# Geometric Performance Limits in Two-Plane Muon Detectors

## Implications for Density Contrast Discrimination in Muography

Juan Caimanque, Mauricio Echiburu, Orlando Soto and Juan Carlos Helo

Departamento de Física, Facultad de Ciencias, Universidad de La Serena, Chile.

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# 1. Motivation: from geometry to density discrimination

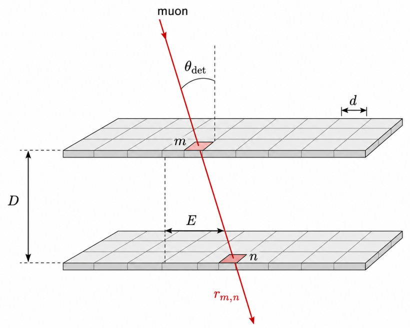
Muon radiography estimates density indirectly from the attenuation of the atmospheric muon flux.

The detector geometry defines:

- reconstructed trajectories,
- angular acceptance,

counting statistics
opacity resolution
density contrast resolution

} linked performance metrics




## Central question

Can a given detector geometry resolve a target density contrast?

# 2. Generic two-plane detector

## Detector geometry

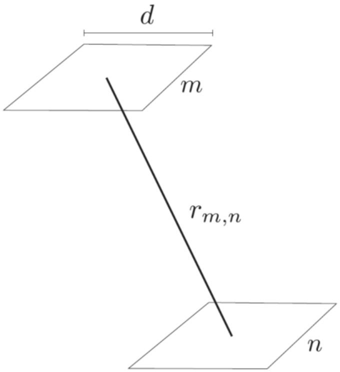
Two parallel detection planes separated by a distance $D$.

Each plane is pixelated. A trajectory is defined by a pair of pixels:

$$m \to n.$$

The pixel size is $d$, and the pixel area is

$$A = d^2.$$



# 3. Trajectories and detector angle

For a trajectory connecting pixels $m$ and $n$, the transverse displacement is

$$E = d\sqrt{h^2 + b^2}.$$

The geometrical length between the two crossed pixels is

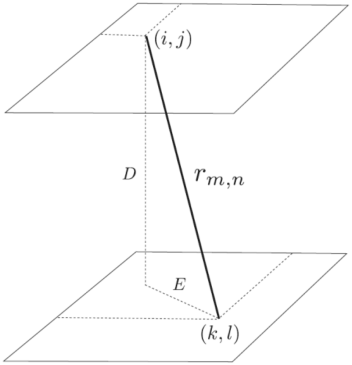
$$r_{m,n} = \sqrt{D^2 + d^2(h^2 + b^2)}.$$

The detector angle is

$$\theta_{\mathrm{det}} = \arctan\left(\frac{E}{D}\right) = \arctan\left(\frac{d\sqrt{h^2 + b^2}}{D}\right).$$

## Key point

Both $D$ and $d$ enter directly into the angular reconstruction.



# 4. Measurement chain and studied sensitivities

The detector geometry propagates through the full muographic chain:

$$(D, d^2) \longrightarrow \theta_{\mathrm{det}} \longrightarrow \Omega \longrightarrow T \longrightarrow N \longrightarrow \gamma \longrightarrow \hat{\rho}$$

We study two basic perturbations:

$$D \to D + \Delta D, \qquad d \to d + \Delta d.$$

$$A = d^2 \to A + \Delta A.$$

## Variables

| | |
|---|---|
| $D$ | plane separation |
| $d$ | pixel size |
| $d^2$ | pixel area |
| $\Omega$ | solid angle |
| $T$ | acceptance |
| $N$ | counts |
| $\gamma$ | opacity |
| $\hat{\rho}$ | reconstructed density |

# 5. Acceptance of a two-plane detector

## Acceptance

$$T(r_{m,n}) = S(r_{m,n})\,\Omega(r_{m,n})$$

## Effective area factor

$$S(r_{m,n}) = N_P A = N_P d^2$$

$S(r_{m,n})$ accounts for the pixel area and the number of pixel-pair combinations associated with a discrete direction.

## Multiplicity

$$N_P = (n_{\mathrm{bars}} - |h|)(n_{\mathrm{bars}} - |b|)$$

## Solid angle

$$\Omega(r_{m,n}) = \frac{4d^2 \cos\theta_{\mathrm{det}}}{r_{m,n}^2}$$

## Final expression

$$T = \frac{4N_P d^4 \cos\theta_{\mathrm{det}}}{D^2 + d^2(h^2 + b^2)}$$

# 6. Sensitivity to $D$: angular reconstruction

## Definition of $D$

$D$ is measured along the detector normal, from the center plane of the first detection layer to the center plane of the second detection layer.

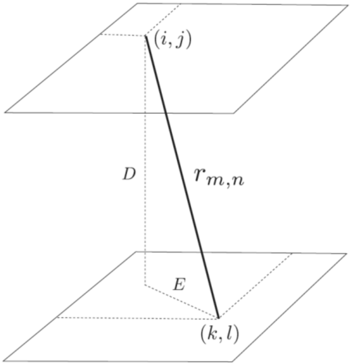
For fixed transverse displacement $E$,

$$\tan\theta_{\rm det} = \frac{E}{D}.$$

A perturbation in the distance between planes gives

$$D \to D + \Delta D, \qquad \theta_{\rm det} \to \theta_{\rm det} + \Delta\theta_{\rm det}.$$

$$\Delta T_D \simeq \left|\frac{\partial T}{\partial D}\right| \Delta D.$$



# 7. Sensitivity to $D$: opacity and density

The opacity along trajectory $i$ is

$$\gamma_i = \sum_k F_{i,k}\rho_k,$$

where $F_{i,k}$ is the path length inside voxel $k$.

If the real geometry is

$$F_{\mathrm{real}} = F + \delta F,$$

but reconstruction uses the ideal matrix $F$, a geometrical bias appears in $\hat{\rho}$.

For an axial perturbation in $D$,

$$\delta\theta \simeq -\frac{\sin\theta\cos\theta}{D}\delta D.$$

The relative density effect is

$$\epsilon_D(\theta) \equiv \left|\frac{\delta\rho}{\rho}\right| \simeq \sin^2\theta\frac{\delta D}{D}.$$

**Bridge to contrast**

$$C_{\rho,\min}(\theta) = k\,\epsilon_D(\theta)$$

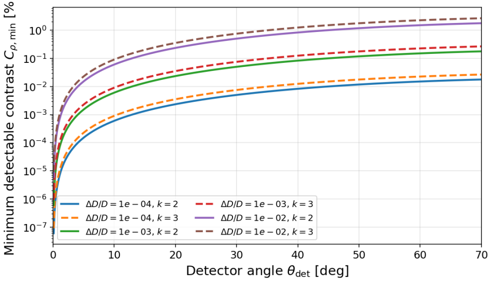
# 8. $D$-induced density contrast threshold

The minimum density contrast detectable from the $D$-induced uncertainty is

$$C_{\rho,\min}(\theta) = k \sin^2\theta \frac{\Delta D}{D}.$$

Here,

$$C_\rho = \frac{|\rho_{\mathrm{target}} - \rho_{\mathrm{host}}|}{\rho_{\mathrm{host}}}.$$




## Interpretation

If $C_\rho < C_{\rho,\min}$, the density contrast cannot be reliably discriminated.

## Role of $k$

$k$ sets the discrimination strength: $k = 2$ is a minimal separation criterion, while $k = 3$ is a more conservative threshold.

# 9. Pixel area $d^2$: acceptance, counts and exposure time

The pixel area is

$$A = d^2.$$

Since

$$T = \frac{4 N_P d^4 \cos\theta_{\rm det}}{D^2 + d^2(h^2 + b^2)},$$

the pixel size strongly affects the acceptance.

The expected number of counts scales as

$$N \propto T_{\rm exp}\, \Phi(\theta)\, T(r_{m,n}).$$

In order of magnitude,

$$N \propto \frac{d^4}{D^2} \cos\theta\, \Phi(\theta)\, T_{\rm exp}.$$

For fixed $N$, $D$, $\theta$, and $\Phi(\theta)$,

$$\frac{T_{\rm obs}(d)}{T_{\rm obs}(d_0)} = \left(\frac{d_0}{d}\right)^4.$$

# 10. Relative observation time versus pixel size

For a fixed target number of counts,

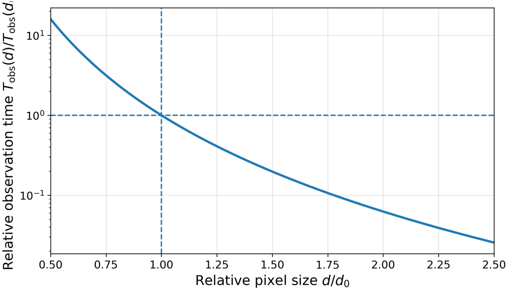
$$T_{\mathrm{obs}}(d) \propto \frac{1}{d^4}.$$

Normalized to a reference pixel size $d_0$,

$$\frac{T_{\mathrm{obs}}(d)}{T_{\mathrm{obs}}(d_0)} = \left(\frac{d_0}{d}\right)^4.$$

## Interpretation

Increasing $d$ reduces the exposure time required for statistics, but it also worsens angular granularity.



# 11. Pixel area $d^2$: resolution-statistics trade-off

## Geometrical contribution

$$\sigma_\theta(\theta) \simeq \frac{\cos^2\theta}{D}\frac{d}{\sqrt{6}}$$

$$\left|\frac{\delta\rho}{\rho}\right|_{\mathrm{geom}} \lesssim \frac{d}{\sqrt{6}D}\sin\theta\cos\theta$$

## Statistical contribution

$$\left|\frac{\delta\rho}{\rho}\right|_{\mathrm{stat}} \approx R\frac{1}{\sqrt{N}}$$

$$\left|\frac{\delta\rho}{\rho}\right|_{\mathrm{stat}} \propto R\frac{D}{d^2}\frac{1}{\sqrt{\cos\theta\,\Phi(\theta)\,T_{\mathrm{exp}}}}$$

$$\epsilon_{\mathrm{total}} = \sqrt{\epsilon^2_{\mathrm{geom}} + \epsilon^2_{\mathrm{stat}}}.$$

# 12. Pixel-size trade-off

In normalized form,

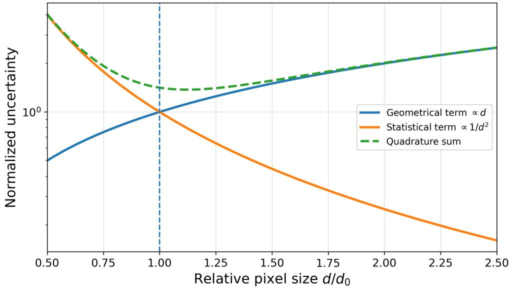
$$\epsilon_{\mathrm{geom}} \propto d, \qquad \epsilon_{\mathrm{stat}} \propto \frac{1}{d^2}.$$

The combined uncertainty can be represented as

$$\epsilon_{\mathrm{total}} = \sqrt{\epsilon_{\mathrm{geom}}^2 + \epsilon_{\mathrm{stat}}^2}.$$

## Design meaning

The useful pixel size is not the one that maximizes counts, but the one that balances statistics and angular resolution.



# 13. Density contrast discrimination

Define the target density contrast relative to the host material as

$$C_\rho = \frac{|\rho_{\mathrm{target}} - \rho_{\mathrm{host}}|}{\rho_{\mathrm{host}}}.$$

The uncertainty can also be expressed relative to the host density:

$$\epsilon_\rho = \frac{\sigma_\rho}{\rho_{\mathrm{host}}}.$$

A practical discrimination criterion is

$$C_\rho > k\epsilon_\rho.$$

## Interpretation of $k$

For $k = 2$, the density contrast must be at least twice the relative uncertainty with respect to $\rho_{\mathrm{host}}$. For $k = 3$, the contrast must be three times larger, giving a more robust discrimination.

## Physical meaning

If $C_\rho \leq k\epsilon_\rho$, the target density cannot be reliably separated from the host material.

# 14. Conclusions

- A two-plane detector defines trajectories through $D$, $d$, $A = d^2$, $\Omega$, and $T$.
- The distance $D$ controls angular reconstruction and affects opacity and density estimation.
- The $D$-induced density uncertainty defines a minimum detectable density contrast.
- The pixel area $d^2$ controls both counting statistics and angular granularity.
- Larger pixels improve statistics but degrade angular resolution.
- Smaller pixels improve angular granularity but require longer exposure times.
- The relevant performance metric is density contrast discrimination relative to $\rho_{\rm host}$.

## Final remark

A third plane may improve performance by adding geometrical redundancy, reducing ambiguous trajectories and improving event selection.