Why is Tsallis distribution so universal?

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BTSLXX meeting, June 11-12 2026

Back to early 2000’s

A company around Technical University Budapest (BME)

- **T.S. Biró**
  - Associate Professor, KFKI RMKI
  - Széchenyi Professor at BME
- **G. Györgyi**, assoc. prof. at ELTE
- **A. Jakovác,** research associate at BME

Inspiring talks, about all kind of topics, in particular:

why Tsallis distribution is so universal

Historical framework

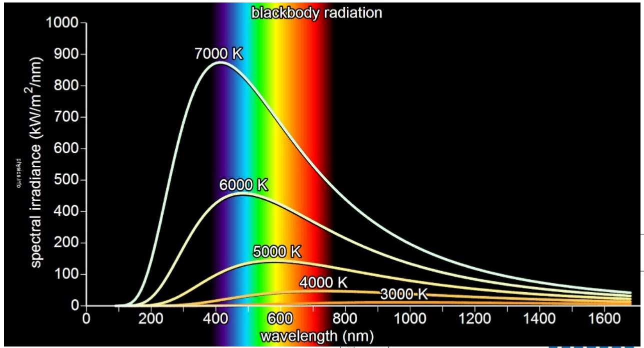
pre-20th century science

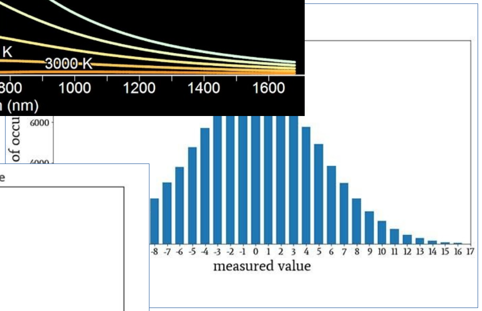
- world is causal and deterministic
- fate of tomorrow lies in today:
  - dynamical variables $X_n(t)$
  - equations (Markovian, diff. equation) $X_n(t+dt)=X_n(t)+F_n(X(t))$
- application: exact equations (Newton, Maxwell, Einstein, etc.)
- problems
  - philosophical (free will vs. determinism, reversibility-irreversibility)
  - practical (how to compute, measurement errors)

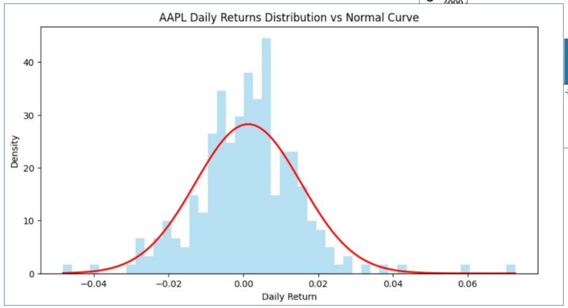
Historical framework

20th century science discovered noise

- blackbody radiation
- Brownian motion
- Johnson–Nyquist noise, shot noise
- measurement errors
- quantum mechanics
- stock market



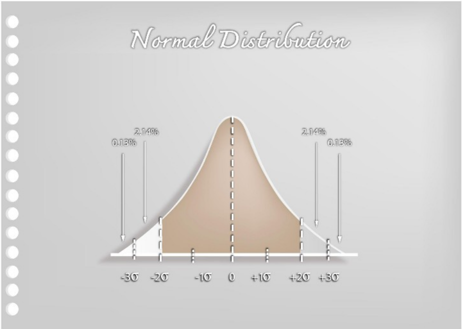




A lot of mathematicians, physicists
worked on understanding the nature of noise
(Boltzmann, Einstein, Langevin, Fokker, Planck, Kolmogorov)

Historical framework

universality observation: exponential / Gaussian fluctuations
(in correct dynamical variables)



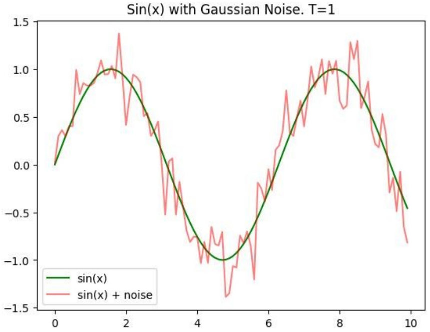
Consequences:

- statistical mechanics, Boltzmann distribution

$$E(f)=\frac{1}{Z}\sum_{n\in\text{states}} f_n e^{-\beta E_n}, \qquad Z=\sum_{n\in\text{states}} e^{-\beta E_n}$$

- error propagation, chi-squared method (MSE), statistics, random matrix theory, information theory, Shannon entropy
- noise reduction (Wiener, Kalman filters)
- susceptibility
- Black-Scholes model for pricing / risk assesement

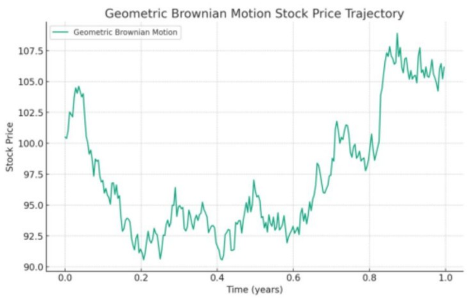
Why is Gaussian universal?



types of dynamical variables

- **singled out variables** (relevant, macrostates) $x_i(t)$
  - carry important information
  - not unique (context, scale) → *fixed points, renormalization*
- **environmental variables** (slowly changing) $A(t)$
- **other variables** (irrelevant, microstates) $y_\alpha(t)$
  - very lot of these variables
  - small effect one-by-one for the evolution of the relevant variables
  - we don’t/cannot follow full time evolution
  - take them into account statistically (random variables)

Why is Gaussian universal?



why is Gaussian so universal?

- evolution equation → linearize in irrelevant variables

$$dx = F(x, A, y) = F(x, A, 0) + \sum_\alpha \left.\frac{\partial F}{\partial y_\alpha}\right|_{y=0} y_\alpha + \dots$$

- **central limit theorem**: sum of i.i.d. random variables is Gaussian (if second moment exists)
- effect of irrelevant variables → adding up Gaussians linearly
  - can lead to stable distribution $P(E) \sim e^{-p^2/(2\sigma_{eq}^2)} \sim e^{-E/T}$

$$\eta_{n+1} = (1 - \gamma\, dt)\eta_n + \sigma \xi \sqrt{dt} \;\Rightarrow\; \sigma_{n+1}^2 = (1 - \gamma\, dt)^2 \sigma_n^2 + \sigma^2 dt \;\Rightarrow\; \sigma_{eq}^2 = \frac{\sigma^2}{2\gamma}$$

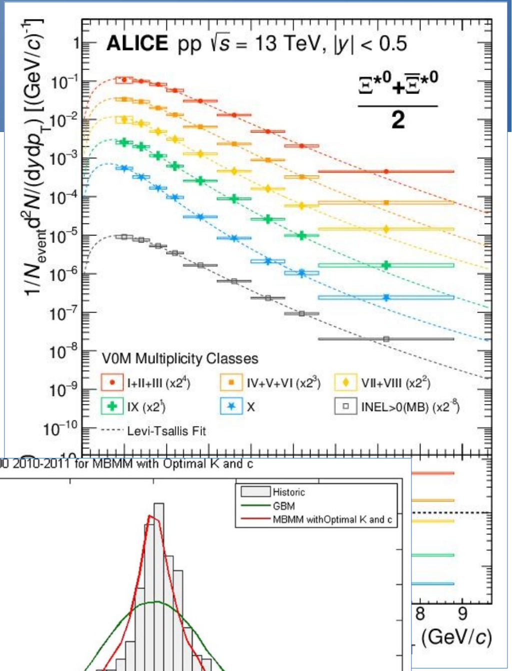
  - temperature, fluctuation-dissipation theorem

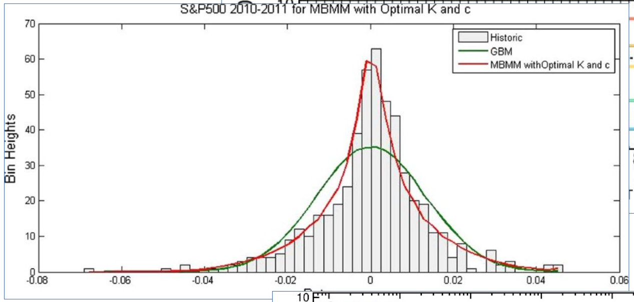
Historical framework

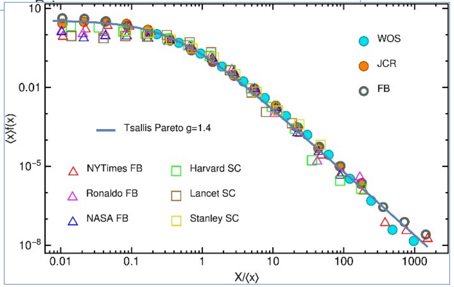
21th century discovered fat tail distributions

- heavy ion collisions $\alpha \approx 6-10$
  - strongly interacting matter blackbody radiation
  - mutiple fat tails (G Bíró, GG Barnaföldi, TS Biró, K Ürmössy, Á Takács Entropy 19 (3), 88)
- stock market $\alpha \approx 4$, ’black swan’ effect
- citations $\alpha \approx 2-4$ (Z Néda, L Varga, TS Biró - PLoS One, 2017)

observation: Tsallis-Pareto distribution universal







Why is Tsalllis distribution universal?

why is Tsallis-Pareto distribution universal?

- ideas:
  - central limit theorem corrections
    (Umarov, Tsallis, Steinberg, Milan j. math. 76, 2008)
  - superstatistics (temperature distribution)
    (C. Beck, E. G. D. Cohen, cond-mat/0205097)
  - constrained phasespace
    (TS Biró, P Ván, GG Barnaföldi, K Ürmössy, 2014)
  - stationarity, but non-equilibrium (Z Néda, L Varga, TS Biró, PLOS, 2017)
  - observed degrees of freedom differ from fundamental ones
    (MM Homor, A Jakovác, PRD, 2018)
  - **multiplicative noise** (TS Biró, A Jakovác, PRL 2005)

Why is Tsalllis distribution universal?

multiplicative noise

- maybe non-linearity is still important?

$$d\eta = -\gamma\eta\, dt + \sigma\xi_1\sqrt{dt} - D\eta\xi_2\sqrt{dt}$$

  - can be interpreted as **stochastic damping** $\gamma \Rightarrow \gamma + D\phi$
    (TS Biró, A Jakovác, PRL; TS Biro, G Györgyi, A Jakovác, G Purcsel, JphysG)
  - can be substituted by one (quadratically) colored noise

$$d\eta = -\gamma\eta\, dt + \bar{\sigma}(\eta)\xi\sqrt{dt}, \qquad \bar{\sigma}(\eta)^2 = \sigma^2 + D^2\eta^2 - 2D\sigma\langle\xi_1\xi_2\rangle\eta$$

  - result is non Gaussian, but analytically solvable!

Why is Tsalllis distribution universal?

- evolution of the distribution function $P(t,\eta)$ (Fokker-Planck-Kolmogorov equation)

$$\partial_t P = \partial_\eta(\mu P) + \frac{1}{2}\partial_\eta^2(\bar{\sigma}^2 P)$$

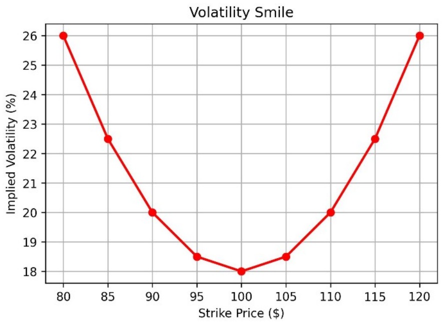
- static solution (for uncorrelated case)

$$P(\eta) \sim \frac{1}{\sigma^2 + D^2\eta^2} e^{-\int_0^\eta \frac{2\gamma x\, dx}{\sigma^2 + D^2 x^2}} \sim \left(1 + (q-1)\frac{E}{T}\right)^{-\frac{q}{q-1}}$$

$$q = 1 + \frac{D^2}{\gamma},\; E = \frac{\eta^2}{2},\; 2\gamma T = \sigma^2$$

Why is Tsalllis distribution universal?

- generalization: $P(x) \sim e^{-\int_0^{\eta} \frac{2\bar{\mu}(x)}{\sigma^2(x)} dx} \quad \Rightarrow \quad P(E) \sim e^{-\int \frac{dE}{T(E)}}$
  - generalized FDT $\bar{\mu}(x) = \mu(x) + \frac{d\sigma^2(x)}{dx}, \quad T = \frac{\sigma^2 E'}{2\bar{\mu}}$



- other idea: stochastic volatility → finance
  - Heston, 1993 → solvable, no power law tail, doesn't solve the „smile problem"
  - SABR model, 2003 → lognormal-normal mixture („fatter" tail), solves smile problem, but not analytically solvable

Why is Tsalllis distribution universal?

remarks

- „any distribution function can be generated" → true, but we found a simple first-term expansion reproducing fat tail!
- no arbitrary moments are computable in Tsallis-Pareto
  - true, but this means that this is an effective statistics for effective degrees of freedom
  - also T(E) function → equipartition
  - effective degrees of freedom in QCD (TS Biro, A Jakovac PRD 2014)
  - MC simulation for strong matter black body radiation (MM Homor, A Jakovác, PRD, 2018)

Conclusion

Conclusion

- a nice explanation of non-Gaussianity
- always very inspiring to discuss with Tamás

🎁 🎂 HAPPY BIRTHDAY! 🎂 🎉