NNLOCAL - Episode III

Revenge of the integrals

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The story so far

Episode I by Gábor [https://indico.wigner.hu/event/1657/]:

- Standard Model great but not the final answer!
- Precision is key!
- Consider a hadron-hadron collision with the production of a colorless state X and m jets (e.g. Higgs + jet production @ LHC). Because of **QCD factorization**, the cross section of such a process can be written as

$$\hat{\sigma}(p_A, p_B) = \sum_{a,b} \int_0^1 \mathrm{d}x_a \, f_{a/A}(x_a, \mu_F^2) \int_0^1 \mathrm{d}x_b \, f_{b/B}(x_b, \mu_F^2) \, \sigma_{ab}(p_a, p_b; \mu_F^2)$$

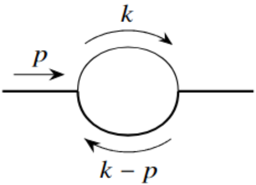
$$\sigma_{ab}(p_a, p_b; \mu_F^2) = \sum_{k=0}^{\infty} \sigma_{ab}^{\mathrm{N}^k\mathrm{LO}}(p_a, p_b; \mu_R^2, \mu_F^2)$$

At N^kLO: k additional partons emitted compared to Born-level

The story so far

Higher-order computations can generate **2** types of singularities with different origins

- UV singularities [hard momenta]

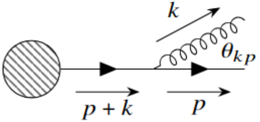


$$\int \frac{\mathrm{d}^4 k}{k^2[(k-p)^2 - m^2]} \xrightarrow{k\to\infty} \int \frac{\mathrm{d}^4 k}{(k^2)^2}$$

$\rightarrow$ Only relevant for **virtual** contributions

The story so far

- IR singularities [soft and/or collinear momenta]



$$\mathcal{M} \sim \frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} = \frac{1}{E_p E_k} \frac{1}{1-\cos\theta_{kp}}$$

$\rightarrow$ Relevant for both **virtual** and **real** contributions

The story so far

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} + \int_{m+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+1} + \int_{m} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2}\right) J_m$$

Double real (RR)

- Tree-level squared MEs with $(m+2)$-parton kinematics
- No loops, so no explicit poles
- MEs diverge as one or two partons become unresolved
- Phase space integral divergent, poles up to $\mathcal{O}(\varepsilon^{-4})$

Real-virtual (RV)

- One-loop squared MEs with $(m+1)$-parton kinematics
- Explicit poles to $\mathcal{O}(\varepsilon^{-2})$
- MEs diverge as one parton becomes unresolved
- Phase space integral divergent, poles up to $\mathcal{O}(\varepsilon^{-2})$

Double virtual (VV)

- Two-loop squared MEs with m-parton kinematics
- Explicit poles to $\mathcal{O}(\varepsilon^{-4})$
- Divergences from unresolved partons screened by jet function
- Phase space integral is finite

- UV divergences treated by renormalization
- IR divergences treated by subtraction

CoLoRFulNNLO

The story so far

Subtract **approximate cross sections** that match the point-wise singularity structure of the partonic cross sections (based on **IR factorization formulae**)

$$\int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} \to \int_{m+2} \Big\{ \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}} J_m \Big\}$$

- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1}$ cancels the singularities coming from a single unresolved emission
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2}$ cancels the singularities coming from a double unresolved emission
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}}$ needed to avoid double subtraction in regions of phase space where single and double limits overlap

The story so far

Of course, what was subtracted needs to be added back! Furthermore, the subtraction terms need to be **integrated** over the momenta of the unresolved emission

$$\int_{m+2}\left\{\mathrm{d}\sigma_{ab}^{\mathrm{RR}}J_{m+2}-\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1}J_{m+1}-\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2}J_m+\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}}J_m\right\}$$
$$+\int_{m+1}\left(\int_1\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1}\right)J_{m+1}+\int_m\left(\int_2\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2}-\int_2\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}}\right)J_m$$

These integrations are performed **analytically**!

Episode II by Pooja [https://indico.wigner.hu/event/1659/]:

- Analytic computation of the (42) master integrals for A_2 for color-singlet production ($m = 0$)
- Based on **IBP reductions** and **differential equations**
- A prophecy:

✦ Details on direct integration method : **Sam Van Thurenhout's talk in the future** .

The future is now!

A large subset of the subtraction terms is integrated directly by setting up a **parametric representation**. Here we focus on A_{12} (again for $m = 0$).

Integrating the A_{12} subtraction terms

From the precise definitions of the A_{12} subtraction terms, it turns out that there are 104 basic integrals to compute!

Luckily, it turns out that the computations follow a certain fixed recipe

Plan: Give an overview of the steps, using the integration of $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ as a guide.

The $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ subtraction term

Generically in A_{12}:

$$CT = (8\pi\alpha_s\mu^{2\varepsilon})^2 \,\mathrm{Sing}_2^{(0)}\, \underbrace{|\mathcal{M}^{(0)}_{\bar{a}\bar{b},X}(\{\bar{p}\}_X; \bar{p}_a, \bar{p}_b)|^2}_{\text{2 partons removed}}$$

To start, we need to define the momentum mapping that characterizes the factorized matrix element

$$|\mathcal{M}^{(0)}_{ab,X+2}(\{p\}_{X+2}, p_a, p_b)|^2 \to |\mathcal{M}^{(0)}_{\hat{\hat{a}}\hat{\hat{b}},X}(\{\hat{\hat{p}}\}_X; \hat{\hat{p}}_a, \hat{\hat{p}}_b)|^2$$

First, there is a collinear mapping $\{p\}_{X+2} \to \{\hat{p}\}_{X+1}$,

$$\begin{aligned}
\hat{p}_a^\mu &= \xi_{a,s} p_a^\mu, \\
\hat{p}_b^\mu &= \xi_{b,s} p_b^\mu, \\
\hat{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_n^\nu, \qquad\qquad n \in F,\; n \neq s \\
\hat{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_X^\nu.
\end{aligned}$$

The $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ subtraction term

$\Lambda(P,\hat{P})$: Proper Lorentz transformation that takes $P \to \hat{P}$ with $P^2 = \hat{P}^2$

$$P^\mu = Q^\mu - p_s^\mu,$$
$$\hat{P}^\mu = \hat{Q}^\mu = (\xi_{a,s}p_a + \xi_{b,s}p_b)^\mu.$$

with $Q = p_a + p_b$. A particular choice is

$$\xi_{a,s} = \sqrt{\frac{s_{ab} - s_{bs}}{s_{ab} - s_{as}}\frac{s_{ab} - s_{sQ}}{s_{ab}}},$$
$$\xi_{b,s} = \sqrt{\frac{s_{ab} - s_{as}}{s_{ab} - s_{bs}}\frac{s_{ab} - s_{sQ}}{s_{ab}}}$$

where $s_{ij} = 2p_i \cdot p_j$.

The $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ subtraction term

Then, the set of double-hatted momenta appearing in the factorized matrix element is obtained by re-applying the collinear mapping above, i.e. $\{\hat{p}\}_{X+1} \to \{\hat{\hat{p}}\}_X$ with

$$
\begin{aligned}
\hat{\hat{p}}_a^\mu &= \xi_{\hat{a},\hat{r}}\hat{p}_a^\mu, \\
\hat{\hat{p}}_b^\mu &= \xi_{\hat{b},\hat{r}}\hat{p}_b^\mu, \\
\hat{\hat{p}}_n^\mu &= \Lambda(\hat{P},\hat{\hat{P}})^\mu{}_\nu\,\hat{p}_n^\nu, \qquad\qquad n \neq r \\
\hat{\hat{p}}_X^\mu &= \Lambda(\hat{P},\hat{\hat{P}})^\mu{}_\nu\,\hat{p}_X^\nu\,.
\end{aligned}
$$

The functional form of $\xi_{\hat{a}/\hat{b},\hat{r}}$ is as $\xi_{a/b,s}$ with $a, b \to \hat{a}, \hat{b}$ and $s \to \hat{r}$.

The $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ subtraction term

In terms of these momenta we then define the subtraction term to be

$$\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF} \sim (8\pi\alpha_s\mu^{2\varepsilon})^2 \frac{\mathcal{F}}{x_{a,s}s_{as}x_{\hat{a},\hat{r}}s_{\hat{a}\hat{r}}} P^{(0)}_{f_{as}f_s}(x_{a,s};\varepsilon) P^{(0)}_{f_{asr}f_r}(x_{\hat{a},\hat{r}};\varepsilon) \times |\mathcal{M}^{(0)}_{\hat{\hat{a}}\hat{\hat{b}},X}(\{\hat{\hat{p}}\}_X,\hat{\hat{p}}_a,\hat{\hat{p}}_b)|^2$$

$$\mathcal{F} = \left[1 - \frac{s_{\hat{r}Q}}{s_{\hat{a}Q}}\right]^2 \left[1 - \frac{s_{\hat{r}\hat{Q}}}{s_{\hat{a}\hat{Q}}}\right]^{-2},$$

$$x_{a,s} = 1 - \frac{s_{sQ}}{s_{ab}}, \qquad x_{\hat{a},\hat{r}} = 1 - \frac{s_{\hat{r}Q}}{s_{\hat{a}Q}}$$

$P^{(0)}_{f_{as}f_s}$ are the standard IF collinear splitting functions

Integrating the $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$ subtraction term

We now need to integrate the subtraction term over the phase space of the unresolved emissions

$$\left[\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}\right] = \int_2 \mathrm{d}\phi_{X+2}(\{p\};Q)\,\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$$

- Φ: Partonic flux factor
- ω: Averaging over initial-state colors and spins

$$\omega(q) = 2N_c\,,\quad \omega(g) = 2(N_c^2-1)(1-\varepsilon)$$

Let us now start setting up our integration recipe.

Step I: Set up a parametric representation of the integral

All momentum mappings employed to define the subtraction terms lead to an **exact factorization** of the real emission phase space

$\rightarrow$ Convolution of the reduced phase space of mapped momenta with an integration measure for the unresolved emissions

In our example:

$$\mathrm{d}\phi_{X+2}(\{p\}_{X+2}; Q) = \int_0^1 \mathrm{d}\xi_a \, \mathrm{d}\xi_b \int_0^1 \mathrm{d}\hat{\xi}_a \, \mathrm{d}\hat{\xi}_b \, \mathrm{d}\phi_X(\{\hat{\hat{p}}\}_X; \hat{\hat{Q}}) \\ \times \mathrm{d}\phi_{if}(\hat{p}_r, \hat{\xi}_a, \hat{\xi}_b) \, \mathrm{d}\phi_{if}(p_s, \xi_a, \xi_b)$$

Step I: Set up a parametric representation of the integral

$$\mathrm{d}\phi_{if}(\hat{p}_r, \hat{\xi}_a, \hat{\xi}_b) = \frac{\mathrm{d}\hat{p}_r}{(2\pi)^{d-1}} \delta_+(\hat{p}_r^2) \delta(\xi_{\hat{a},\hat{r}} - \hat{\xi}_a) \delta(\xi_{\hat{b},\hat{r}} - \hat{\xi}_b) \,,$$

$$\mathrm{d}\phi_{if}(p_s, \xi_a, \xi_b) = \frac{\mathrm{d}p_s}{(2\pi)^{d-1}} \delta_+(p_s^2) \delta(\xi_{a,s} - \xi_a) \delta(\xi_{b,s} - \xi_b)$$

Let us focus our attention on $\mathrm{d}\phi_{if}(p_s, \xi_a, \xi_b)$. It will be convenient in what follows to work in the rest frame of Q, oriented such that p_a points in the z-direction. So, in d-dimensional polar coordinates, we write, assuming all partons to be massless,

$$p_a^\mu = \frac{\sqrt{s_{ab}}}{2}(1, \mathbf{0}_{d-2}, 1) \,,$$

$$p_b^\mu = \frac{\sqrt{s_{ab}}}{2}(1, \mathbf{0}_{d-2}, -1) \,,$$

$$p_s^\mu = E_s(1, \_\_\_, \cos\vartheta) \,.$$

Step I: Set up a parametric representation of the integral

In this frame, the corresponding invariants are

$$
\begin{aligned}
s_{as} &= \sqrt{s_{ab}} E_s (1 - \cos\vartheta)\,, \\
s_{bs} &= \sqrt{s_{ab}} E_s (1 + \cos\vartheta)\,, \\
s_{Qs} &= 2\sqrt{s_{ab}} E_s\,,
\end{aligned}
$$

while the parameters of the momentum mapping are

$$
\xi_{a,s} = \sqrt{\frac{s_{ab} - \sqrt{s_{ab}} E_s (1 + \cos\vartheta)}{s_{ab} - \sqrt{s_{ab}} E_s (1 - \cos\vartheta)} \frac{s_{ab} - 2\sqrt{s_{ab}} E_s}{s_{ab}}}
$$

$$
\xi_{b,s} = \sqrt{\frac{s_{ab} - \sqrt{s_{ab}} E_s (1 - \cos\vartheta)}{s_{ab} - \sqrt{s_{ab}} E_s (1 + \cos\vartheta)} \frac{s_{ab} - 2\sqrt{s_{ab}} E_s}{s_{ab}}}\,.
$$

Step I: Set up a parametric representation of the integral

The phase-space measure can then be written as

$$\mathrm{d}^d p_s = \frac{E_s^{d-3}}{2(2\pi)^{d-1}} \mathrm{d}E_s \mathrm{d}(\cos\vartheta) \sin^{d-4}\vartheta \, \mathrm{d}\Omega_{d-2}$$

and hence

$$\begin{aligned}
\mathrm{d}\phi_{if}(p_s, \xi_a, \xi_b) = {} & \frac{\mathrm{d}E_s \, E_s^{1-2\varepsilon}}{2(2\pi)^{3-2\varepsilon}} \mathrm{d}\Omega_{d-2} \, \mathrm{d}(\cos\vartheta) \sin^{-2\varepsilon}\vartheta \, \theta(E_s) \\
& \times \delta\left(\xi_a - \sqrt{\frac{\sqrt{s_{ab}} - E_s(1+\cos\vartheta)}{\sqrt{s_{ab}} - E_s(1-\cos\vartheta)} \frac{\sqrt{s_{ab}} - 2E_s}{\sqrt{s_{ab}}}}\right) \\
& \times \delta\left(\xi_b - \sqrt{\frac{\sqrt{s_{ab}} - E_s(1-\cos\vartheta)}{\sqrt{s_{ab}} - E_s(1+\cos\vartheta)} \frac{\sqrt{s_{ab}} - 2E_s}{\sqrt{s_{ab}}}}\right)
\end{aligned}$$

Step I: Set up a parametric representation of the integral

Next, we turn the integration over $(E_s, \cos\vartheta)$ into an integration over $(\xi_{a,s}, \xi_{b,s})$. Using the definitions of the latter we have

$$E_s = \frac{\sqrt{s_{ab}}}{2}(1 - \xi_{a,s}\xi_{b,s}),$$
$$\cos\vartheta = \frac{(\xi_{a,s} - \xi_{b,s})(1 + \xi_{a,s}\xi_{b,s})}{(\xi_{a,s} + \xi_{b,s})(1 - \xi_{a,s}\xi_{b,s})}.$$

Of course the transformation induces a **Jacobian**

$$J = 2\sqrt{s_{ab}}\frac{\xi_{a,s}\xi_{b,s}(1 + \xi_{a,s}\xi_{b,s})}{(\xi_{a,s} + \xi_{b,s})^2(1 - \xi_{a,s}\xi_{b,s})}.$$

Step I: Set up a parametric representation of the integral

We then finally find

$$
\begin{aligned}
\mathrm{d}\phi_{if}(p_s,\xi_a,\xi_b) = {} & \frac{\mathrm{d}\Omega_{d-2}}{2(2\pi)^{3-\varepsilon}}\mathrm{d}\xi_{a,s}\,\mathrm{d}\xi_{b,s}\,s_{ab}^{1-\varepsilon}\frac{\xi_{a,s}\xi_{b,s}(1+\xi_{a,s}\xi_{b,s})}{(\xi_{a,s}+\xi_{b,s})^2} \\
& \times\left[\frac{\xi_{a,s}\xi_{b,s}(1-\xi_{a,s}^2)(1-\xi_{b,s}^2)}{(\xi_{a,s}+\xi_{b,s})^2}\right]^{-\varepsilon}\theta(1-\xi_{a,s}\xi_{b,s}) \\
& \times\delta(\xi_{a,s}-\xi_a)\delta(\xi_{b,s}-\xi_b)
\end{aligned}
$$

Naturally, the analysis of $\mathrm{d}\phi_{if}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)$ is completely analogous, and we have

$$
\begin{aligned}
\mathrm{d}\phi_{if}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b) = {} & \frac{\mathrm{d}\Omega_{d-2}}{2(2\pi)^{3-\varepsilon}}\mathrm{d}\xi_{\hat{a},\hat{r}}\,\mathrm{d}\xi_{\hat{b},\hat{r}}\,s_{\hat{a}\hat{b}}^{1-\varepsilon}\frac{\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}(1+\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})}{(\xi_{\hat{a},\hat{r}}+\xi_{\hat{b},\hat{r}})^2} \\
& \times\left[\frac{\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}(1-\xi_{\hat{a},\hat{r}}^2)(1-\xi_{\hat{b},\hat{r}}^2)}{(\xi_{\hat{a},\hat{r}}+\xi_{\hat{b},\hat{r}})^2}\right]^{-\varepsilon}\theta(1-\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}) \\
& \times\delta(\xi_{\hat{a},\hat{r}}-\hat{\xi}_a)\delta(\xi_{\hat{b},\hat{r}}-\hat{\xi}_b)\,.
\end{aligned}
$$

Step I: Set up a parametric representation of the integral

- Integrand independent of the angles in $\mathrm{d}\Omega_{d-2} \Rightarrow \int \mathrm{d}\Omega_{d-2} = (2\pi)^{1-\varepsilon} S_\varepsilon$ with $S_\varepsilon = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}$
- Integrals over $(\xi_{a,s}, \xi_{b,s})$ and $(\xi_{\hat{a},\hat{r}}, \xi_{\hat{b},\hat{r}})$ are completely trivial due to the delta-distributions: $(\xi_{a,s}, \xi_{b,s}, \xi_{\hat{a},\hat{r}}, \xi_{\hat{b},\hat{r}}) \to (\xi_a, \xi_b, \hat{\xi}_a, \hat{\xi}_b)$

We are finally ready to present the fully parametric representation of the (parton-level) integrated counterterm

$$\left[\mathrm{C}_{asr}^{IFF\,(0,0)} \mathrm{C}_{as}^{IF}\right] \otimes \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}} = \int_0^1 \mathrm{d}\xi_a \, \mathrm{d}\xi_b \int_0^1 \mathrm{d}\hat{\xi}_a \, \mathrm{d}\hat{\xi}_b \, \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\hat{\hat{p}}_a, \hat{\hat{p}}_b) \\ \times \left[\mathrm{C}_{asr}^{IFF\,(0,0)} \mathrm{C}_{as}^{IF}(\xi_a, \xi_b, \hat{\xi}_a, \hat{\xi}_b; \varepsilon)\right]$$

$$\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\hat{\hat{p}}_a, \hat{\hat{p}}_b) = \left[\frac{\alpha_s}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2 \frac{\mathrm{d}\phi_X(\{\hat{\hat{p}}\}_X; \hat{\hat{Q}})}{\omega(\hat{\hat{a}})\omega(\hat{\hat{b}})\Phi(\hat{\hat{p}}_a \cdot \hat{\hat{p}}_b)} |\mathcal{M}^{(0)}_{\hat{\hat{a}}\hat{\hat{b}},X}(\{\hat{\hat{p}}\}_X; \hat{\hat{p}}_a, \hat{\hat{p}}_b)|^2$$

Note that now

$$\hat{\hat{p}}_a = \xi_a \hat{\xi}_a p_a \,, \qquad \hat{\hat{p}}_b = \xi_b \hat{\xi}_b p_b$$

Step I: Set up a parametric representation of the integral

$$\left[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\right] = \frac{\omega(\hat{\hat{a}})}{\omega(a)}\frac{\omega(\hat{\hat{b}})}{\omega(b)}\left[\frac{\xi_a^2\xi_b^2(1-\xi_a^2)(1-\xi_b^2)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}$$
$$\times\frac{\xi_a^3\xi_b^3(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}_a^2)(1-\hat{\xi}_b^2)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon}\frac{\hat{\xi}_a^2\hat{\xi}_b^2(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}$$
$$\times\frac{4s_{ab}^2\mathcal{F}}{s_{as}x_{a,s}s_{\hat{a}\hat{r}}x_{\hat{a},\hat{r}}}P^{(0)}_{f_{as}f_s}(x_{a,s};\varepsilon)P^{(0)}_{f_{ars}f_r}(x_{\hat{a},\hat{r}};\varepsilon)$$

Note that the cross section depends on all 4 integration variables!

$$\xi_a\hat{\xi}_a\to\eta_a,\qquad \xi_b\hat{\xi}_b\to\eta_b$$

$$\Rightarrow\left[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}\right]\otimes\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}} = \int_0^1\mathrm{d}\eta_a\int_0^1\mathrm{d}\eta_b\,\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\eta_a p_a,\eta_b p_b)$$
$$\times\int_{\eta_a}^1\frac{\mathrm{d}\xi_a}{\xi_a}\int_{\eta_b}^1\frac{\mathrm{d}\xi_b}{\xi_b}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\xi_a,\xi_b,\eta_a/\xi_a,\eta_b/\xi_b;\varepsilon)]$$

Step I: Set up a parametric representation of the integral

Hence, the integration over ξ_a and ξ_b can be performed once and for all

$$[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\eta_a,\eta_b;\varepsilon)] = \int_{\eta_a}^{1}\frac{\mathrm{d}\xi_a}{\xi_a}\int_{\eta_b}^{1}\frac{\mathrm{d}\xi_b}{\xi_b} \times [\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\eta_a/\xi_a,\eta_b/\xi_b;\varepsilon)]$$

$$\Rightarrow \left[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\right]\otimes \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}} = \int_0^1 \mathrm{d}\eta_a\int_0^1 \mathrm{d}\eta_b\,\mathrm{d}\sigma(\eta_a p_a,\eta_b p_b) \times [\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\eta_a,\eta_b;\varepsilon)]\,.$$

$P^{(0)}_{f_{as}f_s}(x_{a,s};\varepsilon)P^{(0)}_{f_{ars}f_r}(x_{\hat{a},\hat{r}};\varepsilon)$ is a linear combination of terms of the form

$$\left\{\frac{1}{x_{a,s}},\frac{1}{1-x_{a,s}},1,x_{a,s},x_{a,s}^2\right\}\times\left\{\frac{1}{x_{\hat{a},\hat{r}}},\frac{1}{1-x_{\hat{a},\hat{r}}},1,x_{\hat{a},\hat{r}},x_{\hat{a},\hat{r}}^2\right\}.$$

Hence we in fact have 25* basic integrals

$$\left\{\frac{1}{1-x_{\hat{a},\hat{r}}}\frac{1}{1-x_{a,s}},\frac{x_{a,s}^k}{1-x_{\hat{a},\hat{r}}},\frac{x_{\hat{a},\hat{r}}^l}{1-x_{a,s}},x_{a,s}^k x_{\hat{a},\hat{r}}^l\right\},\qquad k,l = -1,0,1,2\,.$$

Step I: Set up a parametric representation of the integral

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \frac{(\eta_a+\eta_b-\eta_b\xi_a+\eta_a\eta_b\xi_a-\eta_a\xi_b+\eta_a\eta_b\xi_b)^{-1+2\varepsilon}}{(-\xi_a+\eta_a\xi_a-\xi_b+\eta_b\xi_b+\xi_a\xi_b-\eta_a\xi_a\xi_b-\eta_b\xi_a\xi_b+\eta_a\eta_b\xi_a\xi_b)} \\
&\times \frac{(1+\eta_a-\xi_a+\eta_a\xi_a)^{-\varepsilon}(1-\xi_b+\eta_b\xi_b)^{2-\varepsilon}(2-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(1+\eta_b-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(2-\xi_a+\eta_a\xi_a-\xi_b+\eta_b\xi_b)^{-1+2\varepsilon}}{(\eta_a+\eta_b-\eta_a^2\eta_b-\eta_a\eta_b^2-2\eta_b\xi_a+2\eta_a\eta_b\xi_a+\eta_b\xi_a^2-2\eta_a\eta_b\xi_a^2+\eta_a^2\eta_b\xi_a^2-2\eta_a\xi_b+2\eta_a\eta_b\xi_b+\eta_a\xi_b^2-2\eta_a\eta_b\xi_b^2+\eta_a\eta_b^2\xi_b^2)} \\
&\times \Big\{[2-(1-\eta_b)\xi_b-((1-\eta_a)\xi_a(1-(1-\eta_b)\xi_b))][(1-\xi_a)(1-(1-\eta_b)\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)] \\
&\times \big(-\eta_b^2(-1+\xi_b)(1+\xi_b)+(-1+\xi_b)(\xi_a-\xi_b)+\eta_b\xi_b(-1-\xi_a+2\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)\big)\Big\} \\
&\times (1-\eta_a)^{1-2\varepsilon}\eta_a^{-\varepsilon}(1-\eta_b)^{-1-2\varepsilon}\eta_b^{-\varepsilon}(1-\xi_a)^{-\varepsilon}\xi_a^{-\varepsilon}(1-\xi_b)^{-1-\varepsilon}\xi_b^{-1-\varepsilon}(1-\xi_a+\eta_a\xi_a)^{-\varepsilon}(2-\xi_a+\eta_a\xi_a)^{-\varepsilon}
\end{aligned}
$$

Step II: Treat overlapping singularities

$$\int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \frac{(\eta_a+\eta_b-\eta_b\xi_a+\eta_a\eta_b\xi_a-\eta_a\xi_b+\eta_a\eta_b\xi_b)^{-1+2\varepsilon}}{(-\xi_a+\eta_a\xi_a-\xi_b+\eta_b\xi_b+\xi_a\xi_b-\eta_a\xi_a\xi_b-\eta_b\xi_a\xi_b+\eta_a\eta_b\xi_a\xi_b)}$$
$$\times \frac{(1+\eta_a-\xi_a+\eta_a\xi_a)^{-\varepsilon}(1-\xi_b+\eta_b\xi_b)^{2-\varepsilon}(2-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(1+\eta_b-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(2-\xi_a+\eta_a\xi_a-\xi_b+\eta_b\xi_b)^{-1+2\varepsilon}}{(\eta_a+\eta_b-\eta_a^2\eta_b-\eta_a\eta_b^2-2\eta_b\xi_a+2\eta_a\eta_b\xi_a+\eta_b\xi_a^2-2\eta_a\eta_b\xi_a^2+\eta_a^2\eta_b\xi_a^2-2\eta_a\xi_b+2\eta_a\eta_b\xi_b+\eta_a\xi_b^2-2\eta_a\eta_b\xi_b^2+\eta_a\eta_b^2\xi_b^2)}$$
$$\times \Big\{[2-(1-\eta_b)\xi_b-((1-\eta_a)\xi_a(1-(1-\eta_b)\xi_b))][(1-\xi_a)(1-(1-\eta_b)\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)]$$
$$\times \big(-\eta_b^2(-1+\xi_b)(1+\xi_b)+(-1+\xi_b)(\xi_a-\xi_b)+\eta_b\xi_b(-1-\xi_a+2\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)\big)\Big\}$$
$$\times (1-\eta_a)^{1-2\varepsilon}\eta_a^{-\varepsilon}(1-\eta_b)^{-1-2\varepsilon}\eta_b^{-\varepsilon}(1-\xi_a)^{-\varepsilon}\xi_a^{-\varepsilon}(1-\xi_b)^{-1-\varepsilon}\xi_b^{-1-\varepsilon}(1-\xi_a+\eta_a\xi_a)^{-\varepsilon}(2-\xi_a+\eta_a\xi_a)^{-\varepsilon}$$

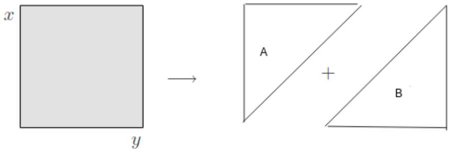
Note that the denominators have non-trivial zeroes. In particular, we need to deal with **overlapping singularities** (e.g. $\xi_a \to 0$ and $\xi_b \to 0$ at the same time)

$$\frac{1}{\eta_a\eta_b\xi_a\xi_b-\eta_a\xi_a\xi_b+\eta_a\xi_a-\eta_b\xi_a\xi_b+\eta_b\xi_b+\xi_a\xi_b-\xi_a-\xi_b}$$

$$\xi_a \to 0 : \frac{1}{\xi_b(\eta_b-1)}, \qquad \xi_b \to 0 : \frac{1}{\xi_a(\eta_a-1)}$$

Step II: Treat overlapping singularities

Disentangle using **sector decomposition** [Heinrich, 2008]



$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, x^{-1-\varepsilon} y^{-\varepsilon} \frac{1}{x+y} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, x^{-1-\varepsilon} y^{-\varepsilon} \frac{1}{x+y} \Big[\underbrace{\theta(x-y)}_{y \to tx} + \underbrace{\theta(y-x)}_{x \to ty} \Big]$$

$$= \int_0^1 \mathrm{d}x \, x^{-1-2\varepsilon} \int_0^1 \mathrm{d}t \, \frac{t^{-\varepsilon}}{1+t} + \int_0^1 \mathrm{d}y \, y^{-1-2\varepsilon} \int_0^1 \mathrm{d}t \, \frac{t^{-1-\varepsilon}}{1+t}$$

$\Rightarrow$ Representation in terms of **factorized singularities**

For our lovely integral: Single step of SD does not suffice

$\Rightarrow$ Need to **iterate**!

Step II: Treat overlapping singularities

At the end of the iteration: All singularities factorized

$\rightarrow$ Need to be regularized: Set up **subtractions**!

$$\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon} f(x) \rightarrow \underbrace{\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon}[f(x) - f(0)]}_{\text{regular } \textbf{per construction}} + \underbrace{\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon} f(0)}_{\frac{f(0)}{\varepsilon}}.$$

After this: Integral can be evaluated **numerically** for checks

$\rightarrow$ Monte Carlo, e.g. using `Vegas` [Lepage, 1978, Lepage, 1980]

$$\mathcal{I}(1/10, 2/10; \varepsilon) = \frac{0.361395}{\varepsilon^2} + \frac{8.11945}{\varepsilon} + 45.1866\,.$$

To continue with the analytic integration, it would be nice to have an idea of the **function space** we expect.

Intermezzo: What is the expected function space?

Generically, the integrals we encounter are **multidimensional integrations of multivariate rational functions**

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{N(x,t_1,t_2,t_3,\ldots)}{D(x,t_1,t_2,t_3,\ldots)}$$

Let's start with the t_1 integration. After **partial fractioning**, we need to compute

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1^n}\,, \qquad \int_0^1 \frac{\mathrm{d}t_1}{(t_1 - f(x,t_2,\ldots))^n}$$

- Straightforward for $n > 1$: Rational function
- Non-trivial for $n = 1$: Need to introduce a new function, the **logarithm**

$$\int \frac{\mathrm{d}t_1}{t_1} = \log(t_1) + C\,, \int \frac{\mathrm{d}t_1}{t_1 - f(x,t_2,\ldots)} = \log\left(t_1 - f(x,t_2,\ldots)\right) + C$$

Intermezzo: What is the expected function space?

Next we consider the t_2 integration. After **partial fractioning** we need to compute integrals of the form

$$\int \mathrm{d}t_2 \, \frac{\log(t_2)}{(t_2 - f(x, t_3, \ldots))^n}$$

- For $n > 1$: Rational functions + logarithms, e.g.

$$\int \mathrm{d}t_2 \, \frac{\log(t_2)}{(t_2 - a)^3} = -\frac{1}{2a}\left(\frac{a\log(t_2)}{(a - t_2)^2} - \frac{1}{a - t_2} + \frac{\log(a - t_2) - \log(t_2)}{a}\right) + C$$

- Non-trivial for $n = 1$: Need to introduce a new function, the **dilogarithm**

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\mathrm{d}t}{t} \log(1 - t)$$

Intermezzo: What is the expected function space?

This reasoning continues for the subsequent integrations as well. Hence we introduce the **classical polylogarithms**

$$\mathrm{Li}_n(x) = \int_0^x \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t)\,, \qquad \mathrm{Li}_1(x) \equiv -\log(1-x)$$

$\rightarrow$ Obey some nice functional relations

- $\mathrm{Li}_n(1/z)$ is **always** related to $\mathrm{Li}_n(z)$, e.g.

$$\mathrm{Li}_2(1/z) = -\mathrm{Li}_2(z) - \frac{1}{2}\log^2(-z) - \zeta_2\,, \qquad \zeta_n = \sum_{i=1}^{\infty}\frac{1}{i^n}$$

- The classical polylogarithm of a **square** is related to the sum of polylogarithms of the **roots**

$$\mathrm{Li}_n(z^2) = 2^{n-1}\left(\mathrm{Li}_n(z) + \mathrm{Li}_n(-z)\right)$$

Intermezzo: What is the expected function space?

The logarithms and classical polylogarithms discussed above are special cases of **generalized polylogarithms** (GPLs) [Goncharov, 1998]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t)\,, \quad G(z) \equiv G(;z) = 1$$

$$G(0;z) = \log z\,, \qquad G(a;z) = \log\left(1-\frac{z}{a}\right)$$

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t) = -G(\underbrace{0,\ldots,0}_{n-1},1;z)$$

$\vec{a}_n = (a_1,\ldots,a_n)$: Weight vector (each separate $a_i =$ **letter**)

Weight: Nr. of integrations

$\rightarrow$ Obey some nice functional relations

Intermezzo: What is the expected function space?

- Shuffle relation

$$G(\vec{a}_m; z)G(\vec{b}_n; z) = \sum_{\vec{c}_{m+n}=\vec{a}_m \sqcup\!\sqcup \vec{b}_n} G(\vec{c}_{m+n}; z)$$

e.g.

$$G(a; z)G(b, c; z) = G(a, b, c; z) + G(b, a, c; z) + G(b, c, a; z)$$

- Fibration basis w.r.t. $(x_1, \ldots, x_n)$

$$\sum_{I=\{i_1,\ldots,i_n\}} c_I \, G(\vec{a}_{1,i_1}; x_1) \ldots G(\vec{a}_{n,i_n}; x_n)$$

such that each weight-vector $\vec{a}_{k,i_k}$ is **independent** of x_k. E.g.

$$G(1+x; 1-y) \xrightarrow{(x,y)} -G(-1; x) + G(0; y) + G(-y; x)$$

$$G(1+x; 1-y) \xrightarrow{(y,x)} -G(-1; x) + G(0; x) + G(-x; y)$$

All this and much more implemented in the `PolyLogTools` package

[Duhr and Dulat, 2019]

Step III: Preparation of the integrand

As the integration kernels for GPLs are linear, we need to **factorize** higher-order polynomials in the denominators & arguments of GPLs

$\rightarrow$ Typically quadratic or quartic
$\Rightarrow$ Expressions can involve non-integer powers of the integration variables

Rationalize the roots!
Construct suitable transformation of the integration variable to remove the root altogether, automated in the `RationalizeRoots` package

[Besier et al., 2020]

$$\sqrt{\eta_a\eta_b - (1-\eta_b)\xi_b(\xi_b - \eta_b(\xi_b - 2))} : \xi_b \rightarrow \frac{2\eta_b(\eta_a t + \eta_b - 1)}{\eta_b\left(\eta_a t^2 - 2\right) + \eta_b^2 + 1}$$

$$\Rightarrow \sqrt{\eta_a\eta_b}\,\frac{\eta_b(t(2-\eta_a t) - 2\eta_b t + \eta_b - 2) + 1}{\eta_a\eta_b t^2 + (\eta_b - 1)^2}$$

Step III: Preparation of the integrand

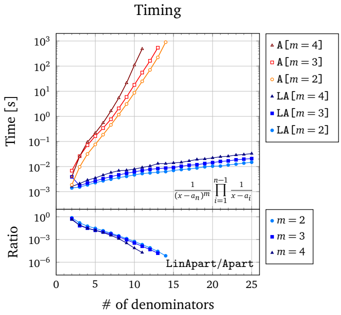
Next, we perform a **partial fraction decomposition** with respect to the integration variable. Surprisingly, this can be a bottleneck in our computations!

$$
\begin{aligned}
f(x_a, x_b; y) = \Big[&(-4+y)(1-y+x_by)(2-y+x_by)(4-y+x_by)(1-x_a-y+x_by)^3 \\
&\times(-1+x_a-y+x_by)(-4-4x_b-y+x_by)(-4x_b-y+x_by) \\
&\times(-4x_a-4x_b-y+x_by)(4x_a-4x_b-y+x_by)(2+2x_b-y+x_by)^3 \\
&\times(6+2x_b-y+x_by)(2-4x_a+2x_b-y+x_by)(2+4x_a+2x_b-y+x_by) \\
&\times(-1+x_a-x_ay+x_ax_by)(1+x_a-x_ay+x_ax_by)(-2+2x_a-x_ay+x_ax_by) \\
&\times(2+2x_a-x_ay+x_ax_by)(-x_b+x_ax_b-x_ay+x_ax_by)^3 \\
&\times(-4+2x_a+2x_ax_b-x_ay+x_ax_by)(4+2x_a+2x_ax_b-x_ay+x_ax_by) \\
&\times(1-2x_a+x_a^2-y-x_ay+x_by+x_ax_by) \\
&\times\left(2x_b-2x_ax_b+x_ay-x_by-x_ax_by+x_b^2y\right)^3\Big]^{-1}
\end{aligned}
$$

- `Apart`: $t > 10^9\,\mathrm{s}$

- Lead to the development of LinApart [Chargeishvili et al., 2025]
- Based on a **closed-form expression** for the decomposition following from the residue theorem
- Significant speed-ups w.r.t. available tools such as Apart
  → E.g. for $f(x_a, x_b; y)$ above: $t \sim 10^{-2}$ s
- Codes (Mathematica + C) publicly available!

[https://github.com/fekeshazy/LinApart]



Step IV: Perform the integration

Use the `GIntegrate` command of `PolyLogTools`, which computes the **primitive function** of the integrand

$\rightarrow$ **Algorithmically**, based on IBP identities

$\rightarrow$ **Only** if the weight-vector is independent of the integration variable!

$\rightarrow$ Hence need to go to a **fibration basis**!

Step V: Repeat

We typically have 1 or 2 integration variables

$$[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)] = \int_{\eta_a}^{1}\frac{\mathrm{d}\xi_a}{\xi_a}\int_{\eta_b}^{1}\frac{\mathrm{d}\xi_b}{\xi_b} \times [\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\xi_a,\xi_b,\eta_a/\xi_a,\eta_b/\xi_b;\varepsilon)]$$

The result of the integration has poles to $\mathcal{O}(\varepsilon^{-2})$ and needs to be computed to $\mathcal{O}(\varepsilon^{0})$. However, even the pole part does not fit on 1 page, so we limit ourselves to some comments here

- The $\mathcal{O}(\varepsilon^{-2})$ part is just **rational**
- The $\mathcal{O}(\varepsilon^{-1})$ part contains weight-1 GPLs (i.e. **logarithms**) involving (roots of) η_a and η_b.
- Finally, the $\mathcal{O}(\varepsilon^{0})$ part is the most complicated, with GPLs up to weight 2.
- The analytic result was tested **numerically** for different values of (η_a,η_b)

Step VI: Regulate endpoint singularities

Note that we are not quite done with the integration!

$$\left[\mathrm{C}_{asr}^{IFF\,(0,0)}\mathrm{C}_{as}^{IF}\right] \otimes \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}} = \int_0^1 \mathrm{d}\eta_a \int_0^1 \mathrm{d}\eta_b \, \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\eta_a p_a, \eta_b p_b) \times \left[\mathrm{C}_{asr}^{IFF\,(0,0)}\mathrm{C}_{as}^{IF}(\eta_a, \eta_b; \varepsilon)\right].$$

Care needs to be taken with the interpretation of this object, as the integrand generically suffers from endpoint singularities!

$\rightarrow$ Regularized by subtraction: Subtract all offending limits and add back integrated expressions

Step VI: Regulate endpoint singularities

- Single limits: $\eta_a \to 1$ or $\eta_b \to 1$

$$\boldsymbol{L}_a[\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \sum_i (1-\eta_a)^{c_i + d_i \varepsilon} g_i(\eta_b; \varepsilon)$$

$$[\boldsymbol{L}_a][\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \sum_i \frac{g_i(\eta_b; \varepsilon)}{c_i + d_i \varepsilon}$$

- Double limit: $\eta_a \to 1$ *and* $\eta_b \to 1$

$$\boldsymbol{L}_{ab}[\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \sum_i (1-\eta_a)^{a_i + b_i \varepsilon} (1-\eta_b)^{c_i + d_i \varepsilon} g_i(\eta_a/\eta_b; \varepsilon)$$

$$[\boldsymbol{L}_{ab}][\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \ldots$$

- Overlaps: $\eta_a \to 1$ or $\eta_b \to 1$ of $\boldsymbol{L}_{ab}$

$$\boldsymbol{L}_a \boldsymbol{L}_{ab}[\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \sum_i (1-\eta_a)^{a_i + b_i \varepsilon} (1-\eta_b)^{c_i + d_i \varepsilon} g_i(\varepsilon)$$

$$[\boldsymbol{L}_a \boldsymbol{L}_{ab}][\mathrm{CT}(\eta_a, \eta_b; \varepsilon)] = \sum_i \frac{g_i(\varepsilon)}{a_i + b_i \varepsilon} (1-\eta_b)^{c_i + d_i \varepsilon}$$

Step VI: Regulate endpoint singularities

Computation of

$$[\boldsymbol{L}_{ab}][\mathrm{CT}(\eta_a,\eta_b;\varepsilon)] = \int_0^1 \mathrm{d}\eta_a \int_0^1 \mathrm{d}\eta_b \sum_i (1-\eta_a)^{a_i+b_i\varepsilon}(1-\eta_b)^{c_i+d_i\varepsilon} \times g_i(\eta_a/\eta_b;\varepsilon)$$

non-trivial!

Overlapping singularities $\Rightarrow$ SD

- Partial fraction
- Fibration basis
- Integrate
- GPL relations for simplifications

Step VI: Regulate endpoint singularities

The limits can be computed using **expansion by regions** [Beneke and Smirnov, 1998]

1. Determine regions with non-trivial behaviour when some asymptotic limit is approached
2. Taylor expand in each region in the appropriate variable
3. Integrate the expanded integrands **over the full integration range**

The first step is the most difficult one. We use the `Mathematica` package `asy2.m` [Pak and Smirnov, 2011, Jantzen et al., 2012] for the automatic determination of the regions.

Step VI: Regulate endpoint singularities

$$
\begin{aligned}
&\left[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}\right]\otimes \mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}} = \int_0^1 \mathrm{d}\eta_a \int_0^1 \mathrm{d}\eta_b \Big\{[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\eta_a p_a,\eta_b p_b)\\
&-\boldsymbol{L_a}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(p_a,\eta_b p_b)\\
&-\boldsymbol{L_b}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\eta_a p_a,p_b) - \Big(\boldsymbol{L_{ab}}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\\
&-\boldsymbol{L_a}\boldsymbol{L_{ab}}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)] - \boldsymbol{L_b}\boldsymbol{L_{ab}}[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\Big)\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(p_a,p_b)\\
&+[\boldsymbol{L_a}][\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(p_a,\eta_b p_b)\\
&+[\boldsymbol{L_b}][\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(\eta_a p_a,p_b) + \Big([\boldsymbol{L_{ab}}][\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)])\\
&-[\boldsymbol{L_a}\boldsymbol{L_{ab}}][\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\\
&-[\boldsymbol{L_b}\boldsymbol{L_{ab}}][\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\eta_a,\eta_b;\varepsilon)]\Big)\mathrm{d}\sigma_{\hat{\hat{a}}\hat{\hat{b}}}(p_a,p_b)\Big\}
\end{aligned}
$$

The A_{12} integration recipe

The integration of all A_{12} (and also others) subtraction terms follows the steps outlined above

1. Set up a parametric representation of the integral
   $\rightarrow$ Typically leads to complicated **multivariate rational function**
2. Treat overlapping singularities
3. Prepare the integrand for integration in terms of GPLs
   - Factor higher-order polynomials and rationalize non-integer powers
   - Partial fraction
   - Fibration basis
4. Integrate
5. Repeat steps 3-4 for all integration variables
6. Regulate endpoint singularities

Summary

- **Precision** is key!
- Treat higher-order kinematic divergences using subtraction: `CoLoRFulNNLO`
- Application to color-singlet production in hadron-hadron collisions is now **within reach**
- All integrated subtraction terms have been computed **analytically**
    - ◇ Subset of the integrations based on IBP reduction and differential equations
    - ◇ Rest done **directly** using our **integration recipe**

Summary

- Implemented in publicly available code NNLOCAL [Del Duca et al., 2025]
- For now, the code is proof-of-concept: Gluon fusion Higgs production in HEFT
- Quark channels in double real radiation complete, full result will be available in the near future

https://github.com/nnlocal

End[1]

Thank you for your attention!

[1]Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

LinApart

Consider some proper rational function $f(x) = \frac{x^l}{Q(x)}$. We consider the decomposition over $\mathbb{C}$ such that

$$Q(x) = \prod_{i=1}^{n} (x - a_i)^{m_i}.$$

- $\{a_i\}_{i=1}^n$: Distinct roots of the polynomial $Q(x)$
- m_i: Multiplicity of the i-th root

Then

$$f(x) = \sum_{i=1}^{n} \left(\frac{c_{i1}}{x - a_i} + \frac{c_{i2}}{(x - a_i)^2} + \cdots + \frac{c_{im_i}}{(x - a_i)^{m_i}} \right)$$

with c_{ij} is the **residue** of $g_{ij}(x) = (x - a_i)^{j-1} f(x)$ at a_i

$$c_{ij} = (g_{ij}, a_i) = \frac{1}{(m_i - j)!} \lim_{x \to a_i} \frac{\mathrm{d}^{m_i - j}}{\mathrm{d}x^{m_i - j}} \Big((x - a_i)^{m_i} f(x) \Big)$$

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Note that

$$(x - a_i)^{m_i} f(x) = x^l \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(x - a_k)^{m_k}}$$

is independent of $a_i \Rightarrow$ simply set $x \to a_i$ to write c_{ij} directly in terms of the roots,

$$c_{ij} = \frac{1}{(m_i - j)!} \frac{\mathrm{d}^{m_i - j}}{\mathrm{d}a_i^{m_i - j}} a_i^l \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(a_i - a_k)^{m_k}} .$$

Hence, $f(x)$ can be expressed as

$$\begin{aligned} f(x) = & \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_{ij}}{(x - a_i)^j} \\ = & \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{1}{(x - a_i)^j} \frac{1}{(m_i - j)!} \frac{\mathrm{d}^{m_i - j}}{\mathrm{d}a_i^{m_i - j}} a_i^l \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(a_i - a_k)^{m_k}} . \end{aligned}$$

Note: $(x - a_i)^{-j}$ is related to the $(j-1)$-st derivative of $(x - a_i)^{-1}$ with respect to a_i,

$$\frac{1}{(x-a_i)^j} = \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}a_i^{j-1}}\frac{1}{x-a_i}\,.$$

Substituting, one sees that the summation over j corresponds to the general Leibniz rule for the $(m_i - 1)$-st derivative of a product of two functions,

$$f(x) = \sum_{i=1}^{n} \frac{1}{(m_i-1)!}\frac{\mathrm{d}^{m_i-1}}{\mathrm{d}a_i^{m_i-1}}\left(\frac{a_i^l}{x-a_i}\prod_{\substack{k=1\\k\neq i}}^{n}\frac{1}{(a_i-a_k)^{m_k}}\right)\,.$$

$\rightarrow$ Can be implemented directly in high-level language (e.g. `Wolfram Mathematica`)

LinApart

For low-level language (e.g. `C`): Resolve differentiation

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\prod_{j=1}^{n} h_j(x) = \sum_{j_1+\cdots+j_n=m} \binom{m}{j_1 \ldots j_n} \prod_{l=1}^{n} \frac{\mathrm{d}^{j_l} h_l(x)}{\mathrm{d}x^{j_l}}$$

$$\Rightarrow f(x) = \sum_{i=1}^{n} \sum_{j_{-1}+j_0+j_1+\cdots+\hat{j}_i+\cdots+j_n=m_i-1} \binom{l}{j_{-1}} \frac{a_i^{l-j_{-1}}}{(x-a_i)^{j_0+1}} \times \prod_{\substack{k=1\\k\neq i}}^{n} \binom{m_k+j_k-1}{j_k} \frac{(-1)^{j_k}}{(a_i-a_k)^{m_k+j_k}}$$

$\hat{j}_i$: j_i is removed from the set of indices

LinApart

So far dealt with **proper** rational functions. So what about **improper** ones? Take $f(x) = \frac{x^l}{Q(x)}$ with $l \geq \deg Q$. We write

$$f(x) = x^{l-(m-1)} \frac{x^{m-1}}{Q(x)}, \qquad l \geq m$$

$m = \deg Q(x) = \sum_{i=1}^{n} m_i$: Degree of the denominator $Q(x)$

Second factor = proper rational function **by construction** $\Rightarrow$ can apply formula derived above

The resulting expression contains only rational functions of the form $g(x) = \frac{x^p}{(x-a)^q}$, and we implement the polynomial division symbolically

$$g(x) = \sum_{i=0}^{p-q} \binom{p-1-i}{q-1} a^{p-q-i} x^i + \sum_{i=p-q+1}^{p} \binom{p}{i} a^i (x-a)^{p-q-i}$$

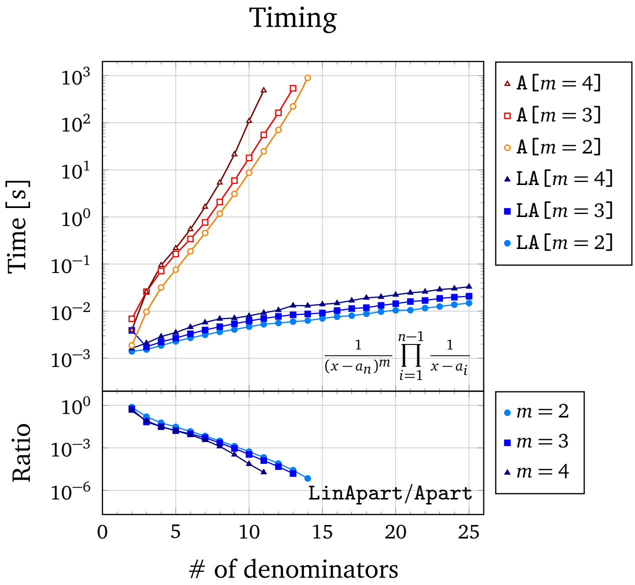
LinApart

So what do we **gain** with our implementation?

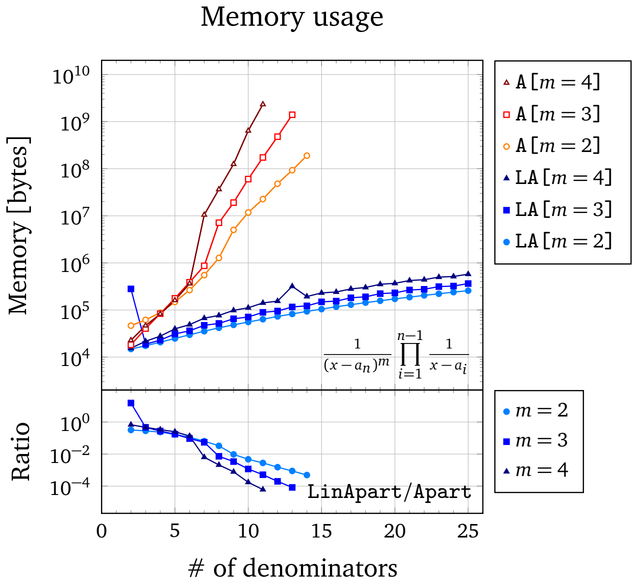
How do we measure **complexity** of a rational function?

1. The number of distinct denominator factors.
2. The complexity of each individual denominator. In fact, even considering only linear denominators of the form $x - a_i$, the roots a_i may be functions of further variables and symbolic constants.
3. The multiplicity of the denominator factors.
4. The polynomial order of the numerator.

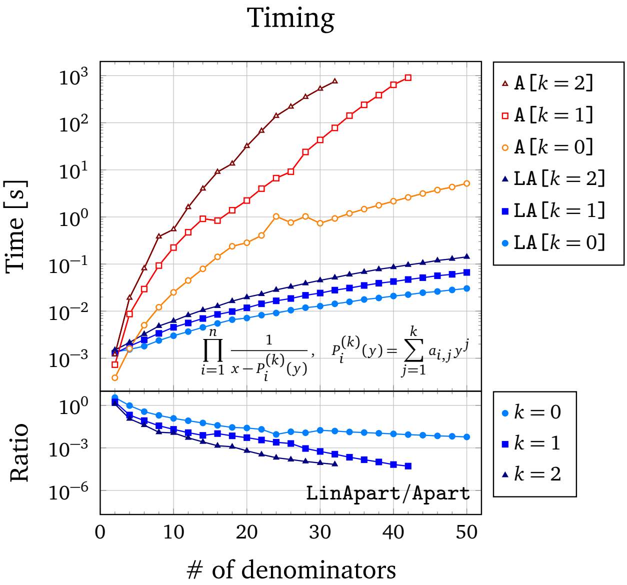
LinApart



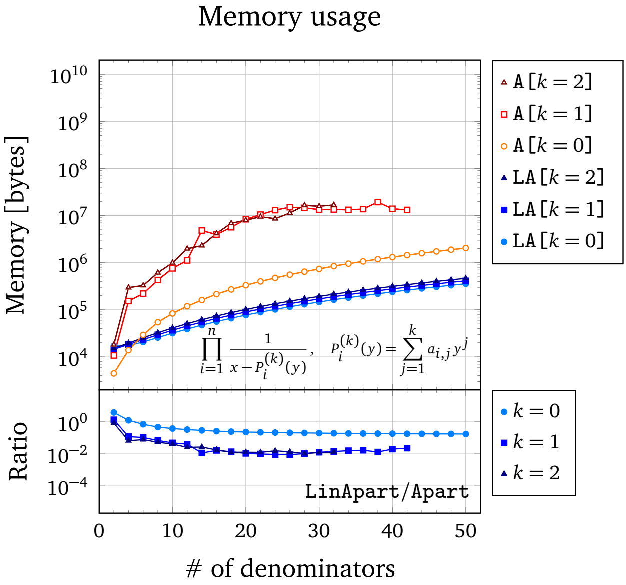
Timing
Time [s]
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
A [$m = 4$]
A [$m = 3$]
A [$m = 2$]
LA [$m = 4$]
LA [$m = 3$]
LA [$m = 2$]
$\frac{1}{(x-a_n)^m}\prod_{i=1}^{n-1}\frac{1}{x-a_i}$
Ratio
10^0
10^{-3}
10^{-6}
LinApart/Apart
$m = 2$
$m = 3$
$m = 4$
0
5
10
15
20
25
# of denominators


Memory usage
Memory [bytes]
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
A [$m = 4$]
A [$m = 3$]
A [$m = 2$]
LA [$m = 4$]
LA [$m = 3$]
LA [$m = 2$]
$\frac{1}{(x-a_n)^m}\prod_{i=1}^{n-1}\frac{1}{x-a_i}$
Ratio
10^{0}
10^{-2}
10^{-4}
LinApart/Apart
$m = 2$
$m = 3$
$m = 4$
0
5
10
15
20
25
# of denominators

LinApart


Timing
Time [s]
Ratio
# of denominators
A [k = 2]
A [k = 1]
A [k = 0]
LA [k = 2]
LA [k = 1]
LA [k = 0]
k = 0
k = 1
k = 2
LinApart/Apart
$\prod_{i=1}^{n} \frac{1}{x - P_i^{(k)}(y)}, \quad P_i^{(k)}(y) = \sum_{j=1}^{k} a_{i,j} y^j$

LinApart



Memory usage
Memory [bytes]
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
$\prod_{i=1}^{n} \frac{1}{x - P_i^{(k)}(y)}, \quad P_i^{(k)}(y) = \sum_{j=1}^{k} a_{i,j} y^j$
A [$k=2$]
A [$k=1$]
A [$k=0$]
LA [$k=2$]
LA [$k=1$]
LA [$k=0$]
Ratio
10^{0}
10^{-2}
10^{-4}
LinApart/Apart
$k=0$
$k=1$
$k=2$
0
10
20
30
40
50
# of denominators

LinApart

$$
\begin{aligned}
f(x_a, x_b; y) = \Big[&(-4+y)(1-y+x_b y)(2-y+x_b y)(4-y+x_b y)(1-x_a-y+x_b y)^3 \\
&\times (-1+x_a-y+x_b y)(-4-4x_b-y+x_b y)(-4x_b-y+x_b y) \\
&\times (-4x_a-4x_b-y+x_b y)(4x_a-4x_b-y+x_b y)(2+2x_b-y+x_b y)^3 \\
&\times (6+2x_b-y+x_b y)(2-4x_a+2x_b-y+x_b y)(2+4x_a+2x_b-y+x_b y) \\
&\times (-1+x_a-x_a y+x_a x_b y)(1+x_a-x_a y+x_a x_b y)(-2+2x_a-x_a y+x_a x_b y) \\
&\times (2+2x_a-x_a y+x_a x_b y)(-x_b+x_a x_b-x_a y+x_a x_b y)^3 \\
&\times (-4+2x_a+2x_a x_b-x_a y+x_a x_b y)(4+2x_a+2x_a x_b-x_a y+x_a x_b y) \\
&\times (1-2x_a+x_a^2-y-x_a y+x_b y+x_a x_b y) \\
&\times \big(2x_b-2x_a x_b+x_a y-x_b y-x_a x_b y+x_b^2 y\big)^3 \Big]^{-1}
\end{aligned}
$$

- Apart: $t > 10^9$ s
- LinApart: $t \sim 10^{-2}$ s

Codes (Mathematica + C) publicly available! [https://github.com/fekeshazy/LinApart]

Where is the plus-distribution?

$$\int_0^1 \mathrm{d}\eta_a \Big\{ \mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2 - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2$$

$$+ \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)\Big\}$$

$$= \int_0^1 \mathrm{d}\eta_a \Big\{ [\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon) + \{\mathcal{I}(\eta_a,\eta_b;\varepsilon) - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)\}]|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2$$

$$- \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2 + \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)\Big\}$$

$$= \int_0^1 \mathrm{d}\eta_a\, \underbrace{\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)\Big\{|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2 - \mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\Big\}}_{[\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)]_+ |\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2}$$

$$+ \int_0^1 \mathrm{d}\eta_a\, \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)$$

$$+ \int_0^1 \mathrm{d}\eta_a\, [\mathcal{I}(\eta_a,\eta_b;\varepsilon) - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)]\,|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2$$

Determination of the regions

In the context of **loop integrals**, [Pak and Smirnov, 2011] starts from the α-representation of the integral, which symbolically corresponds to

$$\mathcal{I} \sim \int \left(\prod_{j=1}^{n} \mathrm{d}x_j \, x_j^{\nu_j} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \mathcal{U}^a \mathcal{F}^b \,.$$

- $\mathcal{U}$ and $\mathcal{F}$: Standard **Symanzik polynomials**
- Consider scaling behaviour of product polynomial $\mathcal{U}\mathcal{F}$
- Each term corresponds to point $\rho^{p_0} x_1^{p_1} \dots x_n^{p_n} \to (p_0, p_1, \dots, p_n)$
- [Pak and Smirnov, 2011] shows that the regions correspond to the points of the **convex hull** of $\{\mathcal{U}\mathcal{F}\}$
- The latter problem is well-known and solved by the divide-and-conquer `quickhull` algorithm [Barber et al., 1996]
- The method was generalized to more general **parametric integrals** in [Jantzen et al., 2012].

Simplifications using PSLQ

The result of $\boldsymbol{IL_{ab}}$ is just a *number*. E.g. for our $\frac{1}{1-x_{\hat{a},\hat{r}}}\frac{1}{1-x_{a,s}}$ master integral we have

$$\boldsymbol{IL_{ab}}I(\varepsilon) = \frac{1}{16\varepsilon^4} - \frac{\zeta_2}{4\varepsilon^2} + \frac{\mathcal{E}}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$

with

$$\begin{aligned}
\mathcal{E} = &-\frac{1}{16}\log^3(2) + \frac{3}{16}G(0;2)\log^2(2) - \frac{1}{16}G(2;1)\log^2(2) - \frac{7}{24}G(0,0;2)\log(2) \\
&+ \frac{1}{2}\zeta_2\log(2) - \frac{1}{6}G\left(-\frac{1}{2};1\right)G(0;2)G(2;1) + \frac{1}{8}G(-2;1)G(0;2)G\left(-\frac{1}{2};1\right) \\
&- \frac{1}{8}G(0;2)G(-2,-2;1) - \frac{1}{48}G(0;2)G(-2,0;1) + \frac{1}{6}G(0;2)G(-2,2;1) \\
&- \frac{1}{8}G\left(-\frac{1}{2};1\right)G(-2,2;1) - \frac{1}{24}G(0;2)G(0,-2;1) + \frac{1}{48}G(-2;1)G(0,0;2) \\
&+ \frac{1}{24}G(2;1)G(0,0;2) - \frac{1}{48}G\left(-\frac{1}{2};1\right)G(0,0;2) - \frac{1}{24}G(0;2)G(0,2;1) \\
&+ \frac{1}{12}G\left(-\frac{1}{2};1\right)G(0,2;1) + \frac{1}{8}G(2;1)G\left(0,-\frac{1}{2};1\right) + \frac{1}{6}G(0;2)G(2,-2;1) \\
&- \frac{1}{24}G(0;2)G(2,0;1) + \frac{1}{12}G\left(-\frac{1}{2};1\right)G(2,0;1) - \frac{1}{8}G(0;2)G(2,2;1) \\
&+ \frac{1}{24}G\left(-\frac{1}{2};1\right)G(2,2;1) + \frac{1}{48}G(0;2)G\left(-\frac{1}{2},0;1\right) + \frac{1}{24}G(2;1)G\left(-\frac{1}{2},0;1\right) \\
&+ \frac{1}{8}G(-2,-2,2;1) + \frac{1}{48}G(-2,0,0;1) - \frac{1}{24}G(-2,0,2;1) - \frac{1}{24}G(-2,2,0;1) \\
&- \frac{1}{24}G(-2,2,2;1) + \frac{1}{24}G(0,-2,0;1) - \frac{1}{12}G(0,-2,2;1) + \frac{1}{12}G(0,0,-2;1) \\
&+ \frac{3}{16}G(0,0,0;2) - \frac{1}{12}G(0,0,2;1) - \frac{1}{12}G(0,2,-2;1) - \frac{1}{8}G(0,2,-1;1) \\
&+ \frac{1}{24}G(0,2,0;1) + \frac{1}{8}G(0,2,2;1) - \frac{1}{24}G(2,-2,0;1) - \frac{1}{24}G(2,-2,2;1) \\
&- \frac{1}{12}G(2,0,-2;1) - \frac{1}{8}G(2,0,-1;1) + \frac{1}{24}G(2,0,0;1) + \frac{1}{8}G(2,0,2;1) \\
&+ \frac{1}{12}G(2,2,-2;1) + \frac{1}{8}G(2,2,-1;1) - \frac{1}{12}G(2,2,2;1) - \frac{1}{48}G\left(-\frac{1}{2},0,0;1\right) \\
&+ \frac{1}{8}G(0,2;1)\log(3) + \frac{1}{8}G(2,-1;1)\log(3) - \frac{1}{8}G(2,2;1)\log(3) - \frac{1}{8}G(2;1)\,\mathrm{Li}_2\left(-\frac{1}{2}\right) \\
&- \frac{35}{48}G(0;2)\zeta_2 - \frac{17}{48}G(2;1)\zeta_2 - \frac{1}{16}\log(3)\zeta_2 - \frac{21\zeta_3}{16}\,.
\end{aligned}$$

Simplifications using PSLQ

Find more compact form of $\mathcal{E}$ by applying **PSLQ** [Ferguson and Bailey, 1992]

- Evaluate $\mathcal{E}$ to high precision, say 100 digits
- Apply PSLQ using

$$\Big\{\zeta_3, \mathrm{Li}_3\left(\frac{1}{2}\right), \log^3(2), \log^2(2)\log(3), \log(2)\log^2(3), \log^3(3), \zeta_2\log(2), \zeta_2\log(3)\Big\}$$

as a basis. This gives

$$\mathcal{E} = -\frac{21\zeta_3}{16}$$

which agrees up to (at least) 200 digits.

$$\boldsymbol{IL_{ab}I}(\varepsilon) = \frac{1}{16\varepsilon^4} - \frac{\zeta_2}{4\varepsilon^2} - \frac{21\zeta_3}{16} - 5\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{19\zeta_4}{32} + \frac{5}{4}\zeta_2\log^2(2) - \frac{35}{8}\zeta_3\log(2) - \frac{5}{24}\log^4(2)$$

References I

[Barber et al., 1996] Barber, C. B., Dobkin, D. P., and Huhdanpaa, H. (1996).
The quickhull algorithm for convex hulls.
ACM Trans. Math. Softw., 22(4):469–483.

[Beneke and Smirnov, 1998] Beneke, M. and Smirnov, V. A. (1998).
Asymptotic expansion of Feynman integrals near threshold.
Nucl. Phys. B, 522:321–344.

[Besier et al., 2020] Besier, M., Wasser, P., and Weinzierl, S. (2020).
RationalizeRoots: Software Package for the Rationalization of Square Roots.
Comput. Phys. Commun., 253:107197.

References II

[Chargeishvili et al., 2025] Chargeishvili, B., Fekésházy, L., Somogyi, G., and Van Thurenhout, S. (2025).
LinApart: Optimizing the univariate partial fraction decomposition.
Comput. Phys. Commun., 307:109395.

[Del Duca et al., 2025] Del Duca, V., Duhr, C., Fekeshazy, L., Guadagni, F., Mukherjee, P., Somogyi, G., Tramontano, F., and Van Thurenhout, S. (2025).
NNLOCAL: completely local subtractions for color-singlet production in hadron collisions.
JHEP, 05:151.

[Duhr and Dulat, 2019] Duhr, C. and Dulat, F. (2019).
PolyLogTools — polylogs for the masses.
JHEP, 08:135.

References III

[Ferguson and Bailey, 1992] Ferguson, H. and Bailey, D. (1992).
A polynomial time, numerically stable integer relation algorithm.

[Goncharov, 1998] Goncharov, A. B. (1998).
Multiple polylogarithms, cyclotomy and modular complexes.
Math. Res. Lett., 5:497–516.

[Heinrich, 2008] Heinrich, G. (2008).
Sector Decomposition.
Int. J. Mod. Phys. A, 23:1457–1486.

[Jantzen et al., 2012] Jantzen, B., Smirnov, A. V., and Smirnov, V. A. (2012).
Expansion by regions: revealing potential and Glauber regions automatically.
Eur. Phys. J. C, 72:2139.

References IV

[Lepage, 1978] Lepage, G. P. (1978).
A New Algorithm for Adaptive Multidimensional Integration.
J. Comput. Phys., 27:192.

[Lepage, 1980] Lepage, G. P. (1980).
VEGAS - an adaptive multi-dimensional integration program.
Technical report, Cornell Univ. Lab. Nucl. Stud., Ithaca, NY.

[Pak and Smirnov, 2011] Pak, A. and Smirnov, A. (2011).
Geometric approach to asymptotic expansion of Feynman integrals.
Eur. Phys. J. C, 71:1626.