On the construction of hyperboloidal initial data without logarithmic singularities I.

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“hyperboloidal initial data without logarithmic singularities”:

Gravitational field in the radiation regime: the expected behavior

- In the early 60’s, Bondi, Sachs, and Penrose proposed a set of boundary conditions that are appropriate for gravitational fields in the radiation regime.
- A somewhat simplified way to introduce their conditions **is to assume** the existence of “**asymptotically quasi-Minkowskian coordinates**” $(x^\nu) = (t, x, y, z)$ in which

$$g_{\mu\nu} - \eta_{\mu\nu} = \frac{f^{(1)}_{\mu\nu}(t-r,\theta,\phi)}{r} + \frac{f^{(2)}_{\mu\nu}(t-r,\theta,\phi)}{r^2} + \dots$$

  where $\eta_{\mu\nu}$ is the Minkowski metric $diag(-1,1,1,1)$, while (r,θ,ϕ) stand for the standard spherical coordinates on $\mathbb{R}^3$.
- The expansion above has to hold at, say, fixed value of $t-r$, while $r \to \infty$.
- As we will see later, there are results that support these expectations.
- In the mid-1990s, it was discovered that the above formula **may not give** the generic asymptotic behavior for radiative vacuum solutions.

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$$g_{\mu\nu} - \eta_{\mu\nu} = \omega\, f^{(1)}_{\mu\nu}(u,\theta,\phi) + \omega^2\, f^{(2)}_{\mu\nu}(u,\theta,\phi) + \dots$$

where the replacements $\omega = r^{-1}$ and $u = t - r$ were used,

- Using the l'Hopital rule:

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- But

$$\partial_\omega\big(\omega \log \omega\big) = \log \omega + 1$$

which is unbounded in the $\omega \to 0$ limit!

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- **Spacetime:** (M, g_{ab}) smooth manifold M ... with a smooth metric g_{ab} ...
- **Einstein’s equations:** $E_{ab} = G_{ab} - \mathscr{G}_{ab} = 0$ $\nabla^a \mathscr{G}_{ab} = 0$
  - In a more conventional setup: $[R_{ab} - \frac{1}{2} g_{ab} R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}$ where the energy-momentum tensor is T_{ab} and Λ is the cosmological constant: $\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}$ (!) matter fields satisfying their field equations...
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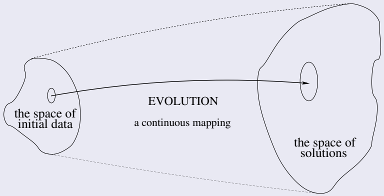

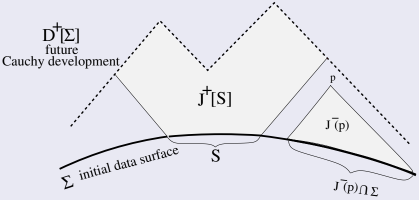

“hyperboloidal initial data without logarithmic singularities”:

- Since there is **no fixed background in GR**, the topology of M is not necessarily $\mathbb{R}^4$
  Cauchy problem: M is constructed together with the metric.

The constraints are projections: $n^a n^b E_{ab} = 0$ & $\pi_i{}^a n^b E_{ab} = 0$

$(\mathscr{G}_{ab} = 0)$ $\quad {}^{(3)}R + \left(K_{ij}h^{ij}\right)^2 - K_{ij}K^{ij} = 0 \quad \& \quad D^i\left[K_{ij} - h_{ij}\left(K_{ef}h^{ef}\right)\right] = 0 \quad D_i \ldots$

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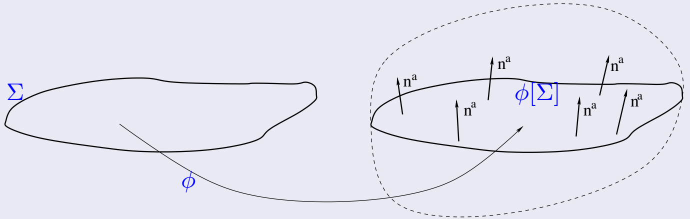

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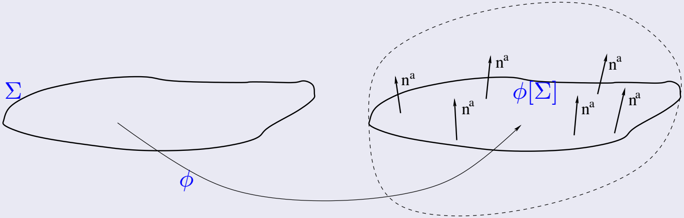

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- Could we work cleverly with the boundary at infinity?
- In 1963, Penrose introduced such **a geometric treatment** of generic, isolated, self-gravitating systems that replaces the $r \to \infty$ limit with an $\omega \to 0$ limit.
- To understand this, let us first look at Schwarzschild spacetime as a simple example.

$$ds^2 = -\left(1 - \tfrac{2\,M}{r}\right) dt^2 + \left(1 - \tfrac{2\,M}{r}\right)^{-1} dr^2 + r^2\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

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$$d\widetilde{s}^2 = \Omega^2 ds^2 = -\left(w^2 - 2Mw^3\right) du^2 + 2\,dudw + \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

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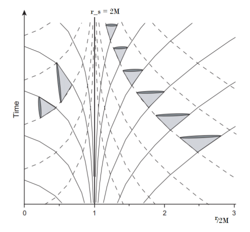

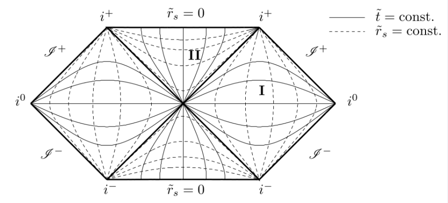

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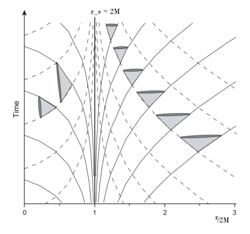

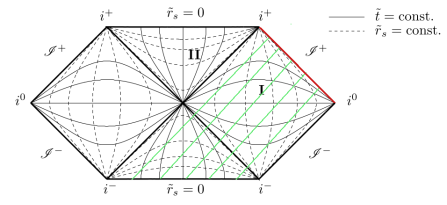

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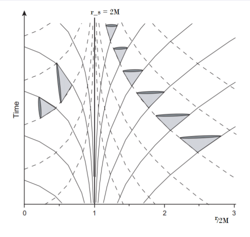

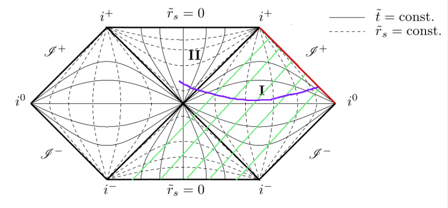

Asymptotically simple spacetimes & conformal compactification

- Consider a **smooth** spacetime (M, g) representing an isolated self-gravitating system.

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How large is the space of asymptotically simple spacetimes?

- Are the applied conditions compatible with the asymptotic behavior of a "sufficiently large" class of physically realistic spacetimes?
- One approach is to use the **hyperboloidal initial value problem**, where the **initial data** are prescribed **on a hypersurface** Σ **with boundary** $\partial\Sigma$, which is conceived as a hypersurface in an asymptotically simple spacetime. This hypersurface intersects future null infinity $\mathscr{I}^+$ in $\partial\Sigma$, and is spacelike everywhere.
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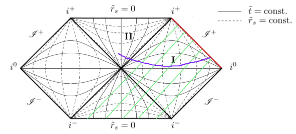

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$$ h_{ij} = \phi^4\,\widetilde{h}_{ij} \quad \& \quad K_{ij} - \tfrac{1}{3}\,h_{ij}K = \phi^{-2}\,\widetilde{K}_{ij} \quad \& \quad \widetilde{K}_{ij} = \widetilde{K}^{[L]}_{ij} + \widetilde{K}^{[TT]}_{ij} $$

- Andersson and Chruściel proved that **even if the free data is smooth**,
  - (1) **in general**, the constrained fields have poly-logarithmic expansions in $\omega \sim r^{-1}$, where r denotes the "distance" from the isolated system

$$ \mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1\omega + \mathcal{C}_2\omega^2 + \ldots \qquad \mathcal{C} = \mathcal{C}_0 + \textstyle\sum_{i=1}^{\infty}\omega^i\left[\mathcal{C}_i + \sum_{j=1}^{\mathcal{N}_j}\mathcal{C}^{[log]}_{i,j}\log^j\omega\right] $$

  - (2) **in non-generic cases**, the initial data can be smooth (i.e., free of log-terms) if **certain tensorial expressions** derived from the free data **vanish at** $\partial\Sigma$.

“hyperboloidal initial data with logarithmic singularities”

Solving the constraints:

- For those interested in **solving the conformal field equations**, the first challenge is acquiring as large variety of hyperboloidal initial data sets as possible and also **characterizing the space of initial data**: in the vacuum case for (h_{ij}, K_{ij}) on Σ

$$ {}^{(3)}R + \left(K_{ij}h^{ij}\right)^2 - K_{ij}K^{ij} = 0 \quad \& \quad D^i\left[K_{ij} - h_{ij}\left(K_{ef}h^{ef}\right)\right] = 0 $$

- It is an underdetermined system, 4 equations for the 12 variables: (h_{ij}, K_{ij})

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Facing the problems:

- These conclusions were disappointing because they implied that the initial data constructed by the conformal method is **generally not regular enough** for use in Friedrich's existence theorems.
- **Andersson and Chruściel** also showed that **if the initial data involves logarithmic terms**, then **the evolving metric will also contain** log-terms.
- $\Longrightarrow$ In general, we cannot decompose the physical metric into a sufficiently smooth non-physical metric and a sufficiently smooth conformal factor.
- Given the definition of asymptotically simple spacetimes, **it is natural to ask** whether Einstein's equations can be used to verify the smoothness assumptions.
- The **question is not** whether C^∞ should be replaced by C^k for sufficiently large k in the context of the physical spacetime. Rather, **the question is** whether solutions to the field equations admit conformal extensions of class C^k, where k can be chosen large enough to make the concept of asymptotically simple spacetimes meaningful.

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- Friedrich emphasized that "before we can arrive at a conclusion we need to answer the following questions:
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“hyperboloidal initial data without logarithmic singularities”

- At the stage of this difficulty **Beyer and Ritchie** [CQG,39,145012,(2022)] came up with an interesting result:
  - Assume that there exist smooth global solutions to the **parabolic-hyperbolic form of the constraints** on a ”hyperboloidal initial data surface” Σ. If these solutions **extend regularly up to some finite order** to $\partial\Sigma$, **then they extend smoothly** to $\partial\Sigma$.
  - *“The main thing we establish is that any solution that satisfies these a priori regularity assumptions extends smoothly to $\partial\Sigma$. This is important because it means that such solutions are free of all log-terms in their expansions.”*
  - Beyer and Ritchie introduced an **impressive Fuchsian-type argument**, but
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- $(h_{ab}, K_{ab}) \longleftrightarrow$ spin-weighted variables: $(\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$

“hyperboloidal initial data without logarithmic singularities”

The parabolic-hyperbolic form of the constraints & the spin-weighted variables

- For (h_{ij}, K_{ij}) on Σ
$$ {}^{(3)}R + (K^j{}_j)^2 - K_{ij}K^{ij} = 0 $$
$$ D_j K^j{}_i - D_i K^j{}_j = 0 $$
- ASSUME: Σ can be foliated by a one-parameter family of topological two-spheres
- $\rho : \Sigma \to \mathbb{R}$: $\partial_i \rho$ a.n. vanishes, $\widehat{n}_i \sim \partial_i \rho$, $\widehat{\gamma}_i{}^j = \delta_i{}^j - \widehat{n}_i \widehat{n}^j$, $h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j$
- Choose a flow ρ^a such that $\rho^a \partial_a \rho = 1$ and such that its integral curves intersect each of the $\rho = const$ level surfaces precisely once: $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$
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$$\widehat{\mathbf{N}} = \widehat{\mathbf{N}}_1 \omega + \mathscr{O}(\omega^2) \qquad \mathbf{K} - 2\,\boldsymbol{\kappa} = \mathscr{O}(\omega) \qquad \mathbf{k} = \mathscr{O}(1)$$

$$\mathbf{a} = \omega^{-2} + \mathscr{O}(\omega^{-1}) \qquad \mathbf{b} = \mathscr{O}(\omega^{-1}) \qquad \mathbf{N} = \mathscr{O}(\omega) \qquad \overset{\circ}{\mathbf{K}}_{qq} = \mathscr{O}(\omega^{-1})$$

The main strategy we used in our investigations:

The free data is assumed to be smooth on $\Sigma \cup \partial\Sigma$: $C^\infty\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$

$$\mathbf{N} = \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
$$\mathbf{a} = \omega^{-2} + \mathbf{a}_{(-1)}\,\omega^{-1} + \mathbf{a}_0 + \mathbf{a}_1\,\omega + \mathbf{a}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
$$\mathbf{b} = \mathbf{b}_{(-1)}\,\omega^{-1} + \mathbf{b}_0 + \mathbf{b}_1\,\omega + \mathbf{b}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
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$$\mathring{\mathbf{K}}_{qq} = \mathring{\mathbf{K}}_{qq\,(-1)}\,\omega^{-1} + \mathring{\mathbf{K}}_{qq\,0} + \mathring{\mathbf{K}}_{qq\,1}\,\omega + \mathring{\mathbf{K}}_{qq\,2}\,\omega^2 + \mathscr{O}(\omega^3)$$

Use the most general poly-logarithmic form of the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$:

$$\widehat{\mathbf{N}} = \sum_{i=1}^{\infty}\omega^i\,\Big[\widehat{\mathbf{N}}_i + \sum_{j=1}^{\mathcal{N}_j}\widehat{\mathbf{N}}_{i,j}^{[log]}\log^j\omega\Big], \qquad \mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{\infty}\omega^i\,\Big[\mathbf{K}_i + \sum_{j=1}^{\mathcal{N}_j}\mathbf{K}_{i,j}^{[log]}\log^j\omega\Big]$$

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We determined the **restrictions on the coefficients**, used in the above asymptotic expansions, that follow from the assumptions that the system **admits well-defined Bondi mass and angular momentum**, and that the **parabolic-hyperbolic form** of the constraint equations holds.

The main strategy we used in our investigations:

The free data is assumed to be smooth on $\Sigma \cup \partial\Sigma$: $C^\infty\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$

$$\mathbf{N} = \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathscr{O}(\omega^3)$$
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We determined the **restrictions on the coefficients**, used in the above asymptotic expansions, that follow from the assumptions that the system **admits well-defined Bondi mass and angular momentum**, and that the **parabolic-hyperbolic form** of the constraint equations holds.

Our first main result: Theorem I.

- Choose generic **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ on Σ that **satisfies the falloff conditions** relevant for asymptotically hyperboloidal data.
- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth solutions** of the parabolic-hyperbolic form of the constraints on Σ.
- $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also assumed to possesses the most general poly-logarithmic expansion near $\partial\Sigma$ as indicated above.
- Then the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi mass and angular momentum** **if and only if** all coefficients of **the logarithmic terms vanish up to order four and three** for $\widehat{\mathbf{N}}, \mathbf{K}$ and $\mathbf{k}$, respectively, and, **in addition,**

$$\overset{\circ}{\mathbf{K}}_{qq\,(-1)} = 0\,, \quad \mathbf{b}_{(-1)} = 0\,, \quad \boldsymbol{\kappa}_1 = 0\,.$$

The finiteness of the Bondi mass (energy):

The Bondi mass can be given as the $\rho \to \infty$ limit of the Hawking mass

$$m_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_\rho} \Theta^{(+)}\Theta^{(-)}\,\widehat{\boldsymbol{\epsilon}} \right) \;\&\; \Theta^{(\pm)} = \mathbf{K} \pm \overset{\star}{\mathbf{K}}\widehat{\mathbf{N}}^{-1} \;\&\; \mathcal{A} = \int_{\mathscr{S}_\rho} \widehat{\boldsymbol{\epsilon}} \sim \rho^2$$

The finiteness of the Bondi angular momentum: arXiv: 2401.14251

The Bondi angular momentum cannot be finite, and thus well-defined, unless for all $j = 1, 2, \ldots, \mathcal{N}_j$

$$\mathbf{k}^{[log]}_{1,j} = \mathbf{k}^{[log]}_{2,j} = 0 \qquad \left(J[\phi] = -(8\,\pi)^{-1} \int_{\mathscr{S}_\rho} \phi^a \mathbf{k}_a\,\widehat{\boldsymbol{\epsilon}} \right)$$

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$$\widehat{\mathbf{N}}_1 = 2\,\mathbf{K}_0^{-1} \quad \widehat{\mathbf{N}}_2 = -\left[\,\mathbf{a}_{(-1)}\,\mathbf{K}_0 + 2\,\mathbf{K}_1\,\right]\mathbf{K}_0^{-2}$$

$$\widehat{\mathbf{N}}_3 = \left(2\,(\mathbf{K}_1^2 - 2) + \mathbf{K}_0\,[\,\mathbf{a}_{(-1)}\,\mathbf{K}_1 - 2\,\mathbf{K}_2] - \mathbf{K}_0^2\,(\,2\,\mathbf{a}_0 - \mathbf{a}_{(-1)}^2 - \mathbf{b}_{(-1)}\,\overline{\mathbf{b}_{(-1)}} + \tfrac{1}{2}\,[\,\eth\overline{\mathbf{N}}_1 + \overline{\eth}\mathbf{N}_1]) \right) \mathbf{K}_0^{-3}$$

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and also for all $j = 1, 2, \ldots, \mathcal{N}_j$

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$$\mathbf{K}_{2,i}^{[log]} = \mathbf{K}_0 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\,\widehat{\mathbf{N}}_{4,i}^{[log]} \right) \cdot \left[\,\mathbf{a}_{(-1)}\,\mathbf{K}_0 + 4\,\mathbf{K}_1\,\right]^{-1}$$

$$\widehat{\mathbf{N}}_{3,i}^{[log]} = -2 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\,\widehat{\mathbf{N}}_{4,i}^{[log]} \right) \cdot \left(\mathbf{K}_0 \left[\,\mathbf{a}_{(-1)}\,\mathbf{K}_0 + 4\,\mathbf{K}_1\,\right] \right)^{-1}$$

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The Bondi mass can be given as the $\rho \to \infty$ limit of the Hawking mass

$$m_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_\rho} \Theta^{(+)}\Theta^{(-)}\, \widehat{\boldsymbol\epsilon}\right) \ \& \ \Theta^{(\pm)} = \mathbf{K} \pm \overset{\star}{\mathbf{K}} \widehat{\mathbf{N}}^{-1} \ \& \ \mathcal{A} = \int_{\mathscr{S}_\rho} \widehat{\boldsymbol\epsilon} \sim \rho^2$$

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To obtain the desired restrictions, we substitute the updated form of the asymptotic expansions into the parabolic-hyperbolic system and sort the terms with respect to powers of ρ^{-1} and also of $\log\,\rho$.

Our second main result: Theorem II.

- Choose generic **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$ on Σ that **satisfies the falloff conditions** relevant for asymptotically hyperboloidal data with κ_0 being a strictly positive smooth on $\partial\widetilde{\Sigma}$.
- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth** [i.e., of class $C^\infty((0,\omega_0), C^\infty(\mathbb{S}^2))$], solutions on Σ such that $\widehat{\mathbf{N}} > 0$ there.
- Then, the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also of class $C^\infty([0,\omega_0), C^\infty(\mathbb{S}^2))$ on the whole of $\widetilde{\Sigma} = \Sigma \cup \partial\Sigma$, i.e., **no logarithmic singularities occur**, if and only if the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi mass and angular momentum**, and, **in addition**,

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To obtain the desired restrictions, we substitute the updated form of the asymptotic expansions into the parabolic-hyperbolic system and sort the terms with respect to powers of ρ^{-1} and also of $\log\,\rho$. $\Longrightarrow$ $\widehat{\mathbf{N}}^{[log]}_{1,j} = \widehat{\mathbf{N}}^{[log]}_{2,j} = \widehat{\mathbf{N}}^{[log]}_{3,j} = \widehat{\mathbf{N}}^{[log]}_{4,j} = 0$

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Our second main result: Theorem II.

- Choose generic **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$ on Σ that **satisfies the falloff conditions** relevant for asymptotically hyperboloidal data with $\boldsymbol{\kappa}_0$ being a strictly positive smooth on $\partial\widetilde{\Sigma}$.
- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth** [i.e., of class $C^\infty\big((0,\omega_0), C^\infty(\mathbb{S}^2)\big)$], solutions on Σ such that $\widehat{\mathbf{N}} > 0$ there.
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and also the following two relations

$$\mathbf{a}_{(-1)} = const \quad \& \quad \mathring{\mathbf{K}}_{qq0} = \tfrac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \eth\overline{\eth}\,\boldsymbol{\kappa}_0^{-2}$$

hold on $\partial\widetilde{\Sigma}$.

The key steps in the Fuchsian argument I.

- Using the smoothness properties found in our first theorem, replace the constrained variables in the parabolic-hyperbolic system with their respective Taylor series:
  - $\widehat{\mathbf{N}} \longrightarrow \widehat{\mathbf{N}}_0 + \widehat{\mathbf{N}}_1\,\omega + \widehat{\mathbf{N}}_2\,\omega^2 + \widehat{\mathbf{N}}_3\,\omega^3 + \widehat{\mathbf{N}}_4\,\omega^4 + \omega^4 w_{\widehat{\mathbf{N}}}(\omega)$
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  where $\underline{H}$ is a (lengthy, but explicitly known) vector-valued function that is smooth in each of its arguments, and regularly extends to $\omega = 0$.

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$$\partial_\omega \underline{W}(\omega, p) = \frac{1}{\omega} diag(0, -3, -1)\, \underline{W}(\omega, p) + \underline{H}(\omega, p; \widehat{\mathbf{N}}_4(p), \mathbf{k}_2(p), \mathbf{K}_1(p), \underline{W}(\omega, p), \eth\underline{W}, \bar{\eth}\underline{W}, \eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

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- Combining these results with those of the corresponding hyperboloidal initial value problem [Friedrich,Frauendiener,Kroon,...] we can conclude that the Cauchy developments of the corresponding asymptotically hyperboloidal initial data specifications must admit smooth conformal boundary as assumed in the original definition of asymptotically simple spacetimes by Penrose.
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Thanks for your attention