

A recursive construction of joint eigenfunctions for the commuting hyperbolic Calogero-Moser Hamiltonians

(Joint work with M. Hallnäs)

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Introduction

Reminders

- ▶ The hyperbolic (A_{N-1}) Calogero-Moser systems are integrable systems describing N particles on the line with hyperbolic pair interaction.
- ▶ The nonrelativistic quantum version is defined by the Hamiltonian

$$H = -\frac{\hbar^2}{2} \sum_{j=1}^N \partial_{x_j}^2 + g(g - \hbar) \sum_{1 \leq j < l \leq N} V(x_j - x_l),$$

with $\hbar > 0$ (Planck's constant), $g \in \mathbb{R}$ (coupling constant), and pair potential

$$V(x) = \mu^2 / 4 \sinh^2(\mu x / 2), \quad \mu > 0.$$

- ▶ The $N = 2$ Schrödinger equation can be solved via the conical function, a specialization of the Gauss hypergeometric function.

- ▶ Associated **integrable system** (N commuting PDOs):

$$H_1 = -i\hbar \sum_{j=1}^N \partial_{x_j}, \quad H_2 = -\hbar^2 \sum_{1 \leq j_1 < j_2 \leq N} \partial_{x_{j_1}} \partial_{x_{j_2}} - g(g-\hbar) \sum_{1 \leq j < l \leq N} V(x_j - x_l),$$

$$H_k = (-i\hbar)^k \sum_{1 \leq j_1 < \dots < j_k \leq N} \partial_{x_{j_1}} \dots \partial_{x_{j_k}} + \text{l. o.}, \quad k = 3, \dots, N,$$

where l.o. = lower order in partials. Thus, the defining Hamiltonian is given by

$$H = H_1^2/2 - H_2.$$

- ▶ Integrable versions exist for Lie algebras $B_N, \dots, E_8, F_4, G_2$ (Olshanetsky/Perelomov, Oshima) and BC_N (Inozemtsev, Oshima).
- ▶ $N > 2$ eigenfunctions: Harish-Chandra, Heckman/Opdam, Felder/Varchenko, Chalykh,...

Introduction

Goal

- ▶ By proceeding recursively in N , construct joint eigenfunctions $\Psi_N(x, p)$ of the Hamiltonians H_k :

$$H_k(x)\Psi_N(x, p) = S_k(p)\Psi_N(x, p), \quad k = 1, \dots, N,$$

where

$$S_k(p) = \sum_{1 \leq j_1 < \dots < j_k \leq N} p_{j_1} \cdots p_{j_k}.$$

- ▶ For convenience, we rewrite $\Psi_N(x, p)$ as

$$\begin{aligned} \Psi_N(g; (x_1, \dots, x_N), (p_1, \dots, p_N)) &= W_N(g/\hbar; \mu x/2)^{1/2} \\ &\times F_N(g/\hbar; (\mu x_1/2, \dots, \mu x_N/2), (2p_1/\hbar\mu, \dots, 2p_N/\hbar\mu)) \end{aligned}$$

with

$$W_N(\lambda; t) \equiv \prod_{1 \leq j < k \leq N} [4 \sinh^2(t_j - t_k)]^\lambda.$$

Introduction

Main results

- Assuming $\operatorname{Re} \lambda \geq 1$, $u \in \mathbb{R}^N$ and $|\operatorname{Im} t_j| < \pi/2$, we obtain

$$F_N(\lambda; t, u) = \int_{\mathbb{R}^{N(N-1)/2}} \prod_{n=1}^{N-1} \frac{\prod_{1 \leq j < k \leq n} [4 \sinh^2(t_{nj} - t_{nk})]^\lambda}{n! \prod_{j=1}^{n+1} \prod_{k=1}^n [2 \cosh(t_{n+1,j} - t_{nk})]^\lambda} \\ \times \exp \left(i \sum_{n=1}^N u_n \left(\sum_{j=1}^n t_{nj} - \sum_{j=1}^{n-1} t_{n-1,j} \right) \right) \prod_{n=1}^{N-1} \prod_{j=1}^n dt_{nj},$$

where

$$t_{Nj} \equiv t_j, \quad j = 1, \dots, N.$$

- This integral can also be written

$$\exp \left(i u_N \sum_{j=1}^N t_j \right) \prod_{n=1}^{N-1} \int_{t_{nn} < \dots < t_{n1}} \exp \left(i(u_n - u_{n+1})(t_{n1} + \dots + t_{nn}) \right) \\ \times \frac{\prod_{1 \leq j < k \leq n} [2 \sinh(t_{nj} - t_{nk})]^{2\lambda}}{\prod_{j=1}^{n+1} \prod_{k=1}^n [2 \cosh(t_{n+1,j} - t_{nk})]^\lambda} \prod_{j=1}^n dt_{nj}.$$

Introduction

Tools

A crucial ingredient is an explicit description of the eigenvalue equations for F_N .

- ▶ The starting point consists of the **Lax matrix**

$$\mathcal{L}(t, u)_{jk} \equiv \delta_{jk} u_j + (1 - \delta_{jk}) \frac{i\lambda}{\sinh(t_j - t_k)}$$

and the diagonal matrix

$$\mathcal{E}(t) \equiv \text{diag}(w_1(t), \dots, w_N(t))$$

with

$$w_j(t) \equiv -i\lambda \sum_{k \neq j} \coth(t_j - t_k).$$

- ▶ We let $:\hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t):$ denote the **normal-ordered** PDOs obtained from the symmetric functions

$$\Sigma_k(\mathcal{L}(t, u) + \mathcal{E}(t)) \equiv \sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=k}} \det(\mathcal{L}(t, u) + \mathcal{E}(t))_I$$

by performing the substitutions

$$u_j \rightarrow -i\partial_{t_j}, \quad j = 1, \dots, N.$$

- ▶ The Hamiltonians $\mathcal{H}_k(\lambda; t) \equiv (2/\hbar\mu)^k H_k(\lambda\hbar; 2t/\mu)$ are given by **(S. R.)**

$$\mathcal{H}_k(\lambda; t) = W(t)^{1/2} : \hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t) : W(t)^{-1/2}.$$

- ▶ It follows that $F_N(t, u)$ should satisfy the eigenvalue equations

$$:\hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t) : F_N(t, u) = S_k(u) F_N(t, u), \quad k = 1, \dots, N.$$

Another key ingredient is given by so-called kernel functions.

- ▶ Given a pair of operators $H_1(v)$ and $H_2(w)$, a **kernel function** is a function $\Psi(v, w)$ satisfying

$$H_1(v)\Psi(v, w) = H_2(w)\Psi(v, w).$$

Here, v and w may vary over spaces of different dimension.

- ▶ There exist **elementary** kernel functions that connect the PDOs $:\hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t)$ to a sum of PDOs in variables $s_1, \dots, s_{N-\ell}$. (**Langmann** for $k=2$, **Hallnäs/S. R.** for $k>2$.)
- ▶ For $\ell = 1$ this connection can be used to set up a recursive scheme yielding the above **explicit** integral representations of the **joint eigenfunctions** F_N .
- ▶ For $\lambda = 1/2$ recursive H -eigenfunctions were previously found by **Gerasimov/Kharchev/Lebedev**, and for $\lambda = -1, -2, \dots$ by **Felder/Veselov**. (Relation unclear to date.)

$N = 2$ case

From $N = 1$ to $N = 2$

- ▶ For $N = 1$ we set

$$F_1(t, u) \equiv \exp(itu),$$

which obviously satisfies

$$-i\partial_t F_1(t, u) = uF_1(t, u).$$

- ▶ Now consider

$$F_2(\lambda; t, u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2)$$

with kernel function

$$\mathcal{K}_2^\#(\lambda; t, s) \equiv \prod_{j=1}^2 [2 \cosh(t_j - s)]^{-\lambda}.$$

- ▶ If $\operatorname{Re} \lambda > 0$ and $u \in \mathbb{R}^2$, then the integrand decays exponentially as $|s| \rightarrow \infty$. It has singularities only at

$$s = t_j \pm \frac{i\pi}{2}(2n+1), \quad j = 1, 2, \quad n \in \mathbb{N}.$$

- ▶ Hence $F_2(\lambda; t, u)$ is well defined as long as

$$\operatorname{Re} \lambda > 0, \quad u \in \mathbb{R}^2,$$

and $t \in \mathbb{C}^2$ satisfies

$$|\operatorname{Im} t_j| < \pi/2, \quad j = 1, 2.$$

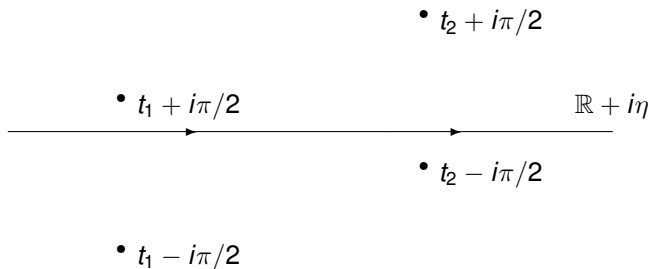
$N = 2$ case

Holomorphy

- ▶ $F_2(\lambda; t, u)$ has analytic continuation in (λ, t) to

$$\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 0\} \times \{t \in \mathbb{C}^2 | |\operatorname{Im}(t_1 - t_2)| < \pi\}.$$

- ▶ Follows via contour shifts:



where we can choose $\eta = \operatorname{Im}(t_1 + t_2)/2$.

- ▶ Can allow $u \in \mathbb{C}^2$ such that $|\operatorname{Im}(u_1 - u_2)| < 2\operatorname{Re} \lambda$.

$N = 2$ case

Eigenfunction property

We claim that $F_2(\lambda; t, u)$ is a joint eigenfunction of the PDOs

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) := -i(\partial_{t_1} + \partial_{t_2}),$$

$$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) := -\partial_{t_1} \partial_{t_2} + \lambda \coth(t_1 - t_2)(\partial_{t_1} - \partial_{t_2}) + \lambda^2.$$

- ▶ **Key point:** eigenfunction identity

$$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^\sharp(t, \mathbf{s}) = 0,$$

and kernel identity

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^\sharp(t, \mathbf{s}) =: \hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(-s) : \mathcal{K}_2^\sharp(t, \mathbf{s}) = i\partial_s \mathcal{K}_2^\sharp(t, \mathbf{s}).$$

- ▶ By analyticity, need only consider $t \in \mathbb{R}$ and $\lambda > 0$ (say).

To establish the eigenfunction property for : $\hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t)$: (e. g.), we use the following 7 steps.

1. Recall that

$$F_2(\lambda; t, u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).$$

2. Act with : $\hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t)$:, and shift through plane wave:

$$e^{iu_2(t_1+t_2)} : \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E} + u_2 \mathbf{1}_2)(t) : \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).$$

3. Note the expansion

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E} + u_2 \mathbf{1}_2)(t) : := : \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : + 2u_2.$$

4. Act with PDO under the integral sign and invoke kernel identity:

$$e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds F_1(s, u_1 - u_2) : \hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(-s) : \mathcal{K}_2^\#(\lambda; t, s) + 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).$$

5. Recall that : $\hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(-s) := i\partial_s$, and integrate by parts:

$$e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) : \hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(s) : F_1(s, u_1 - u_2) \\ + 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).$$

6. Use eigenfunction property for F_1 :

$$(u_1 - u_2 + 2u_2) e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).$$

7. Conclude that

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : F_2(t, u) = S_1^{(2)}(u) F_2(t, u),$$

where $S_1^{(2)}(a_1, a_2) \equiv a_1 + a_2$.



$N = 2$ case

A bound

- ▶ Let $u \in \mathbb{R}^2$. For $\operatorname{Re} \lambda > 0$ and $|\operatorname{Im}(t_1 - t_2)| < \pi$, we have the F_2 -bound

$$|F_2(\lambda; t, u)| < C(\lambda, |\operatorname{Im}(t_1 - t_2)|) \times \exp(-\operatorname{Im}(t_1 + t_2)(u_1 + u_2)/2) \frac{\operatorname{Re}(t_1 - t_2)}{\sinh(\operatorname{Re} \lambda \operatorname{Re}(t_1 - t_2))}.$$

- ▶ This bound readily follows from the integral evaluation

$$\int_{\mathbb{R}} \frac{ds}{\prod_{\delta=+,-} 2 \cosh(s + \delta z/2)} = \frac{z}{2 \sinh z}.$$

Recursion scheme

Kernel function

- ▶ The function

$$\mathcal{K}_N^\#(\lambda; t, s) \equiv \prod_{j=1}^N \prod_{k=1}^{N-1} [2 \cosh(t_j - s_k)]^{-\lambda}, \quad N > 1,$$

satisfies the **eigenfunction identity**

$$: \hat{\Sigma}_N^{(N)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_N^\#(t, s) = 0,$$

and the **kernel identities**

$$(: \hat{\Sigma}_k^{(N)}(\mathcal{L} + \mathcal{E})(t) : - : \hat{\Sigma}_k^{(N-1)}(\mathcal{L} + \mathcal{E})(-s) :) \mathcal{K}_N^\#(t, s) = 0, \quad k < N.$$

- ▶ This connection between the N and $N - 1$ variable cases can be used to recursively construct the joint eigenfunctions F_N of the N PDOs $: \hat{\Sigma}_k^{(N)}(\mathcal{L} + \mathcal{E})(t) :, k = 1, \dots, N$.

Recursion scheme

Formal structure

- ▶ Assume the function $F_{N-1}(t, u)$, $t, u \in \mathbb{C}^{N-1}$, has been constructed.
- ▶ Then $F_N(t, u)$, $t, u \in \mathbb{C}^N$, is **formally** given by

$$F_N(t, u) \equiv \frac{e^{iu_N \sum_{j=1}^N t_j}}{(N-1)!} \int_{\mathbb{R}^{N-1}} ds W_{N-1}(s) \mathcal{K}_N^\#(t, s) \\ \times F_{N-1}(s, (u_1 - u_N, \dots, u_{N-1} - u_N)).$$

- ▶ Have shown $F_N(\lambda; t, u)$ is well defined for $\operatorname{Re} \lambda \geq 1$ and $t \in \mathbb{C}^N$ such that $|\operatorname{Im} t_j| < \pi/2$ (and $u \in \mathbb{R}^N$), and continues analytically to

$$\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 1\} \times \{t \in \mathbb{C}^N | \max_{1 \leq j < k \leq N} |\operatorname{Im}(t_j - t_k)| < \pi\}.$$

A_{N-1} Heckman-Opdam hypergeometric function

- ▶ $F_{A_{N-1}}(\tilde{\lambda}, k; h)$ depends on three types of parameters:
 - ▶ coupling parameter $k \in \mathbb{C}$,
 - ▶ eigenvalue vector $\tilde{\lambda} \in \mathbb{C}^N$,
 - ▶ a quantity h , which can be viewed as a diagonal $N \times N$ matrix with $\det(h) = 1$.
- ▶ For $t \in \mathbb{C}^N$ such that $\sum t_j = 0$, let

$$h(t) = \text{diag}(e^{2t_1}, \dots, e^{2t_N}).$$

- ▶ Comparing eigenvalue equations and normalisations, we deduced

$$F_{A_{N-1}}(iu/2, \lambda; h(t)) = \frac{F_N(\lambda; t, u)}{F_N(\lambda; 0, u)},$$

where $\sum t_j = \sum u_j = 0$.

Relativistic generalisation

Reminders

- ▶ Given by $2N$ commuting **analytic difference operators**

$$A_{k,\delta} = \sum_{|I|=k} \prod_{\substack{m \in I \\ n \notin I}} \frac{s_\delta(x_m - x_n - ib)}{s_\delta(x_m - x_n)} \prod_{m \in I} \exp(-ia_{-\delta} \partial_{x_m}),$$

where $k = 1, \dots, N$, $\delta = +, -$, and

$$s_\delta(z) \equiv \sinh(\pi z / a_\delta).$$

- ▶ Physical picture: The imaginary periods and shift step sizes are determined by the length parameters

$$a_+ \equiv 2\pi/\mu, \quad (\text{interaction length}),$$

$$a_- \equiv \hbar/mc, \quad (\text{Compton wave length}),$$

with \hbar Planck's constant, m particle mass and c speed of light.

Relativistic generalisation

Tools

- ▶ The **hyperbolic gamma function**

$$G(a_+, a_-; z) \equiv \exp(ig(a_+, a_-; z)),$$

with

$$g(z) \equiv \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2 \sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y} \right),$$

for $|\operatorname{Im} z| < (a_+ + a_-)/2$.

- ▶ The **kernel function**

$$S_N^\#(b; x, y) \equiv \prod_{j=1}^N \prod_{k=1}^{N-1} \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)},$$

which satisfies the $2N$ identities

$$A_{k,\delta}^{(N)}(x) S_N^\#(b; x, y) = \left(A_{k,\delta}^{(N-1)}(-y) + A_{k-1,\delta}^{(N-1)}(-y) \right) S_N^\#(b; x, y).$$

Relativistic generalisation

Sketch of results

- ▶ For $N = 1$ we set

$$J_1(x, y) \equiv \exp(i\alpha xy), \quad \alpha = \frac{2\pi}{a_+ a_-}.$$

- ▶ For $N > 1$ we construct joint eigenfunctions $J_N(x, y)$, $x, y \in \mathbb{C}^N$, **recursively** according to

$$J_N(x, y) = e^{i\alpha y_N(x_1 + \dots + x_N)} \int_{\mathbb{R}^{N-1}} dz W_{N-1}(z) S_N^\#(x, z) \\ \times J_{N-1}(z, (y_1 - y_N, \dots, y_{N-1} - y_N))$$

with

$$W_M(z) \equiv \prod_{1 \leq m < n \leq M} w(z_m - z_n),$$

$$w(z) \equiv 1/c(z)c(-z), \quad c(z) \equiv G(z + ia - ib)/G(z + ia).$$

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