity to study the properties of hot and dense quark gluon plasma (QGP) [1]. At the CERN Large Hadron Collider (LHC),

and had assumptions that the entropy four-current contains only linear terms in dissipative quantities. Consequently we have Fourier-Navier-Stokes equations which might lead to noncausality and propagate viscous and thermal signals with

References

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Motivation and Conclusions

- Relativistic *second order* dissipative fluid dynamics (e.g., Israel-Stewart formalism) is a very important scientific achievement of the last three decades.
- It has inspired many authors to apply its methodology to the study of heavy ion collisions and astrophysics.
- In short it furnishes equations which are closed by imposing the entropy principle up to second order, with respect to equilibrium.
- First-order theories (the Navier-Stokes equations) break down at relativistic speeds. Issues: Causality and stability problems in first-order theories.
- The second-order terms (e.g., relaxation times) help solve issues in first-order theories.

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- The reluctance to exploit higher-order terms in relativistic dissipative fluid dynamics arises due to the complexity of calculations.
- However, pursuing these terms is essential for several reasons:
  (1) Second-order approaches better connect relativistic and classical cases.
  (2) Higher-order terms depend on lower-order ones, potentially impacting equilibrium conditions.
  (3) Couplings between key dissipative processes (e.g., heat conduction and viscosity) are only fully realized at the third order, making these terms crucial for accurate modeling.
- Despite the complexity, the inclusion of these terms greatly improves the understanding of fluid dynamics.

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The objective of relativistic dissipative fluid dynamics for one component fluid is the determination of the 14 fields of

$N^{\mu}(x^{\beta})$	net charge density — net charge flux vector
$T^{\mu\nu}(x^{\beta})$	stress — energy — momentum tensor

$T^{\mu\nu}$ is assumed symmetric so that it has 10 independent components.

The 14 fields are determined from the field equations (fluid dynamical equations)

$\partial_{\mu} N^{\mu} = 0$	net charge (e.g., baryon, strangeness, etc) conservation
$\partial_{\nu} T^{\mu\nu} = 0$	energy – momentum conservation
$\partial_{\lambda} F^{\mu\nu\lambda} = P^{\mu\nu}$	balance law of fluxes

$F^{\mu\nu\lambda}$ is completely symmetric tensor of fluxes and $P^{\mu\nu}$ is its production density such that

$$F^{\mu\nu}_{\ \ \nu} = m^2 N^{\mu} \qquad \text{and} \qquad P^{\nu}_{\nu} = 0$$

We then have a set of 14 independent equations (net charge conservation (1); energy-momentum conservation (4); balance of fluxes (9))

However, the dynamic equations cannot serve as the field equations for the thermodynamic fields N^{μ} and $T^{\mu\nu}$. Because the additional fields $F^{\mu\nu\lambda}$ and $P^{\mu\nu}$ have appeared.

Restriction on the general form of the constitutive functions $F^{\mu\nu\lambda}(N^{\alpha}, T^{\alpha\beta})$ and $P^{\mu\nu}(N^{\alpha}, T^{\alpha\beta})$ is imposed by

- entropy principle —the entropy density–entropy flux vector $S^{\mu}(N^{\alpha}, T^{\alpha\beta})$ is a constitutive quantity which obeys the inequality

$$\partial_{\mu} S^{\mu} \geq 0 \qquad \text{for all thermodynamic process}$$

- requirement of hyperbolicity — ensures that Cauchy problems of our field equations are well-posed and all wave speeds are finite $\Longrightarrow$ our set of field equations should be symmetric hyperbolic

Net charge 4-current $N^\mu = nu^\mu$

$n \equiv \sqrt{N^\mu N_\mu} = u_\mu N^\mu$ net charge density in fluid rest frame,

$u^\mu \equiv \dfrac{N^\mu}{\sqrt{N^\nu N_\nu}}$ the fluid 4-velocity,

$u^\nu u_\nu = 1 \implies u^\mu$ has 3 independent components

Stress–energy–momentum tensor $T^{\mu\nu} = \varepsilon u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + 2q^{(\mu} u^{\nu)} + \pi^{\langle\mu\nu\rangle}$

$\varepsilon \equiv u_\mu u_\nu T^{\mu\nu}$ energy density in fluid rest frame,

$p \equiv p(\varepsilon, n)$ pressure in fluid rest frame,

Π bulk viscous pressure, $(p+\Pi) \equiv -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$

$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ projection tensor onto 3-space, $\Delta^{\mu\nu}u_\nu = \Delta^{\mu\nu}u_\mu = 0$

$g^{\mu\nu} \equiv \mathrm{diag}(+1, -1, -1, -1)$ metric tensor

$q^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta}$ heat flux 4-current,

$q^\mu u_\mu = 0 \implies q^\mu$ has 3 independent components

$\pi^{\langle\mu\nu\rangle} \equiv T^{\langle\mu\nu\rangle}$ shear stress tensor

$\pi^{\langle\mu\nu\rangle}u_\mu = \pi^{\langle\mu\nu\rangle}u_\nu = 0,\quad \pi^{\langle\nu}_{\nu\rangle} = 0 \implies \pi^{\langle\mu\nu\rangle}$ has 5 independent components

Production densities tensor $P^{\mu\nu} = \mathcal{P}_\Pi \Pi \left(\Delta^{\mu\nu} - 3u^\mu u^\nu \right) + 2\mathcal{P}_q q^{(\mu} u^{\nu)} + \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}$

The functions $\mathcal{P}_\Pi$, $\mathcal{P}_q$, $\mathcal{P}_\pi$ are related to the bulk viscosity, heat conductivity and shear viscosity and thus may be determined from measurements of these coefficients

Liu, I-Shih; Müller, I.; Ruggeri, T.; *Relativistic thermodynamics of gases.* Annals of Physics, 169 (1986) 191 - 219

Tensor of fluxes (up to 2nd order)

$$
\begin{aligned}
F^{\mu\nu\lambda} \quad = \quad & \frac{1}{2}\mathcal{F}_1^0 g^{(\mu\nu}u^{\lambda)} + \frac{1}{2}\mathcal{F}_2^0\left(g^{(\mu\nu}u^{\lambda)} - 2u^\mu u^\nu u^\lambda\right) \\
& +\mathcal{F}_1^1\Pi\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_2^1\left(\Delta^{(\mu\nu}q^{\lambda)} - 5u^{(\mu}u^\nu q^{\lambda)}\right) \\
& +\mathcal{F}_3^1\pi^{(\langle\mu\nu\rangle}u^{\lambda)} \\
& +\mathcal{F}_1^2\Pi^2\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_2^2\left(-q^\nu q_\nu\Delta^{(\mu\nu}u^{\lambda)} - 3u^{(\mu}q^\nu q^{\lambda)}\right) \\
& -\mathcal{F}_3^2 q^\alpha q_\alpha\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_4^2\left(3u^\mu\pi^{2\langle\nu\lambda\rangle} - \pi^{2\langle\alpha\alpha\rangle}u^\mu u^\nu u^\lambda\right) \\
& +\mathcal{F}_5^2\pi^{2\langle\alpha\alpha\rangle}\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_6^2\left(q^{(\mu}\pi^{\langle\nu\lambda\rangle)} - 2u^{(\mu}u^\nu\pi^{\langle\lambda)\nu\rangle}q_\nu\right) \\
& +\mathcal{F}_7^2\left(\Delta^{(\mu\nu}\pi^{\langle\lambda)\alpha\rangle)}q_\alpha - 5u^\mu u^\nu\pi^{\langle\lambda)\alpha\rangle}q_\alpha\right) + \mathcal{F}_8^2\Pi u^{(\mu}\pi^{\langle\nu\lambda\rangle)} \\
& +\mathcal{F}_9^2\Pi\left(\Delta^{(\mu\nu}q^{\lambda)} - 5q^{(\mu}u^\nu u^{\lambda)}\right)
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order

Entropy 4-current (up to 3rd order)

$$
\begin{aligned}
S^\mu \;=\;& \mathcal{S}_1^0 u^\mu \\
&+\mathcal{S}_1^1 \Pi u^\mu + \mathcal{S}_2^1 q^\mu \\
&+\left(\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{2\langle\alpha\alpha\rangle}\right) u^\mu \\
&+\mathcal{S}_4^2 \Pi q^\mu + \mathcal{S}_5^2 \pi^{\langle\mu\alpha\rangle} q_\alpha \\
&+\left(\mathcal{S}_1^3 \Pi^3 - \mathcal{S}_2^3 \Pi q_\alpha q^\alpha + \mathcal{S}_3^3 \Pi \pi^{2\langle\alpha\alpha\rangle} + \mathcal{S}_4^3 q_\alpha q_\beta \pi^{\langle\alpha\beta\rangle} + \mathcal{S}_5^3 \pi^{3\langle\alpha\alpha\rangle}\right) u^\mu \\
&+\left(\mathcal{S}_6^3 \Pi^2 - \mathcal{S}_7^3 q_\alpha q^\alpha + \mathcal{S}_8^3 \pi^{2\langle\alpha\alpha\rangle}\right) q^\mu + \mathcal{S}_9^3 \Pi \pi^{\langle\mu\alpha\rangle} q_\alpha + \mathcal{S}_{10}^3 \pi^{2\langle\mu\alpha\rangle} q_\alpha
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order +Third order

Notation

Parentheses around some indices denote symmetrization, while angular brackets around two indices denote skew-symmetrization

$$a^{(\mu\nu)} \equiv \frac{1}{2}\left(a^{\mu\nu} + a^{\nu\mu}\right)$$

$$a^{\langle\mu\nu\rangle} \equiv \left(\Delta_\alpha^{(\mu}\Delta_\beta^{\nu)} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)a^{\alpha\beta}$$

The space-time derivative will be split into time and spatial components as follows

$$\partial_\mu \equiv u_\mu D + \nabla_\mu$$

with $D \equiv u^\alpha \partial_\alpha$ convective (comoving) time derivative
and $\nabla_\mu \equiv \Delta^{\mu\nu}\partial_\nu$ spatial gradient

$$\dot{a}^{\cdots} \equiv Da^{\cdots} = u^\mu \partial_\mu a^{\cdots} \quad \text{convective (comoving) time derivative of } a^{\cdots}$$

$$\theta = \nabla_\mu u^\mu = \partial_\mu u^\mu \quad \text{expansion scalar (divergence of 4-velocity)}$$

Equilibrium

Equilibrium is defined as a process in which production densities vanish and/or the entropy production vanishes

$$\left.\begin{array}{rcl} P^{\mu\nu}_{Eq.} & = & 0 \\ \Xi_{Eq.} & = & 0 \end{array}\right\} \Longrightarrow \Pi_{Eq.} = 0\,, \quad q^{\mu}_{Eq} = 0\,, \quad \pi^{\langle\mu\nu\rangle}_{Eq} = 0$$

$$\begin{array}{rcl} F^{\mu\nu\lambda}_{Eq} & = & \frac{1}{2}\mathcal{F}^0_1 g^{(\mu\nu}u^{\lambda)} + \frac{1}{2}\mathcal{F}^0_2\left(g^{(\mu\nu}u^{\lambda)} - 2u^{\mu}u^{\nu}u^{\lambda}\right) \\ S^{\mu}_{Eq.} & = & s(\varepsilon, n)u^{\mu} \end{array}$$

The energy-momentum tensor reduces to

$$T^{\mu\nu}_{Eq.} = \varepsilon u^{\mu}u^{\nu} - p\Delta^{\mu\nu}$$

In the ideal “perfect” fluid limit one has 5 independent fields ($p(n,e)$(2), u^{μ}(3)) and 5 field equations

14-Fields Theory of Relativistic Dissipative Fluid Dynamics :

In dissipative(non-ideal) fluid dynamics one needs 9 additional equations for the dissipative fluxes. The 14 fields $p(n, \varepsilon)$, Π, u^α, q^α, $\pi^{\langle\alpha\beta\rangle}$ are governed by the following fields equations

$$
\begin{aligned}
\partial_\mu N^\mu &= 0 \\
\Delta_{\alpha\mu}\partial_\nu T^{\mu\nu} &= 0 \\
u_\mu \partial_\nu T^{\mu\nu} &= 0 \\
u_\mu u_\nu \partial_\lambda F^{\mu\nu\lambda} &= -\mathcal{P}_\Pi \Pi \\
\Delta^\mu_\alpha u_\nu \partial_\lambda F^{\alpha\nu\lambda} &= \mathcal{P}_q q^\mu \\
\left(\Delta^{(\mu}_\alpha \Delta^{\nu)}_\beta - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)\partial_\lambda F^{\alpha\beta\lambda} &= \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}
\end{aligned}
$$

For all thermodynamic processes the entropy principle holds

$$\partial_\mu S^\mu \geq 0$$

Dissipative fluxes: Zeroth order: Equilibrium

$$
\begin{aligned}
\Pi &= \Pi_{Eq.} = 0 \\
q^{\alpha} &= q^{\alpha}_{Eq.} = 0 \\
\pi^{\langle\alpha\beta\rangle} &= \pi^{\langle\alpha\beta\rangle}_{Eq.} = 0
\end{aligned}
$$

Dissipative fluxes: First order

$$
\begin{aligned}
\Pi^{(1)} &= \Pi_E = -\zeta \nabla_\alpha u^\alpha \\
q^{\alpha\,(1)} &= q_E^\alpha = \kappa T \Delta^{\alpha\mu}\left(\frac{\nabla_\alpha T}{T} - \dot{u}_\alpha\right) \\
\pi^{\langle\alpha\beta\rangle\,(1)} &= \pi_E^{\langle\alpha\beta\rangle} = 2\eta \Delta^{\alpha\mu}\Delta^{\beta\nu}\nabla_{\langle\alpha} u_{\beta\rangle}
\end{aligned}
$$

Relativistic versions of the laws of Navier-Stokes and Fourier
first derived by Eckart, Landau-Lifshitz.
ζ is the bulk viscosity, κ is the thermal conductivity, η is the shear viscosity

- simple algebraic expressions of dissipative fluxes
- may lead to acausal and unstable equations of motion

Dissipative fluxes: Second order

Müller-Israel-Stewart (MIS) equations: $F^{\mu\nu\lambda}$ linear (first-order) in dissipative fluxes and S^{μ} quadratic (second-order) in dissipative fluxes
Resulting equations causal and hyperbolic

$$\begin{aligned}
\Pi^{(2)} &= \Pi_{MIS} = -\zeta\left[2\mathcal{S}_1^2\dot{\Pi} + \mathcal{S}_4^2\nabla_\alpha q^\alpha\right] \\
&\quad -\zeta\left[\Pi(\dot{\mathcal{S}}_1^2 + \mathcal{S}_1^2\nabla_\alpha u^\alpha) + q^\alpha(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha)\right] \\
q^{\mu\,(2)} &= q^\mu_{MIS} = \kappa T\Delta^{\alpha\mu}\left[2\mathcal{S}_2^2\dot{q}_\alpha + \mathcal{S}_4^2\nabla_\alpha\Pi + \mathcal{S}_5^2\nabla^\beta\pi_{\langle\alpha\beta\rangle}\right] \\
&\quad +\kappa T\Delta^{\alpha\mu}\left[q_\alpha(\dot{\mathcal{S}}_2^2 + \mathcal{S}_2^2\nabla_\nu u^\nu) + \Pi(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha)\right. \\
&\quad \left. +\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}^\beta)\right] \\
\pi^{\langle\mu\nu\rangle\,(2)} &= \pi^{\langle\mu\nu\rangle}_{MIS} = 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[2\mathcal{S}_3^2\dot{\pi}_{\langle\alpha\beta\rangle} + \mathcal{S}_5^2\nabla_{\langle\alpha}q_{\beta\rangle}\right] \\
&\quad +2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^2 + \mathcal{S}_3^2\nabla_\lambda u^\lambda)\right. \\
&\quad \left. +q_{\langle\alpha}(\nabla_{\beta\rangle}\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}_{\beta\rangle})\right]
\end{aligned}$$

Dissipative fluxes

- The terms in red are neglected in the original MIS formulation. Terms of the general form $\Pi\partial_\nu u^\mu$, $\Pi\partial_\lambda n$, $\Pi\partial_\lambda\varepsilon$, $q_\alpha\partial_\nu u^\mu$, $q_\alpha\partial_\lambda n$, $q_\alpha\partial_\lambda\varepsilon$, $\pi_{\langle\alpha\beta\rangle}\partial_\nu u^\mu$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda n$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda\varepsilon$ have been considered non-linear and thus ignored. These terms have been shown to be important in heavy ion collisions. They will be even more important at low energies and high densities.
- Derivations of the equations from kinetic theory reveals terms that are not explicit from phenomenological considerations (e.g., vorticity terms)

Dissipative fluxes: Third order: Bulk equation

$$
\begin{aligned}
\Pi^{(3)} \quad = \quad & -\zeta\Big[3\mathcal{S}_1^3\dot{\Pi} + 2\mathcal{S}_2^3\dot{q}_\lambda q^\lambda + 2\mathcal{S}_3^3\dot{\pi}_{\langle\alpha\beta\rangle}\pi^{\langle\alpha\beta\rangle} \\
& +\mathcal{S}_6^3(\Pi\nabla_\alpha q^\alpha + q^\alpha\nabla_\alpha\Pi) + \mathcal{S}_9^3(\pi^{\langle\alpha\beta\rangle}\nabla_\alpha q_\beta + q_\beta\nabla_\alpha\pi^{\langle\alpha\beta\rangle})\Big] \\
& -\zeta\Big[\Pi^2(\dot{\mathcal{S}}_1^3 + \mathcal{S}_1^3\nabla_\alpha u^\alpha) - q^\alpha q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\alpha u^\alpha) \\
& +\pi^{2\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\alpha u^\alpha) \\
& +\Pi q^\alpha(\nabla_\alpha\mathcal{S}_6^3 - \mathcal{S}_6^3 a_\alpha) + \pi^{\langle\alpha\beta\rangle}q_\beta(\nabla_\alpha\mathcal{S}_9^3 - \mathcal{S}_9^3 a_\alpha)\Big]
\end{aligned}
$$

$$
\begin{aligned}
q^{\mu\,(3)} = \; & \kappa T \Delta^{\alpha\mu}\Big[-\mathcal{S}_2^3(2\Pi\dot{q}_\alpha + q_\alpha\dot{\Pi}) + \mathcal{S}_4^3(2\dot{q}^\beta\pi_{\langle\alpha\beta\rangle} + q^\beta\dot{\pi}_{\langle\alpha\beta\rangle}) \\
& + 2\mathcal{S}_6^3\Pi\nabla_\alpha\Pi - 2\mathcal{S}_7^3 q^\beta\nabla_\alpha q_\beta + \mathcal{S}_9^3(\Pi\nabla^\beta\pi_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\nabla^\beta\Pi) \\
& + 2\mathcal{S}_{10}^3\pi_{\langle\beta\nu\rangle}\nabla_\alpha\pi^{\langle\beta\nu\rangle}\Big] \\
& + \kappa T \Delta^{\alpha\mu}\Big[\Pi q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\nu u^\nu) + q^\beta\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_\nu u^\nu) \\
& + (\Pi^2\nabla_\alpha\mathcal{S}_6^3 - q^\lambda q_\lambda\nabla_\alpha\mathcal{S}_7^3 + \pi^{2\langle\lambda\lambda\rangle}\nabla_\alpha\mathcal{S}_8^3) \\
& + \Pi\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_9^3 - \mathcal{S}_9^3 a^\beta) + \pi^2_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a^\beta) \\
& + \mathcal{S}_7^3 q_\alpha q^\lambda a_\lambda\Big]
\end{aligned}
$$

Dissipative fluxes: Third order: Shear equation

$$
\begin{aligned}
\pi^{\langle\mu\nu\rangle\,(3)} = \; & 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\mathcal{S}_3^3(2\Pi\dot{\pi}_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\dot{\Pi}) + 2\mathcal{S}_4^3\dot{q}_{\langle\alpha}q_{\beta\rangle} \\
& + 3\mathcal{S}_5^3\dot{\pi}_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle} + \mathcal{S}_8^3\pi_{\langle\alpha\beta\rangle}\nabla_\lambda q^\lambda + \mathcal{S}_9^3(\Pi\nabla_{\langle\alpha}q_{\beta\rangle} + q_{\langle\alpha}\nabla_{\beta\rangle}\Pi) \\
& + \mathcal{S}_{10}^3(q_{(\beta}\nabla^\lambda\pi_{\langle\alpha)\lambda\rangle} + \pi_{\langle\lambda(\alpha\rangle}\nabla^\lambda q_{\beta)})\Big] \\
& + 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\Pi\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\lambda u^\lambda) \\
& + q_{\langle\alpha}q_{\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_\lambda u^\lambda) + \pi_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle}(\dot{\mathcal{S}}_5^3 + \mathcal{S}_5^3\nabla_\lambda u^\lambda) \\
& + \pi_{\langle\alpha\beta\rangle}q^\lambda(\nabla_\lambda\mathcal{S}_8^3 - \mathcal{S}_8^3 a_\lambda) + \Pi q_{\langle\beta}(\nabla_{\alpha\rangle}\mathcal{S}_9^3 - \mathcal{S}_9^3 a_{\alpha\rangle}) \\
& + \pi_{\langle\alpha\lambda}q^\lambda(\nabla_{\beta\rangle}\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a_{\beta\rangle})\Big]
\end{aligned}
$$

Entropy from Kinetic Theory

We derive the third order entropy 4-current as well the non-classical coefficients by going beyond Israel-Stewart entropy 4-current expression in kinetic theory. The kinetic expression for entropy, can be written as

$$S^{\mu} = -\int dwp^{\mu}\psi[f(x,p)]\ ,$$

where

$$\psi[f(x,p)] = f(x,p)\left\{\ln[A_0^{-1}f(x,p)] - 1\right\}\ ,$$

and $f(x,p)$ is the out of the equilibrium distribution function. Expanding $\psi(f)$ around $\psi(f^{eq})$ up to third order we get,

$$\begin{aligned}\psi(f) &= \psi(f^{eq}) + \psi'(f^{eq})(f - f^{eq}) + \frac{1}{2}\psi''(f^{eq})(f - f^{eq})^2 \\ &\quad + \frac{1}{6}\psi'''(f^{eq})(f - f^{eq})^3 + ..\ ,\end{aligned}$$

with student Fhumulani Nemulodi, MSc (Physics): *Third order relativistic dissipative fluid dynamics for heavy-ion collisions.*, 2011 Unpublished MSc Dissertation, University of Cape Town,
Extension of the work by W. Israel and J. M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Annals Phys. 118 (1979) 341–372.

Entropy from kinetic theory

$$S^{\mu(1)} = \frac{q^\mu}{T} \ ,$$

$$S^{\mu(2)} = \frac{1}{2}\beta u^\mu \left[\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{\nu\alpha}\pi_{\nu\alpha}\right] + \beta\left[\mathcal{S}_4^2 q^\mu \Pi + \mathcal{S}_5^2 q_\alpha \pi^{\mu\alpha}\right] \ ,$$

$$\begin{aligned} S^{\mu(3)} \quad = \quad & \frac{1}{6}\beta u^\mu \left\{\mathcal{S}_1^3 \Pi^3 + \mathcal{S}_2^3 \Pi q^\alpha q_\alpha + \mathcal{S}_3^3 \Pi \pi^{\nu\alpha}\pi_{\nu\alpha} + \mathcal{S}_4^3 q_\nu q_\alpha \pi^{\nu\alpha} + \mathcal{S}_5^3 \pi_{\nu\alpha}\pi^\nu_\beta \pi^{\alpha\beta}\right\} \\ & -\frac{1}{6}\beta q^\mu \left\{\mathcal{S}_6^3 \Pi^2 + \mathcal{S}_7^3 q^\alpha q_\alpha - \mathcal{S}_8^3 \pi^{\nu\alpha}\pi_{\nu\alpha}\right\} - \beta \mathcal{S}_9^3 \Pi q_\alpha \pi^{\mu\alpha} \\ & +\frac{1}{2}\beta \mathcal{S}_{10}^3 q_\alpha \pi^{\nu\alpha}\pi^\mu_\nu \ , \end{aligned}$$

compare with phenomenology results above, now with relaxation and coupling coefficients calculated from 1-pdfs integrals

Third order coefficients

As function of m/T, in the large temperature limit

$$\left.\begin{aligned} \mathcal{S}_1^2 &= \infty \\ \mathcal{S}_2^2 &= \frac{5}{4p} \\ \mathcal{S}_3^2 &= \frac{3}{4p} \\ \mathcal{S}_4^2 &= \infty \\ \mathcal{S}_5^2 &= \frac{1}{4p} \end{aligned}\right\} \Longrightarrow \text{Second order coefficients known}$$

Third order coefficients

$$\begin{aligned} \mathcal{S}_1^3 &= \infty, \quad \mathcal{S}_2^3 = \infty, \quad \mathcal{S}_3^3 = \infty \\ \mathcal{S}_4^3 &= \frac{6}{p^2}, \quad \mathcal{S}_5^3 = \frac{3}{4p^2} = \frac{\mathcal{S}_3^2}{p}, \quad \mathcal{S}_6^3 = \infty \\ \mathcal{S}_7^3 &= \frac{2}{p^2} = 2\frac{\mathcal{S}_2^2}{p}, \quad \mathcal{S}_8^3 = \frac{27}{32p^2}, \quad \mathcal{S}_9^3 = \infty \\ \mathcal{S}_{10}^3 &= \frac{9}{32p^2} \end{aligned}$$

Third order coefficients

compare with those in

- A. Muronga, *Relaxation and Coupling Coefficients in Third Order Relativistic Fluid Dynamics* , Acta Phys. Polon. Supp. 7 (2014) 197
- Teboho Moloi and AM *Thermodynamic coefficients in third-order relativistic dissipative fluid dynamics*, in preparation

Applications

- Chattopadhyay, C., Jaiswal, A., & Heinz, U. (2018). *Higher-order and non-linear effects in relativistic hydrodynamics.* Physical Review C, 97(3), 034910.
- Diles, S.M., Miranda, A.S., Mamani, L.A.H. et al. *Third-order relativistic fluid dynamics at finite density in a general hydrodynamic frame.* Eur. Phys. J. C 84, 516 (2024).

After simplification and keeping all the terms, the final equations for shear pressure for third order viscous and massless fluids is found to be

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{1}{2}\frac{\pi}{\tau} + \frac{3}{10}\frac{\varepsilon}{\tau} + \frac{5}{8}\frac{\pi}{\varepsilon}\dot{\varepsilon} - \frac{3}{2}\frac{\pi^2}{\varepsilon\tau} + \frac{27}{8}\frac{\pi^2}{\varepsilon^2}\dot{\varepsilon} - \frac{12}{5}\frac{\pi}{\varepsilon}\dot{\pi}. \quad (29)$$

Also in $(1+1)$ dimensional Bjorken flow, the energy and number density equations calculated from Eqs. (8) and (9) are similarly given by

$$\dot{\varepsilon} = -\frac{\varepsilon + P}{\tau} + \frac{\pi}{\tau}, \quad \dot{n} = -\frac{n}{\tau}. \quad (30)$$

The shear differential equation shown in Eq. (28) has an order by order implication on the final output or calculated energy and entropy densities. The effects due to the inclusion of various orders on the solutions of dissipative equations will be discussed in the next section.

THIRD ORDER VISCOUS HYDRODYNAMICS … PHYSICAL REVIEW C **102**, 034902 (2020)

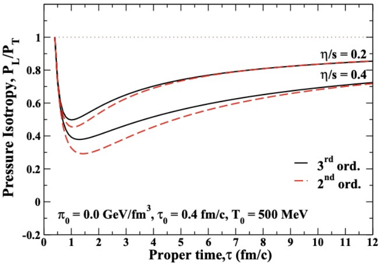


FIG. 1. Pressure isotropy of relativistic fluid using second and third order shear equations with $\pi_0 = 0$.

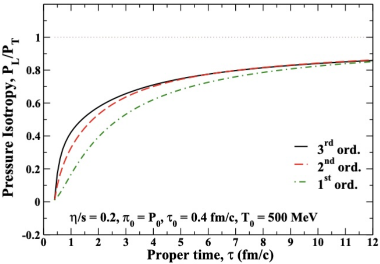


FIG. 2. Pressure isotropy of relativistic fluid using second and third order shear equations with $\pi_0 = P_0$.

Pressure isotropy evolution

MOHAMMED YOUNUS AND AZWINNDINI MURONGA PHYSICAL REVIEW C **102**, 034902 (2020)

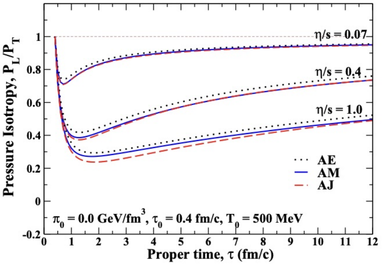


FIG. 3. Comparison of models on time evolution of pressure isotropization for different η/s values. Muronga *et al.* (AM) denotes inclusion of $\partial_i S_3^2$ and $\partial_i S_5^3$ terms in the third order equation.

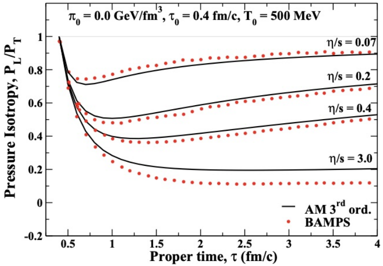


FIG. 5. Comparison pressure isotropy ration from AM third order theory with BAMPS transport calculation for different η/s.

Comparison with other model calculations

Simple scaling solution

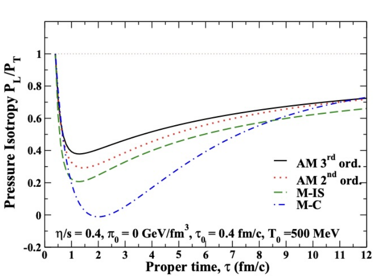


FIG. 7. Time evolution of pressure isotropy for various terms in shear pressure differential equation.

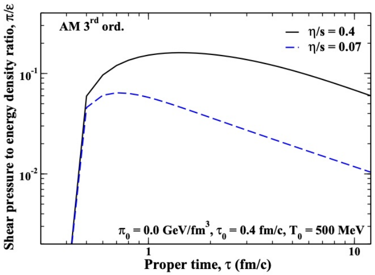


FIG. 8. Time evolution of shear pressure to energy density ratio for various $\frac{\eta}{s}$ values at $T_0 = 500\,\mathrm{MeV}$ and $\tau_0 = 0.4\,\mathrm{fm/c}$.

Pressure isotropy by terms or order

THIRD ORDER VISCOUS HYDRODYNAMICS … PHYSICAL REVIEW C **102**, 034902 (2020)

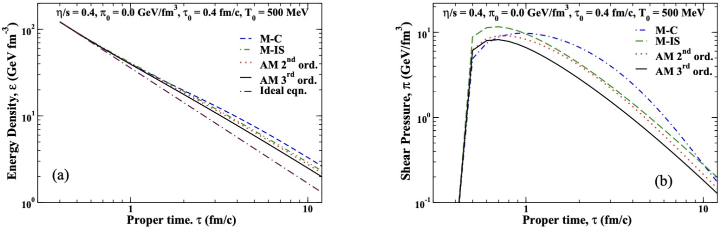
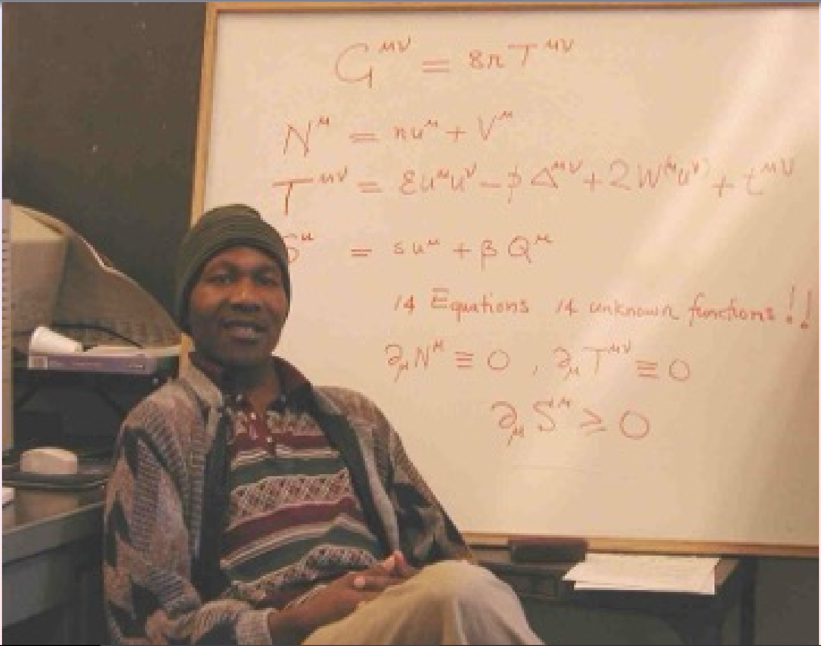


FIG. 6. (a) Energy density solution for Maxwell-Cattaneo–like equations, Muller-Israel-Stewart theory, and Muronga *et al.* second and third order equations. (b) Solutions for shear pressure for the models mentioned.

Energy density and Shear evolution

In closing - from cold Minnesota days. AI/ML might help!



Conclusions

- In a one-component fluid, there are three main mechanisms of entropy production: one associated with dynamic pressure, one due to heat flux, and one related to shear stress.
- For non-negative entropy production, the coefficients corresponding to bulk viscosity, shear viscosity, and heat flux must satisfy specific inequality relations.
- Third-order relativistic fluid dynamics introduces couplings and relaxation times not present in second-order theories.
- The numerous identities encountered in deriving the equations suggest the possibility of constraining lower-order known functions, such as the equation of state and transport, relaxation, and coupling coefficients.
- The equations derived here remain consistent, whether approached through divergence theory or kinetic theory.
- The relaxation and coupling coefficients are not entirely new parameters but can be determined from the equation of state.