On the construction of hyperboloidal initial data without logarithmic singularities

STAR WARS
EPISODE II
ATTACK OF THE SINGULARITIES

Károly Zoltán Csukás

csukas.karoly@wigner.hu

HUN-REN Wigner Research Center for Physics

Gen Relativ Gravit **57**, 96 (2025), arXiv:2503.11804

joint work with István Rácz

Theoretical Department

Isolated systems

Physical intuition

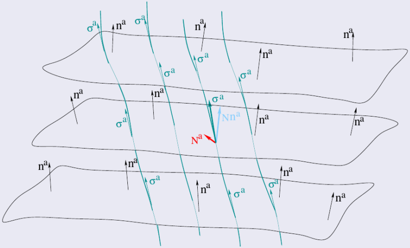
- interesting dynamical system
- neglect effects from outside (except possibily incoming radiation)
- far away from the source we should approach Minkowski
- observer is infinitely far away $\rightarrow$ observables are defined at "infinity"

Asymptotically simple spacetime (Penrose 1963)

- smooth physical spacetime (M, g_{ab})
- smooth unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ such that
  - $\tilde{M}$ has boundary $\mathscr{I}$
  - M can be identified with interior of $\tilde{M}$ in a way that
  - $\tilde{g}_{ab} = \Omega^2 g_{ab}$ in the interior
  - Ω is smooth boundary defining ($\Omega|_{\mathscr{I}} = 0$ and $\mathrm{d}\Omega|_{\mathscr{I}} \neq 0$ with $\Omega > 0$ in the interior)

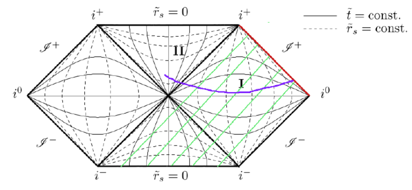
Initial value problem

Initial value problem



- (Σ, h_{ab}, K_{ab}) as a spacelike hypersurface within a to-be-constructed spacetime
- evolution generates both the spacetime M and the metric g_{ab}

Hyperboloidal initial data



- $\tilde{\Sigma}$ with boundary $\partial\tilde{\Sigma}$
- $\omega > 0$ in the interior, $\omega|_{\partial\tilde{\Sigma}} = 0$, $\mathrm{d}\omega|_{\partial\tilde{\Sigma}} \neq 0$
- $\tilde{h}_{ab} = \omega^2 h_{ab}$ extends regularly
- trace of K_{ab} bounded away from 0 near $\partial\tilde{\Sigma}$
- trace-free part of K_{ab} extends regularly

Constraint equations

Projections of Einstein's equation

$$ {}^{(3)}R + K^2 - K_{ab}K^{ab} = 0, \qquad D^b\left[K_{ab} - h_{ab}K\right] = 0 $$

with $K = K_{ab}h^{ab}$ and $D_a h_{bc} = 0$.
Underdetermined system: 4 equations for 12 variables $\rightarrow$
4 constrained variables, 8 freely specifiable

Conformal method (Lichnerowicz, York)

$$ (h_{ab}, K_{ab}) \quad \longleftrightarrow \quad (\phi, \widetilde{h}_{ab}; K, X_a, \widetilde{K}^{[TT]}_{ab}) $$

$$ h_{ab} = \phi^4\,\widetilde{h}_{ab}, \qquad K_{ab} - \tfrac{1}{3}\,h_{ab}K\; = \phi^{-2}\,\widetilde{K}_{ab}, \qquad \widetilde{K}_{ab} = \widetilde{K}^{[L]}_{ab} + \widetilde{K}^{[TT]}_{ab} $$

Elliptic system, boundary conditions.

Evolutionary method (Rácz 2015)

Introduction in a few minutes...
Evolutionary systems, initial data.

Logarithmic singularities

- Andersson and Chruściel 1993, 1994, 1996 (conformal framework): in general the solutions have poly-logarithmic expansion around $\partial\tilde{\Sigma}$ EVEN IF THE FREE DATA SMOOTHLY EXTENDS THERE

$$\mathcal{C} = \mathcal{C}_0 + \sum_{i=1}^{\infty} \omega^i \left[\mathcal{C}_i + \sum_{j=1}^{\mathcal{N}_j} \mathcal{C}_{i,j}^{[log]} \log^j \omega \right]$$

- $\lim_{\omega\to 0} \omega^a \log^b \omega = 0$ for $a > 0$
- even if the extension is continuous its derivative might not be
- is solution with logs relevant for physics?
- are there relevant systems that need logs?
- what are the relevant systems?

Open questions

- Andersson and Chruściel 1993, 1994, 1996
  - constant mean curvature ($K = const$), conformal framework
  - constraints on the limits of some of the free data ensuring smooth extension
  - constraints on the limits of some of the free data ensuring that the development has smooth boundary
- Beyer and Ritchie 2022
  - asymptotically constant mean curvature ($K|_{\mathscr{I}} = 0$), parabolic-hyperbolic formulation
  - if we ensure finite differentiability on $\mathscr{I}$ (and some constraints on free data), then the solution is actually smooth
- Csukás and Rácz 2025
  - parabolic-hyperbolic formulation
  - we relax the conditions on the free data that B&R had
  - we show that if we require finite global energy and angular momentum, then the solutions fulfill the requirement on the finite differentiability

Spin-weighted variables

For $(h_{ab}; K_{ab})$ in Σ

$$
\begin{aligned}
{}^{(3)}R + (K^a{}_a)^2 + K_{ab}K^{ab} &= 0 \\
D_b K^b{}_a - D_a K^b{}_b &= 0
\end{aligned}
$$

ASSUME: Σ can be foliated with $\mathcal{S}_\rho$ 1-parameter family of 2-surfaces

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ASSUME: Σ can be foliated with $\mathcal{S}_\rho$ 1-parameter family of 2-surfaces

- $\widehat{n}_a$, $\widehat{\gamma}_{ab} = h_{ab} - \widehat{n}_a \widehat{n}_b$

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For $(h_{ab}; K_{ab})$ in Σ

$$^{(3)}R + (K^a{}_a)^2 + K_{ab}K^{ab} = 0$$
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- $\widehat{n}_a$, $\widehat{\gamma}_{ab} = h_{ab} - \widehat{n}_a \widehat{n}_b$
- ρ^a st. $\rho^a \partial_a \rho = 1$, and $\widehat{N}$, $\widehat{N}^a$

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- ρ^a st. $\rho^a \partial_a \rho = 1$, and $\widehat{N}$, $\widehat{N}^a$
- introduce coordinates (ϑ, φ) and complex null dyad q^a (wrt. unit sphere metric q_{ab}) on some $\mathcal{S}_\rho$ and Lie-drag them along ρ^a

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For $(h_{ab}; K_{ab})$ in Σ

$$ {}^{(3)}R + (K^a{}_a)^2 + K_{ab}K^{ab} = 0 $$
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$$
\begin{aligned}
\widehat{N} &= \rho^a \widehat{n}_a & \boldsymbol{\kappa} &= \widehat{n}^a \widehat{n}^b K_{ab} \\
\mathbf{N} &= q^a \widehat{\gamma}_{ab} \rho^b & \mathbf{k} &= q^a \widehat{n}^b \widehat{\gamma}_a{}^c K_{bc} \\
\mathbf{a} &= \frac{1}{2} q^a \overline{q}^b \widehat{\gamma}_{ab} & \mathbf{K} &= \widehat{\gamma}^{ab} K_{ab} \\
\mathbf{b} &= \frac{1}{2} q^a q^b \widehat{\gamma}_{ab} & \mathring{\mathbf{K}}_{qq} &= q^a q^b \left(\widehat{\gamma}_a{}^c \widehat{\gamma}_b{}^d K_{cd} - \frac{1}{2} \widehat{\gamma}_{ab} \widehat{\gamma}^{cd} K_{cd} \right)
\end{aligned}
$$

Spin-weighted variables

For $(h_{ab}; K_{ab})$ in Σ

$$ {}^{(3)}R + (K^a{}_a)^2 + K_{ab}K^{ab} = 0 $$
$$ D_b K^b{}_a - D_a K^b{}_b = 0 $$

ASSUME: Σ can be foliated with $\mathcal{S}_\rho$ 1-parameter family of 2-surfaces

- $\widehat{n}_a$, $\widehat{\gamma}_{ab} = h_{ab} - \widehat{n}_a \widehat{n}_b$
- ρ^a st. $\rho^a \partial_a \rho = 1$, and $\widehat{N}$, $\widehat{N}^a$
- introduce coordinates (ϑ, φ) and complex null dyad q^a (wrt. unit sphere metric q_{ab}) on some $\mathcal{S}_\rho$ and Lie-drag them along ρ^a

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\begin{aligned}
\widehat{N} &= \rho^a \widehat{n}_a & \boldsymbol{\kappa} &= \widehat{n}^a \widehat{n}^b K_{ab} \\
\mathbf{N} &= q^a \widehat{\gamma}_{ab} \rho^b & \mathbf{k} &= q^a \widehat{n}^b \widehat{\gamma}_a{}^c K_{bc} \\
\mathbf{a} &= \frac{1}{2} q^a \overline{q}^b \widehat{\gamma}_{ab} & \mathbf{K} &= \widehat{\gamma}^{ab} K_{ab} \\
\mathbf{b} &= \frac{1}{2} q^a q^b \widehat{\gamma}_{ab} & \overset{\circ}{\mathbf{K}}_{qq} &= q^a q^b \left(\widehat{\gamma}_a{}^c \widehat{\gamma}_b{}^d K_{cd} - \frac{1}{2} \widehat{\gamma}_{ab} \widehat{\gamma}^{cd} K_{cd} \right)
\end{aligned}
$$

$$ (h_{ab}; K_{ab}) \to (\widehat{N}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \overset{\circ}{\mathbf{K}}_{qq}) $$

Parabolic-hyperbolic form of constraints

$$(h_{ab}; K_{ab}) \quad \to \quad (\widehat{N}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \overset{\circ}{\mathbf{K}}_{qq})$$

- we can solve for $(\widehat{N}, \mathbf{k}, \mathbf{K})$, free to specify $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ over Σ
  - parabolic PDE for $\widehat{N}$
  - symmetric hyperbolic system for $(\mathbf{k}, \mathbf{K})$
- provide initial data on $\mathcal{S}_0$, then integrate toward $\mathscr{I}^+$
- reduced to scalar equations
- adapted to geometry, not coordinates
- no direct control over the asymptotics of solution

Asymptotically hyperboloidal data

Taking the geometric picture and using ω as a coordinate near $\mathscr{I}$

Falloff conditions

$$
\begin{aligned}
\widehat{N} &= \widehat{N}_1\omega + \mathscr{O}(\omega^2), & \mathbf{N} &= \mathscr{O}(\omega),\\
\mathbf{a} &= \omega^{-2} + \mathscr{O}(\omega^{-1}), & \mathbf{b} &= \mathscr{O}(\omega^{-1}),\\
\mathbf{K} &= \mathbf{K}_0 + \mathscr{O}(\omega), & \mathbf{K} - 2\boldsymbol{\kappa} &= \mathscr{O}(\omega),\\
\mathbf{k} &= \mathscr{O}(1), & \mathring{\mathbf{K}}_{qq} &= \mathscr{O}(\omega^{-1}).
\end{aligned}
$$

If all coefficients satisfy these conditions, and smoothly extend to $\mathscr{I}$, the solutions might still possess logarithmic terms in their asymptotic expansion.

Asymptotic expansion of variables

Insert asymptotic expansion into the expressions and group different powers of ω and $\log\omega$

Free data extends smoothly

$$\begin{aligned}
\mathbf{N} &= \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{a} &= \omega^{-2} + \mathbf{a}_{(-1)}\,\omega^{-1} + \mathbf{a}_0 + \mathbf{a}_1\,\omega + \mathbf{a}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{b} &= \mathbf{b}_{(-1)}\,\omega^{-1} + \mathbf{b}_0 + \mathbf{b}_1\,\omega + \mathbf{b}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\boldsymbol{\kappa} &= \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1\,\omega + \boldsymbol{\kappa}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathring{\mathbf{K}}_{qq} &= \mathring{\mathbf{K}}_{qq\,(-1)}\,\omega^{-1} + \mathring{\mathbf{K}}_{qq\,0} + \mathring{\mathbf{K}}_{qq\,1}\,\omega + \mathring{\mathbf{K}}_{qq\,2}\,\omega^2 + \mathscr{O}(\omega^3)
\end{aligned}$$

Constrained data worst possible case

$$\widehat{N} = \sum_{i=1}^{\infty}\omega^i\left[\widehat{N}_i + \sum_{j=1}^{\mathcal{N}_j}\widehat{N}^{[log]}_{i,j}\log^j\omega\right], \qquad \mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{\infty}\omega^i\left[\mathbf{K}_i + \sum_{j=1}^{\mathcal{N}_j}\mathbf{K}^{[log]}_{i,j}\log^j\omega\right]$$

$$\mathbf{k} = \mathbf{k}_0 + \sum_{i=1}^{\infty}\omega^i\left[\mathbf{k}_i + \sum_{j=1}^{\mathcal{N}_j}\mathbf{k}^{[log]}_{i,j}\log^j\omega\right], \text{ where } \widehat{N}_1 = \boldsymbol{\kappa}_0^{-1}, \mathbf{K}_0 = 2\boldsymbol{\kappa}_0, \mathbf{k}_0 = \boldsymbol{\kappa}_0^{-1}\eth\boldsymbol{\kappa}_0$$

Theorem I

- Choose generic **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ on Σ that **satisfies the falloff conditions** relevant for asymptotically hyperboloidal data.
- Suppose that $(\widehat{N}, \mathbf{K}, \mathbf{k})$ are **smooth solutions** of the parabolic-hyperbolic form of the constraints on Σ.
- $(\widehat{N}, \mathbf{K}, \mathbf{k})$ are also assumed to possesses the most general poly-logarithmic expansion near $\partial\Sigma$ as indicated above.
- Then the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi energy and angular momentum** **if and only if** all coefficients of **the logarithmic terms vanish up to order four and three** for $\widehat{N}, \mathbf{K}$ and $\mathbf{k}$, respectively, and, **in addition,**

$$\overset{\circ}{\mathbf{K}}_{qq\,(-1)} = 0\,, \quad \mathbf{b}_{(-1)} = 0\,, \quad \boldsymbol{\kappa}_1 = 0\,.$$

Hawking energy

$$m_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_\rho} \Theta^{(+)}\Theta^{(-)} \sqrt{\mathbf{d}}\, \mathring{\boldsymbol{\epsilon}} \right)$$

$$\Theta^{(+)}\Theta^{(-)} = \mathbf{K}^2 - \overset{\star}{K}{}^2 \widehat{N}^{-2}, \quad \mathring{\boldsymbol{\epsilon}} = \sin\vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$$

- $\mathbf{K}$ and $\widehat{N}$ are involved
- $\mathcal{A} = \mathcal{O}(\omega^{-2})$, $\sqrt{\mathbf{d}} = \mathcal{O}(\omega^{-2})$ requires $\int_{\mathscr{S}_\rho} \Theta^{(+)}\Theta^{(-)} \sqrt{\mathbf{d}}\, \mathring{\boldsymbol{\epsilon}} = -16\pi + \mathcal{O}(\omega)$
- $\overset{\star}{K} = \mathcal{O}(\omega)$, but $\mathbf{K} = \mathcal{O}(1)$
- we need

$$\Theta^{(+)}\Theta^{(-)} \sqrt{\mathbf{d}} = f_{(-2)}\omega^{-2} + f_{(-1)}\omega^{-1} + f_0 + f_1\omega + \mathcal{O}(\omega^2)$$

with $f_{(-2)} = f_{(-1)} = 0$, $f_0 = -4$, and f_1 will determine the limit

Finite limit of Hawking energy

inserting ansatz into expression gives for all $j = 1, 2, \ldots, \mathcal{N}_j$

$$\begin{aligned}
\widehat{N}_{1,j}^{[log]} &= \widehat{N}_{2,j}^{[log]} = \mathbf{K}_{1,j}^{[log]} = 0 \\
\mathbf{K}_{2,i}^{[log]} &= \mathbf{K}_0 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\, \widehat{N}_{4,i}^{[log]} \right) \cdot \left[\mathbf{a}_{(-1)}\, \mathbf{K}_0 + 4\,\mathbf{K}_1 \right]^{-1} \\
\widehat{N}_{3,i}^{[log]} &= -2 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\, \widehat{N}_{4,i}^{[log]} \right) \cdot \left(\mathbf{K}_0 \left[\mathbf{a}_{(-1)}\, \mathbf{K}_0 + 4\,\mathbf{K}_1 \right] \right)^{-1}
\end{aligned}$$

and

$$\widehat{N}_1 = 2\,\mathbf{K}_0^{-1}, \quad \widehat{N}_2 = -\left[\mathbf{a}_{(-1)}\, \mathbf{K}_0 + 2\,\mathbf{K}_1 \right] \mathbf{K}_0^{-2}$$

$$\begin{aligned}
\widehat{N}_3 = \Big(& 2\,(\mathbf{K}_1^2 - 2) + \mathbf{K}_0 \left[\mathbf{a}_{(-1)} \mathbf{K}_1 - 2\,\mathbf{K}_2 \right] \\
& - \mathbf{K}_0^2 \left(2\,\mathbf{a}_0 - \mathbf{a}_{(-1)}^2 - \mathbf{b}_{(-1)} \overline{\mathbf{b}_{(-1)}} + \tfrac{1}{2} \left[\eth \overline{\mathbf{N}}_1 + \overline{\eth} \mathbf{N}_1 \right] \right) \Big) \mathbf{K}_0^{-3}
\end{aligned}$$

Angular momentum

- details of the construction in arXiv:2401.14251
- for any axial vector field ϕ^a

$$J[\phi] = -(8\,\pi)^{-1} \int_{\mathscr{S}_\rho} (\phi^a \overline{q}_a \mathbf{k} + cc.)\, \sqrt{\mathbf{d}}\, \mathring{\boldsymbol{\epsilon}}$$

- with ${\phi_{(i)}}^a$ being a basis finiteness must hold for each component
- since $\sqrt{\mathbf{d}} = \mathcal{O}(\omega^{-2})$ we need
  - $\mathbf{k}_{1,j}^{[log]} = \mathbf{k}_{2,j}^{[log]} = 0$ for all $j = 1, 2, \ldots, \mathcal{N}_j$, and
  - $\mathbf{k}_0$ and $\mathbf{k}_1$ being either 0 or total divergence (come back later), and
  - $\mathbf{k}_2$ will determine the limit

Constraints

- update ansatz and insert into constraints
  - for all $j = 1, 2, \dots, \mathcal{N}_j$

$$\widehat{N}^{[log]}_{3,j} = \widehat{N}^{[log]}_{4,j} = \mathbf{k}^{[log]}_{3,j} = \mathbf{K}^{[log]}_{2,j} = \mathbf{K}^{[log]}_{3,j} = \mathbf{K}^{[log]}_{4,j} = 0$$

$$\begin{aligned}
&\widehat{N}_0 = 0, && \widehat{N}_1 = {\boldsymbol{\kappa}_0}^{-1}, && \widehat{N}_2 = \dots, && \widehat{N}_3 = \dots, \\
&\mathbf{K}_0 = 2\boldsymbol{\kappa}_0, && \mathbf{K}_1 = \mathbf{K}_1, && \mathbf{K}_2 = \dots, && \\
&\mathbf{k}_0 = \frac{\eth\boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0}, && \mathbf{k}_1 = \dots, && \mathbf{k}_2 = \mathbf{k}_2, && \boldsymbol{\kappa}_1 = 0.
\end{aligned}$$

- for $f_0 = -4$ to hold $\mathring{\mathbf{K}}_{qq(-1)} + \mathbf{b}_{(-1)}\boldsymbol{\kappa}_0 = 0$
- $\mathbf{k}_0$ is already a gradient
- for $\mathbf{k}_1$ to be a gradient we need $\mathring{\mathbf{K}}_{qq(-1)} = 0$
  - subitem1

Theorem II

- Choose generic **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ on Σ that **satisfies the falloff conditions** relevant for asymptotically hyperboloidal data with $\boldsymbol{\kappa}_0$ being a strictly positive smooth function on $\partial\widetilde{\Sigma}$.
- Suppose that $(\widehat{N}, \mathbf{K}, \mathbf{k})$ are **smooth** [i.e., of class $C^\infty\big((0,\omega_0), C^\infty(\mathbb{S}^2)\big)$], solutions of the parabolic-hyperbolic system on Σ such that $\widehat{N} > 0$ there.
- Then, the constrained fields $(\widehat{N}, \mathbf{K}, \mathbf{k})$ are also of class $C^\infty\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ on the whole of $\widetilde{\Sigma} = \Sigma \cup \partial\Sigma$, i.e., **no logarithmic singularities occur**, **if and only if** the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi mass and angular momentum** and also the following two relations

$$\mathbf{a}_{(-1)} = const \quad \text{and} \quad \overset{\circ}{\mathbf{K}}_{qq0} = \tfrac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \eth\eth\,\boldsymbol{\kappa}_0^{-2}$$

  hold on $\partial\widetilde{\Sigma}$.

Algebraic relations

- Using the smoothness properties found in our first theorem, replace the constrained variables in the parabolic-hyperbolic system with their respective Taylor series:
  - $\widehat{N} \longrightarrow \widehat{N}_0 + \widehat{N}_1\,\omega + \widehat{N}_2\,\omega^2 + \widehat{N}_3\,\omega^3 + \widehat{N}_4\,\omega^4 + \omega^4 w_{\widehat{N}}(\omega)$
  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$

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  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$
- $w_{\widehat{N}}(\omega), w_{\mathbf{K}}(\omega), w_{\mathbf{k}}(\omega)$, are of class $C^0\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ and vanish at $\partial\widetilde{\Sigma}$, thus they can represent **higher-order log-terms** that may still occur.

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- All the "coefficients in black" can be **derived from the free data** and the coefficients $(\widehat{N}_4, \mathbf{k}_2, \mathbf{K}_1)$ which represent the **asymptotic degrees of freedom**.

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- $w_{\widehat{N}}(\omega), w_{\mathbf{K}}(\omega), w_{\mathbf{k}}(\omega)$, are of class $C^0\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ and vanish at $\partial\widetilde{\Sigma}$, thus they can represent **higher-order log-terms** that may still occur.
- All the "coefficients in black" can be **derived from the free data** and the coefficients $(\widehat{N}_4, \mathbf{k}_2, \mathbf{K}_1)$ which represent the **asymptotic degrees of freedom**.
- **If the last two algebraic conditions hold**, then the following **Fuchsian-type** (singular) **equation** holds for the vector-valued variable $\underline{W} = (w_{\widehat{N}}, w_{\mathbf{K}}, w_{\mathbf{k}})^T$, comprised of the residuals, for every $p \in \mathscr{S}^2$ and for every $0 < \omega < \omega_0$:

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\,\underline{W}(\omega,p)$$

$$+\,\underline{H}\left(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}\right) \qquad (*)$$

Algebraic relations

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  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$
- $w_{\widehat{N}}(\omega), w_{\mathbf{K}}(\omega), w_{\mathbf{k}}(\omega)$, are of class $C^0\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ and vanish at $\partial\widetilde{\Sigma}$, thus they can represent **higher-order log-terms** that may still occur.
- All the "coefficients in black" can be **derived from the free data** and the coefficients $(\widehat{N}_4, \mathbf{k}_2, \mathbf{K}_1)$ which represent the **asymptotic degrees of freedom**.
- **If the last two algebraic conditions hold**, then the following **Fuchsian-type** (singular) **equation** holds for the vector-valued variable $\underline{W} = (w_{\widehat{N}}, w_{\mathbf{K}}, w_{\mathbf{k}})^T$, comprised of the residuals, for every $p \in \mathscr{S}^2$ and for every $0 < \omega < \omega_0$:

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\,\underline{W}(\omega,p) + \underline{H}\Big(\omega, p; \widehat{N}_4(p), \mathbf{k}_2(p), \mathbf{K}_1(p), \underline{W}(\omega,p), \eth\underline{W}, \bar{\eth}\underline{W}, \eth\bar{\eth}\underline{W}\Big) \qquad (*)$$

Smoothness

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\, \underline{W}(\omega,p) + \underline{H}(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

$$\underline{W}(\omega,p) = diag\left[\omega^{-3},\omega^{-1},1\right] \times \int_0^\omega diag\left[s^3,s,1\right] \times \underline{H}(s,p)\,\mathrm{d}s \qquad (**)$$

Smoothness

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\,\underline{W}(\omega,p)$$
$$+\,\underline{H}(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

$$\underline{W}(\omega,p) = diag\left[\omega^{-3},\omega^{-1},1\right] \times \int_0^\omega diag\left[s^3,s,1\right] \times \underline{H}(s,p)\,\mathrm{d}s \qquad (**)$$

- Since the integrand regularly extends to $s=0$, we can perform the integral transformation by replacing s with the product $\omega\cdot\tau$, which yields

$$\frac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\left[\tau^3,\tau,1\right] \times \underline{H}(\omega\cdot\tau,p)\,\mathrm{d}\tau \qquad (***)$$

Smoothness

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\, \underline{W}(\omega,p) + \underline{H}(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

$$\underline{W}(\omega,p) = diag\left[\omega^{-3},\omega^{-1},1\right] \times \int_0^\omega diag\left[s^3,s,1\right] \times \underline{H}(s,p)\,\mathrm{d}s \qquad (**)$$

- Since the integrand regularly extends to $s=0$, we can perform the integral transformation by replacing s with the product $\omega\cdot\tau$, which yields

$$\frac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\left[\tau^3,\tau,1\right] \times \underline{H}(\omega\cdot\tau,p)\,\mathrm{d}\tau \qquad (***)$$

- Since the integrand on the right-hand side is regular over the entire $\widetilde{\Sigma}$, the left-hand side must also be regular there.

Smoothness

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\, \underline{W}(\omega,p) + \underline{H}(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

$$\underline{W}(\omega,p) = diag\left[\omega^{-3},\omega^{-1},1\right] \times \int_0^\omega diag\left[s^3,s,1\right] \times \underline{H}(s,p)\,\mathrm{d}s \qquad (**)$$

- Since the integrand regularly extends to $s=0$, we can perform the integral transformation by replacing s with the product $\omega\cdot\tau$, which yields

$$\frac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\left[\tau^3,\tau,1\right] \times \underline{H}(\omega\cdot\tau,p)\,\mathrm{d}\tau \qquad (***)$$

- Since the integrand on the right-hand side is regular over the entire $\widetilde{\Sigma}$, the left-hand side must also be regular there.
- This then implies that both terms on the right hand side of (*) are regular on $\widetilde{\Sigma}$, and, in turn, the first order ω-derivative $\partial_\omega\underline{W}$ of the vector-valued variable of the residuals $\underline{W}(\omega,p) = \big(w_{\mathbf{K}}(\omega,p), w_{\mathbf{k}}(\omega,p), w_{\widehat{N}}(\omega,p)\big)^T$ is also regular at $\omega=0$.

Smoothness

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag(0,-3,-1)\, \underline{W}(\omega,p) + \underline{H}(\omega,p;\widehat{N}_4(p),\mathbf{k}_2(p),\mathbf{K}_1(p),\underline{W}(\omega,p),\eth\underline{W},\bar{\eth}\underline{W},\eth\bar{\eth}\underline{W}) \qquad (*)$$

- The solution can then be given as

$$\underline{W}(\omega,p) = diag\left[\omega^{-3},\omega^{-1},1\right] \times \int_0^\omega diag\left[s^3,s,1\right] \times \underline{H}(s,p)\,\mathrm{d}s \qquad (**)$$

- Since the integrand regularly extends to $s=0$, we can perform the integral transformation by replacing s with the product $\omega\cdot\tau$, which yields

$$\frac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\left[\tau^3,\tau,1\right] \times \underline{H}(\omega\cdot\tau,p)\,\mathrm{d}\tau \qquad (***)$$

- Since the integrand on the right-hand side is regular over the entire $\widetilde{\Sigma}$, the left-hand side must also be regular there.
- This then implies that both terms on the right hand side of (*) are regular on $\widetilde{\Sigma}$, and, in turn, the first order ω-derivative $\partial_\omega \underline{W}$ of the vector-valued variable of the residuals $\underline{W}(\omega,p) = \left(w_{\mathbf{K}}(\omega,p), w_{\mathbf{k}}(\omega,p), w_{\widehat{N}}(\omega,p)\right)^T$ is also regular at $\omega=0$.
- By repeating this process inductively we can also prove that the ω-derivatives of the vector-valued variable $\underline{W}(\omega,p)$ up to arbitrary order extend regularly to $\partial\widetilde{\Sigma}$, thereby, the constrained variables $(\widehat{N},\mathbf{K},\mathbf{k})$ extend smoothly to $\partial\widetilde{\Sigma}$.

Energy, angular momentum, and asymptotic freedom

- Bondi energy is given by background and the pair $(\mathbf{K}_1, \widehat{N}_4)$

$$m_B = \frac{1}{32\pi}\int_{\partial\widetilde{\Sigma}} \Big[\frac{\mathbf{K}_1{}^3}{\boldsymbol{\kappa}_0} + 8\,\widehat{N}_4\,\boldsymbol{\kappa}_0{}^3 - 4\,\boldsymbol{\kappa}_0\,\boldsymbol{\kappa}_3 + 12\,\mathbf{a}_1\,\boldsymbol{\kappa}_0{}^2 - 2\,\mathbf{K}_1\,\Big[\boldsymbol{\kappa}_0{}^{-1} + 3\,\mathbf{a}_0\,\boldsymbol{\kappa}_0 - 4\,\boldsymbol{\kappa}_2\Big]$$
$$- \Big\{\boldsymbol{\kappa}_0{}^5\,\mathbf{N}_1{}^{-1/2}\,\overline{\eth}\,\Big(\mathbf{N}_1{}^{3/2}\boldsymbol{\kappa}_0{}^{-4}\mathbf{K}_1\Big) + 2\,\boldsymbol{\kappa}_0\,\mathbf{N}_2{}^2\,\overline{\eth}\,\Big(\mathbf{N}_2{}^{-1}\boldsymbol{\kappa}_0\Big) + \tfrac{1}{4}\,\boldsymbol{\kappa}_0{}^5\,\eth\overline{\eth}\,\Big(\boldsymbol{\kappa}_0{}^{-6}\,\mathbf{K}_1\Big)$$
$$- \tfrac{3}{2}\,\boldsymbol{\kappa}_0{}^{-2}\,\mathbf{K}_1\,\eth\overline{\eth}\,\boldsymbol{\kappa}_0 + \tfrac{1}{4}\,\boldsymbol{\kappa}_0\,\eth\overline{\eth}\,\mathbf{K}_1 + \text{“}cc\text{”}\Big\}\Big]\,\mathring{\boldsymbol{\epsilon}}\,.$$

- angular momentum is given by $\mathbf{k}_2$ and background

$$J_B[\phi] = -(16\,\pi)^{-1}\int_{\partial\widetilde{\Sigma}} [(\phi^a\overline{q}_a)\,\mathbf{k}_2 + (\phi^a q_a)\,\overline{\mathbf{k}}_2]\,\boldsymbol{\epsilon}_q$$

- all asymptotic quantity is given by $(\mathbf{K}_1, \mathbf{k}_2, \widehat{N}_4)$ and background

Setup

- hyperboloidal slice of Kerr
- free data is Kerr
- initial data is Kerr + (not necessarily small) perturbation
- solve for the perturbation directly
- spin-weighted spherical harmonic expansion in the angular sector
- 4th order adaptive stepper in r
- integrate from $r = 1$ to $r = 10^5$
- ${}^{(\triangle)}\mathbf{K}|_{r=1} = 10 \cdot {Y_2}^0$

Kerr

$$^{(0)}\widehat{N} = \frac{\sqrt{M^2 - a^2\sin^2\vartheta}}{r} - \frac{M^4 - a^4\sin^2\vartheta}{2r^3\sqrt{M^2 - a^2\sin^2\vartheta}} + \mathcal{O}(r^{-4})$$

$$^{(0)}\mathbf{N} = -iaM\sin\vartheta\,\frac{M^2 - 2a^2\sin^2\vartheta}{r^3} + \mathcal{O}(r^{-4})$$

$$^{(0)}\mathbf{a} = r^2 + \frac{1}{2}a^2(1+\cos^2\vartheta) + \mathcal{O}(r^{-1})$$

$$^{(0)}\mathbf{b} = -\frac{1}{2}a^2\sin^2\vartheta - \frac{a^2M}{r}\sin^2\vartheta + \mathcal{O}(r^{-2})$$

$$^{(0)}\boldsymbol{\kappa} = \frac{1}{\sqrt{M^2 - a^2\sin^2\vartheta}} + \frac{a^4\sin^2(2\vartheta)}{8r^2(M^2 - a^2\sin^2\vartheta)^{3/2}} + \mathcal{O}(r^{-3})$$

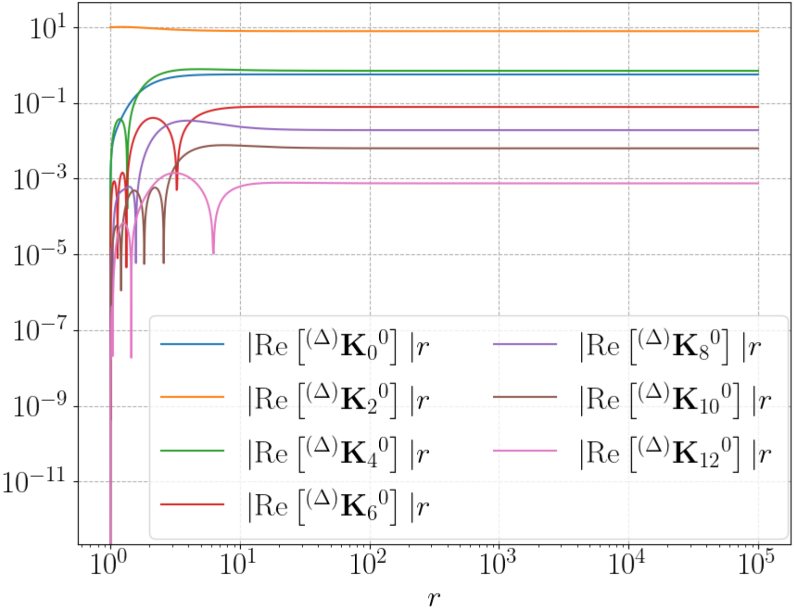
$$^{(0)}\mathbf{k} = -\frac{a^2\cos\vartheta\sin\vartheta}{M^2 - a^2\sin^2\vartheta} + \mathcal{O}(r^{-2})$$

$$^{(0)}\mathbf{K} = \frac{2}{\sqrt{M^2 - a^2\sin^2\vartheta}} + \mathcal{O}(r^{-2})$$

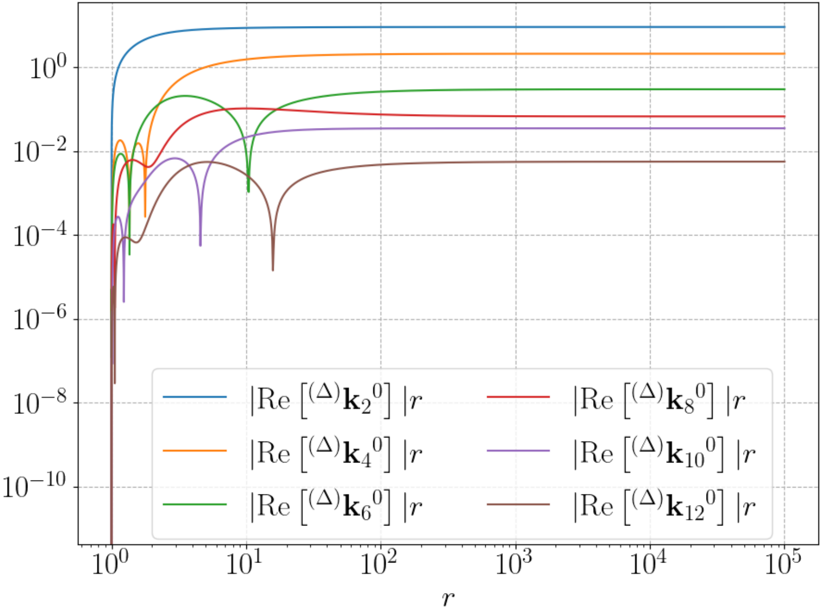
$$^{(0)}\mathring{\mathbf{K}}_{qq} = \frac{a^2\sin^2\vartheta}{\sqrt{M^2 - a^2\sin^2\vartheta}} + \frac{3a^2M\sin^2\vartheta}{r\sqrt{M^2 - a^2\sin^2\vartheta}} + \mathcal{O}(r^{-2})$$

$$\begin{aligned}
\mathbf{N} &= \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{a} &= \omega^{-2} + \mathbf{a}_{(-1)}\,\omega^{-1} + \mathbf{a}_0\\
&\quad + \mathbf{a}_1\,\omega + \mathbf{a}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathbf{b} &= \mathbf{b}_0 + \mathbf{b}_1\,\omega\\
&\quad + \mathbf{b}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\boldsymbol{\kappa} &= \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_2\,\omega^2 + \mathscr{O}(\omega^3)\\
\mathring{\mathbf{K}}_{qq} &= \mathring{\mathbf{K}}_{qq0} + \mathring{\mathbf{K}}_{qq1}\,\omega\\
&\quad + \mathring{\mathbf{K}}_{qq2}\,\omega^2 + \mathscr{O}(\omega^3)
\end{aligned}$$

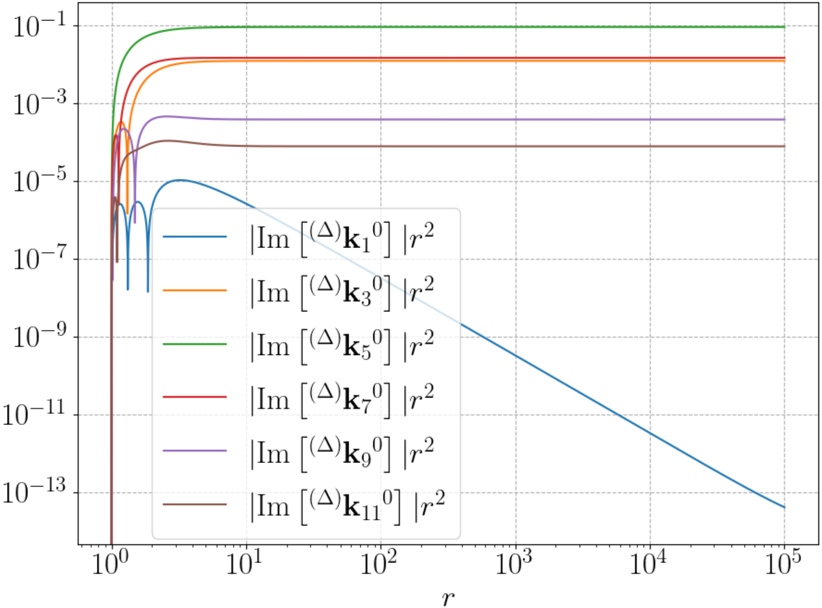
Leading order behaviour of $\mathbf{K}$



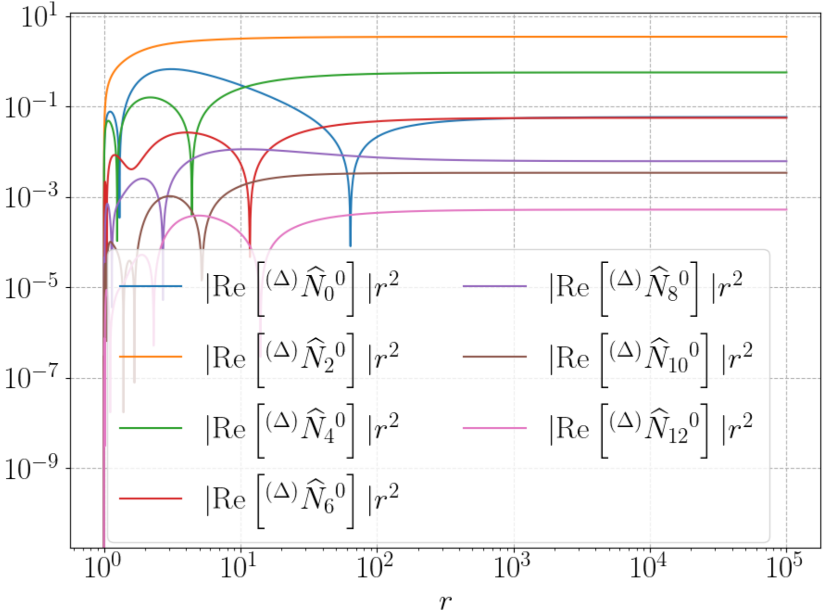
Leading order behaviour of $\mathfrak{K}\mathbf{k}$



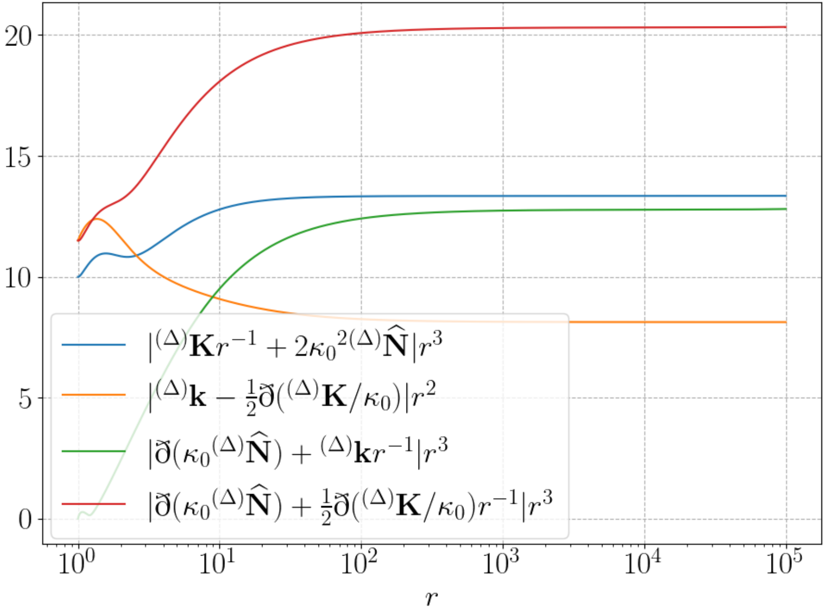
Leading order behaviour of $\Im\mathbf{k}$



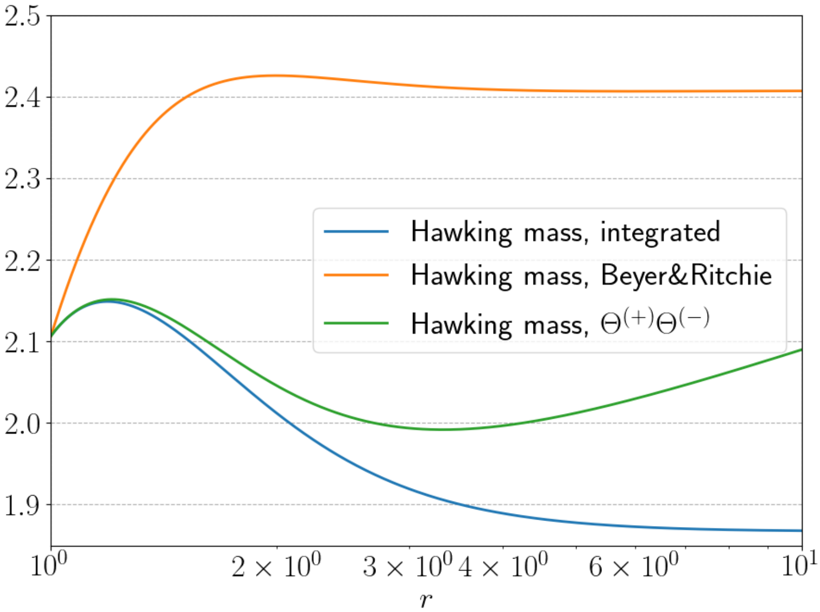
Leading order behaviour of $\widehat{N}$



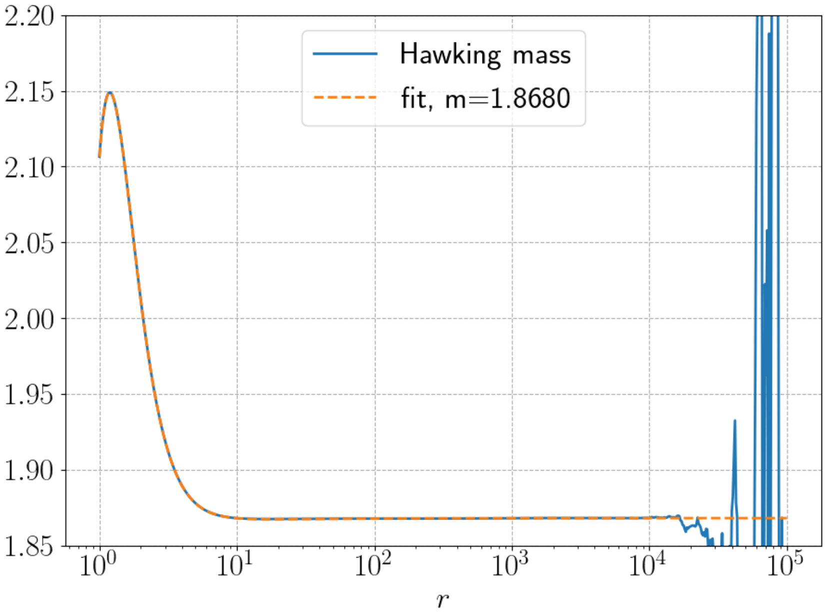
Combinations



Hawking energy comparison



Hawking energy



Summary

- We give a substantial generalization of the results of Beyer&Ritchie.
- Involving the Bondi energy and angular momentum we argue that physically meaningful solutions are smooth (if we use the parabolic-hyperbolic formulation).
- The restrictions from having finite Bondi energy also ensure that the development of the data has smooth conformal boundary.

Accessibility

- arXiv:2503.11804, also OA at Gen Relativ Gravit **57**, 96 (2025)
- https://gitlab.wigner.hu/csukas.karoly/constraintsolver
- code archive: https://doi.org/10.5281/zenodo.14778802
- data and script: https://doi.org/10.5281/zenodo.14779011

Angular momentum