Characteristic Gluing and Conserved Charges in General Relativity

Presented by Zita Csapó

Meeting of Young Astronomers and Astrophysicists – FIKUT NEXTGEN 2026 November 5th, Budapest

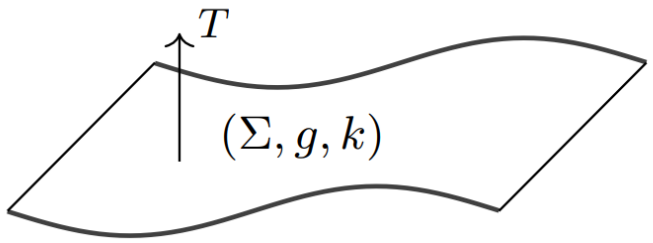
1. Characteristic gluing problem

Initial value formulation of GR

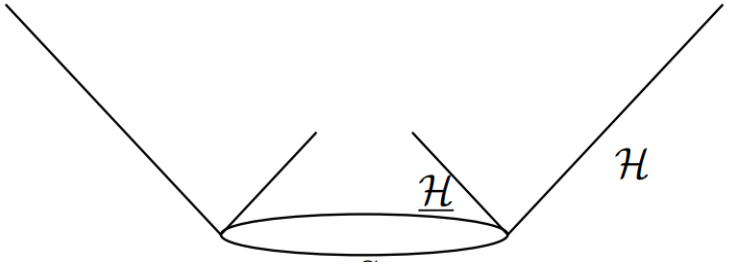
The vacuum Einstein equations can be formulated as an initial value problem

$$R_{\mu\nu} = 0$$

The initial data must admit to the appropriate constraints



Spacelike



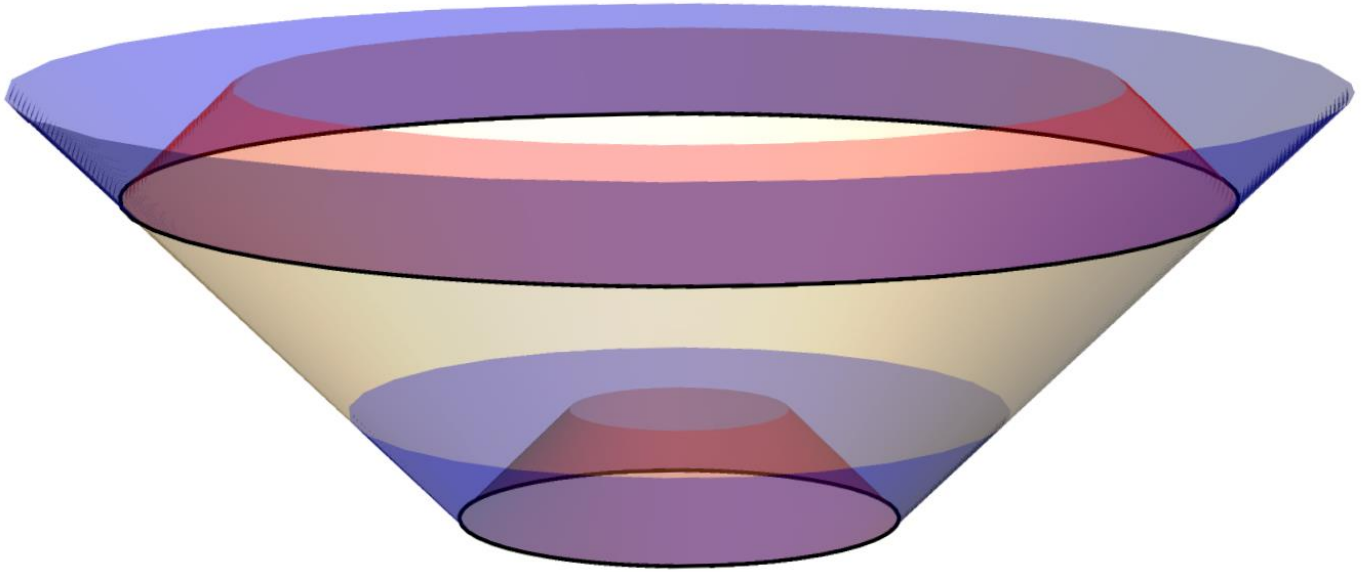
[1]

Null

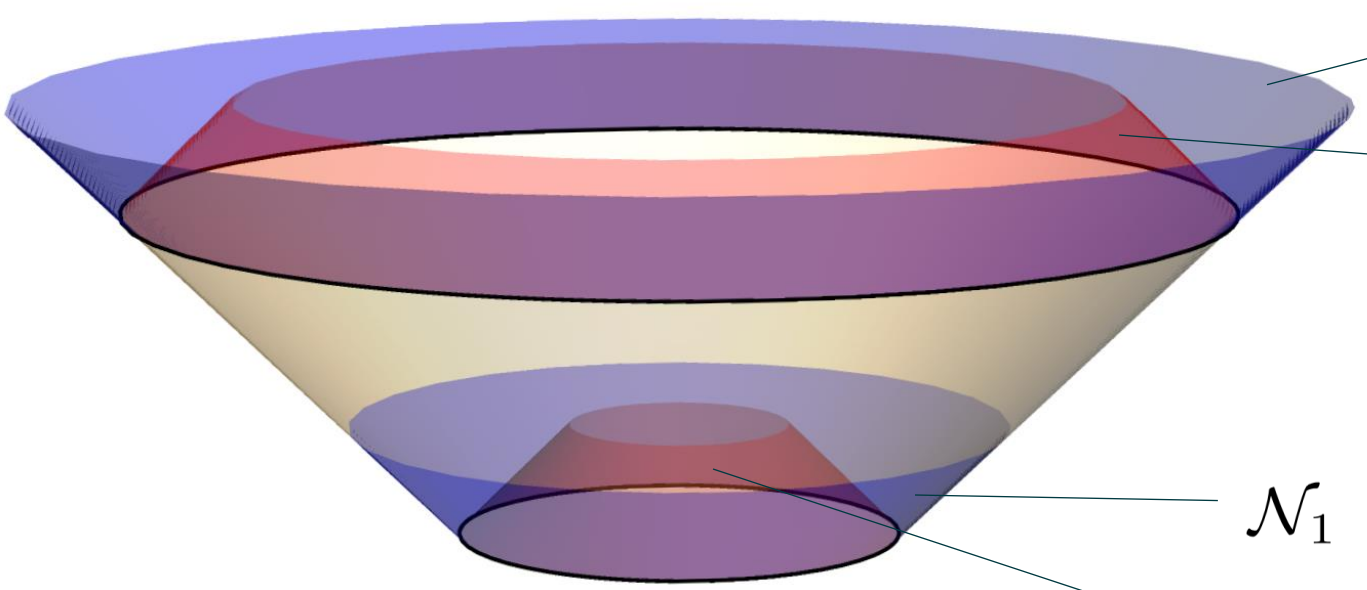
Elliptic constraint equation

Transport type null constraint equations

Characteristic gluing problem



Characteristic gluing problem



$\mathcal{N}_2$

$\underline{\mathcal{N}_2}$

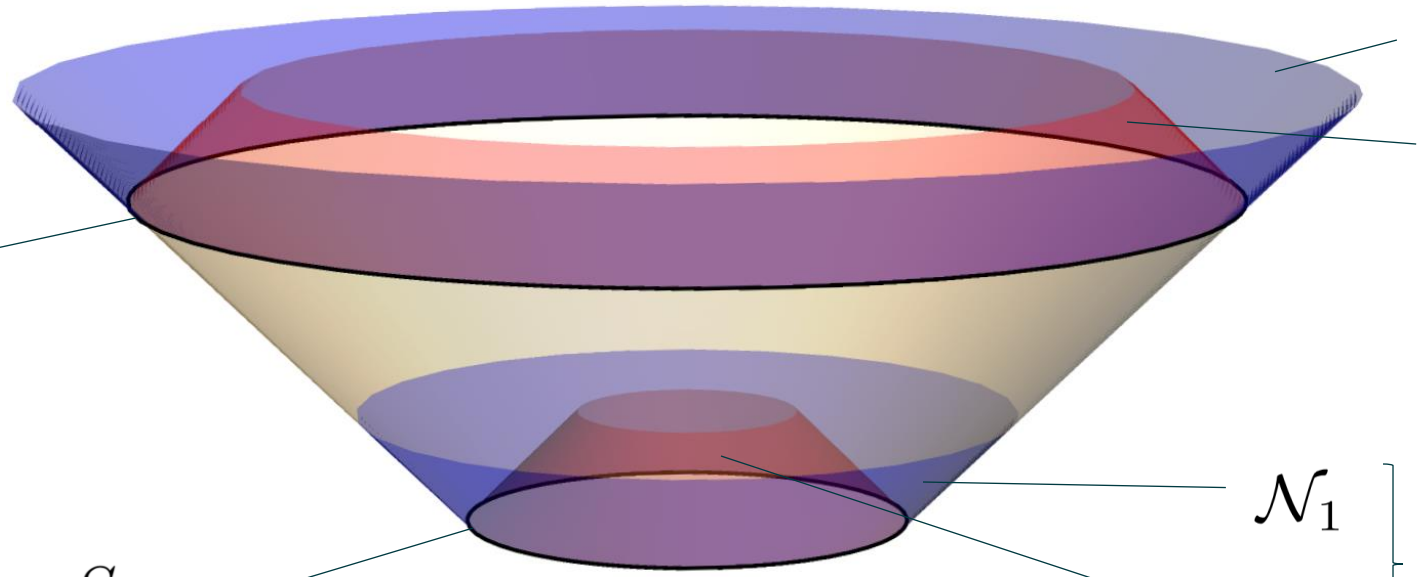
null surfaces in $\mathcal{M}_2$

$\mathcal{N}_1$

$\underline{\mathcal{N}_1}$

null surfaces in $\mathcal{M}_1$

Characteristic gluing problem



sphere & sphere data:

$S_2,\ x_2$

$\mathcal{N}_2$

$\underline{\mathcal{N}}_2$

null surfaces in $\mathcal{M}_2$

$\mathcal{N}_1$

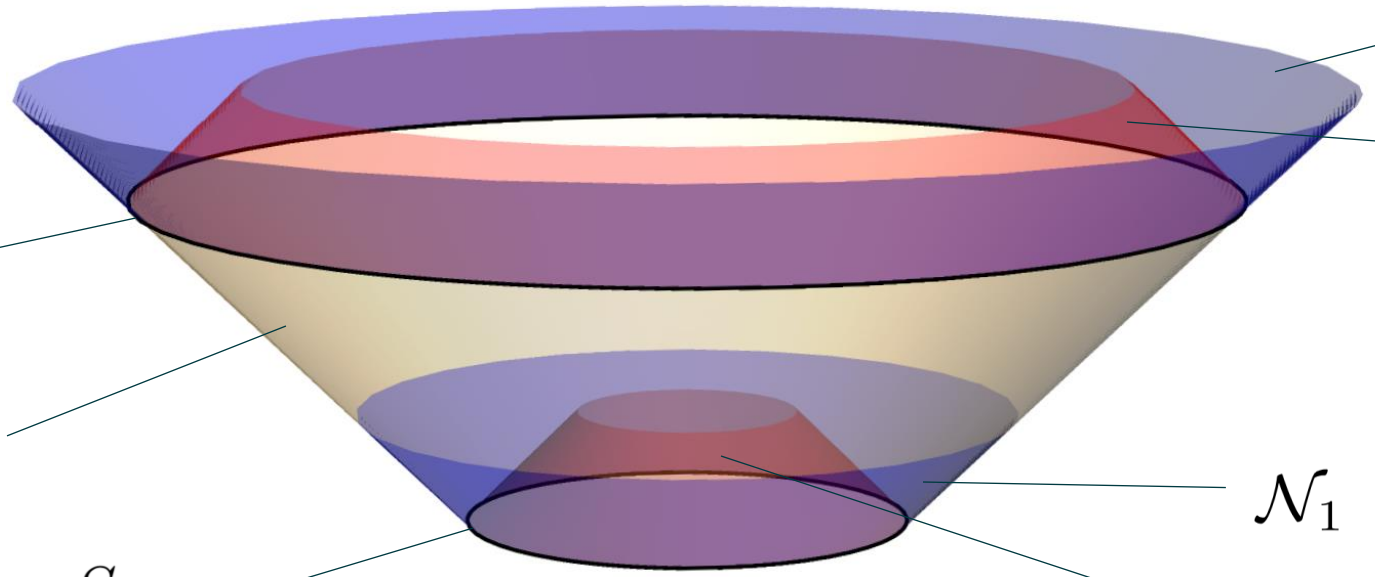
$\underline{\mathcal{N}}_1$

null surfaces in $\mathcal{M}_1$

$S_1,\ x_1$

C^k sphere data: set of all metric components and their derivatives up to kth order, on the sphere

Characteristic gluing problem



sphere & sphere data: $S_2,\ x_2$

gluing surface: $\mathcal{N}$

$S_1,\ x_1$

$\mathcal{N}_2$

$\underline{\mathcal{N}}_2$

null surfaces in $\mathcal{M}_2$

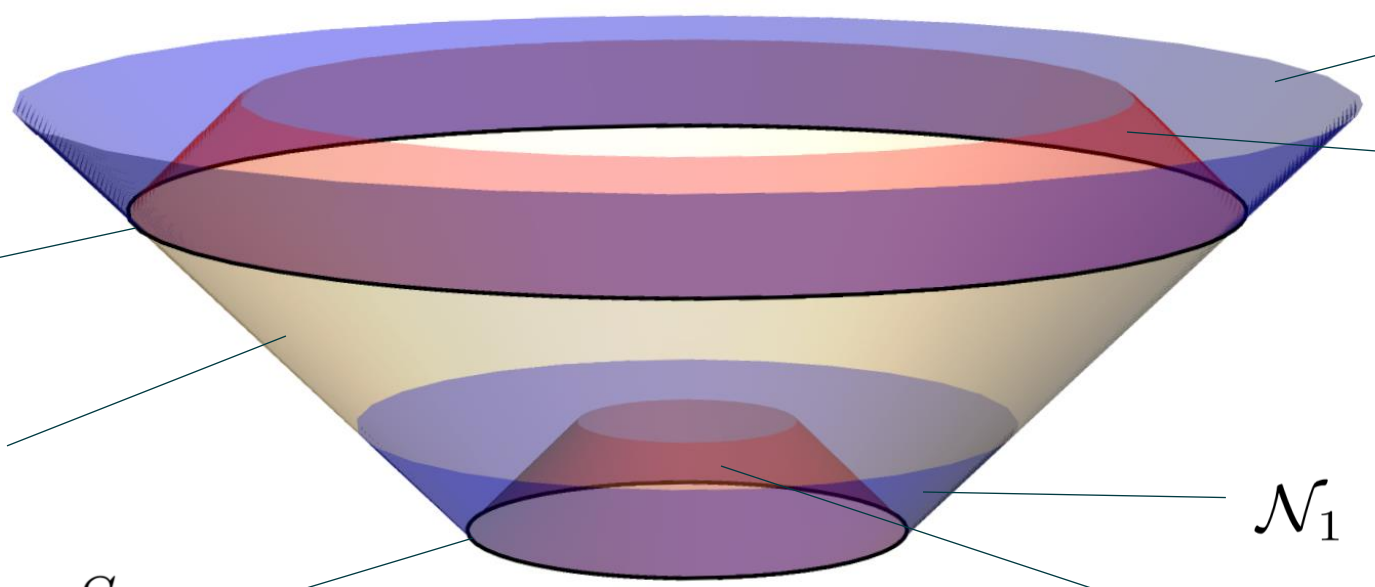
$\mathcal{N}_1$

$\underline{\mathcal{N}}_1$

null surfaces in $\mathcal{M}_1$

C^k sphere data: set of all metric components and their derivatives up to kth order, on the sphere

Characteristic gluing problem

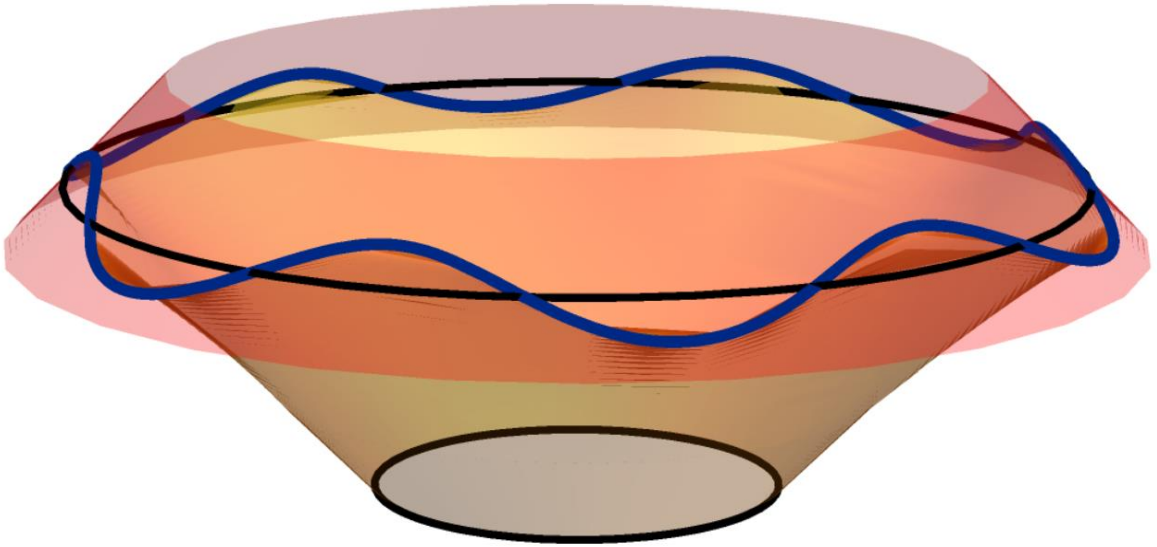


C^k sphere data: set of all metric components and their derivatives up to kth order, on the sphere

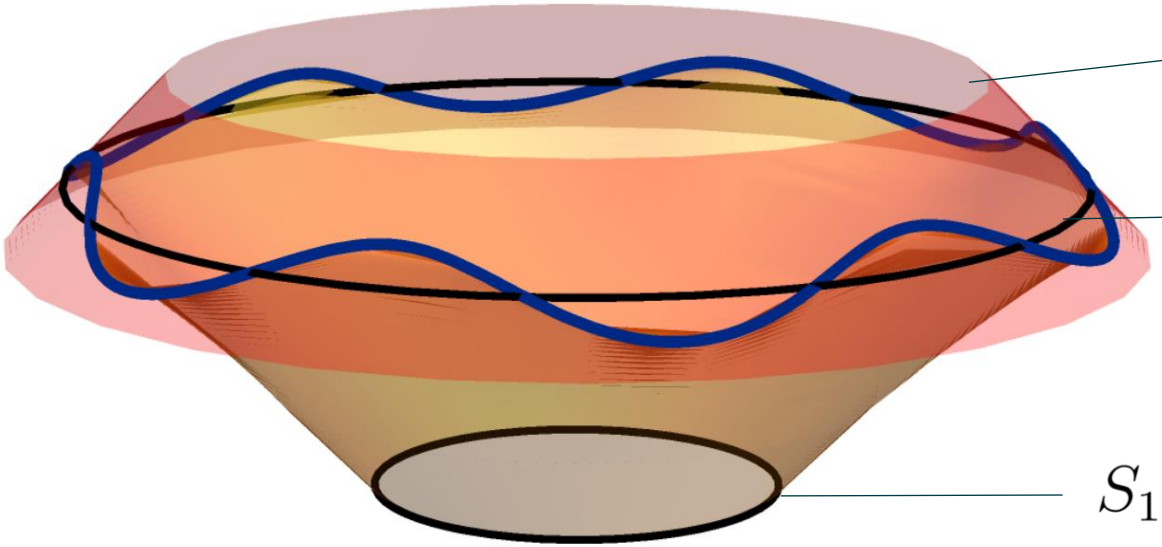
Characteristic gluing: connecting $\mathcal{N}_1$ and $\mathcal{N}_2$ using a surface $\mathcal{N}$ with the dataset x, such that

1. The null constraint equations are satisfied everywhere along $\mathcal{N}_1 \cup \mathcal{N} \cup \mathcal{N}_2$
2. $x|_{S_1} = x_1$ and $x|_{S_2} = x_2$

Perturbative characteristic gluing problem



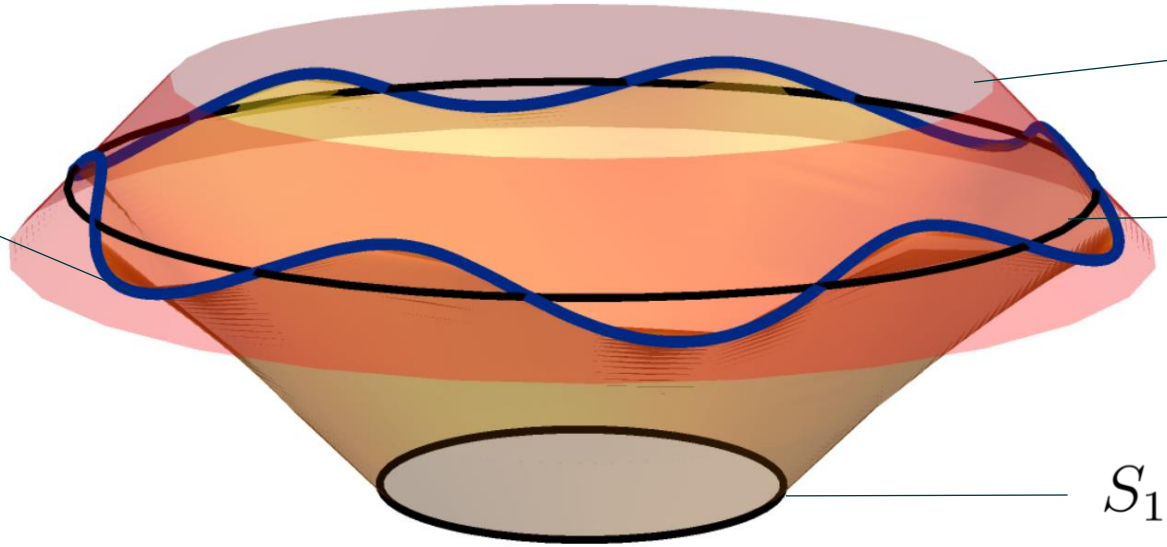
Perturbative characteristic gluing problem



Perturbative characteristic gluing problem

perturbed sphere & sphere data:

$S_2', \; x_2'$



$\mathcal{N}_2$

null surface in $\mathcal{M}_2$

S_2

$S_1, \; x_1$

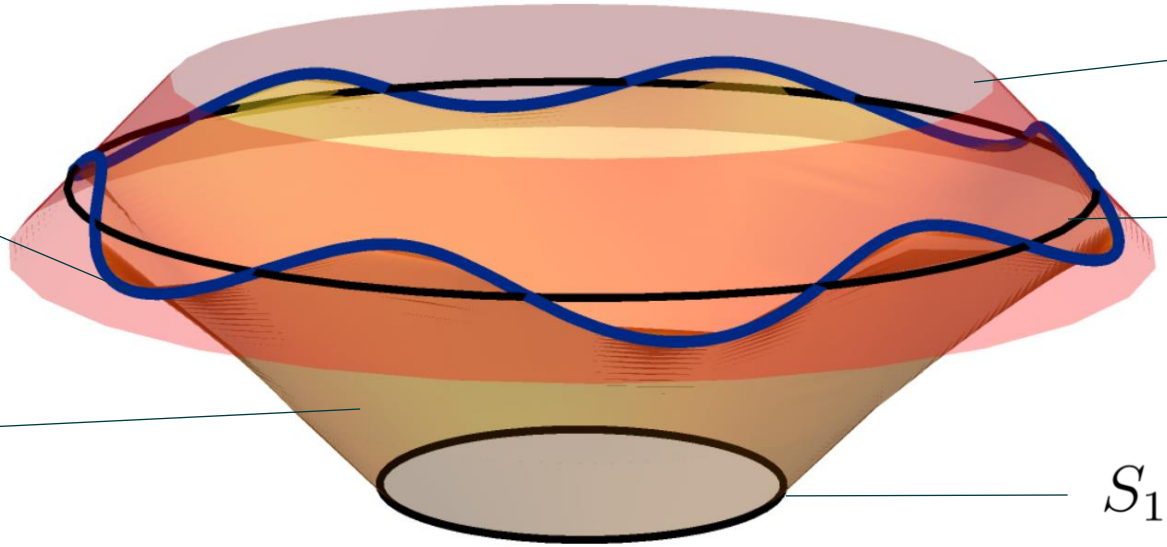
Perturbative characteristic gluing problem

perturbed sphere & sphere data:

$S_2', \; x_2'$

gluing surface:

$\mathcal{N}'$



$\mathcal{N}_2$ null surface in $\mathcal{M}_2$

S_2

$S_1, \; x_1$

Characteristic gluing: connecting $\mathcal{N}_1$ and $\mathcal{N}_2$ using a surface $\mathcal{N}'$ with the dataset x, such that

1. The null constraint equations are satisfied everywhere along $\mathcal{N}_1 \cup \mathcal{N}' \cup \mathcal{N}_2$
2. $x|_{S_1} = x_1$ and $x|_{S_2'} = x_2'$

Obstructions to gluing

One **cannot** always find an x dataset that satisfies the criteria

Obstructions: conservation laws arising from the null constraint equations

Gauge-invariant obstructions: a finite number of charges, that are invariant of the perturbations

Gauge-dependent obstructions: an infinite number of charges, which can be removed with suitably chosen sphere perturbations

2. Conserved radial charges

Bondi metric

Describes a general asymptotically flat spacetime in the $x^\mu = (u,\ r,\ x^A)$ coordinates:

$$g_{\mu\nu}dx^\mu dx^\nu = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big)$$

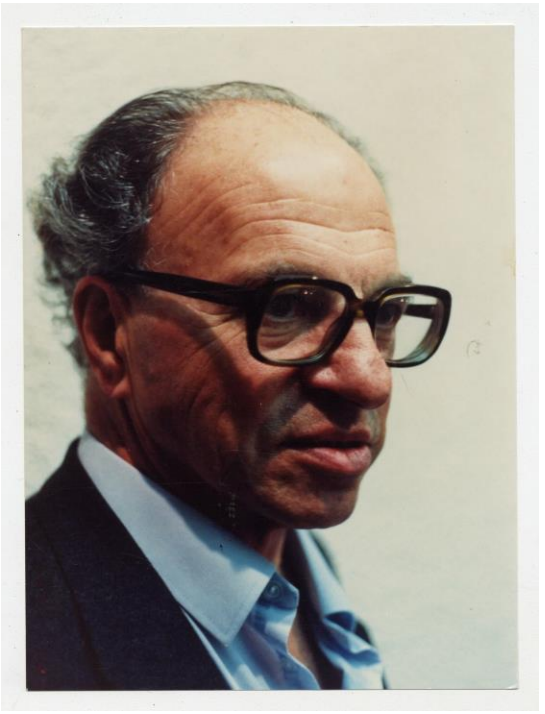
A spacetime is asymptotically flat at null infinity if it admits to the expansion:

$$\gamma_{AB} = \mathring{h}_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(r^{-2}) \ ,$$

$$U^A = -\frac{\mathring{D}_B C^{AB}}{2r^2} + \frac{1}{r^3}\Big(-\frac{2}{3}N^A + \frac{1}{16}\mathring{D}_A(C^{BC}C_{BC}) + \frac{1}{2}C^{AB}\mathring{D}^C C_{BC}\Big) + \mathcal{O}(r^{-4}) \ ,$$

$$V = r - 2M + \mathcal{O}(r^{-1}) \ ,$$

$$\beta = -\frac{1}{32}\frac{C^{AB}C_{AB}}{r^2} + \mathcal{O}(r^{-3}) \ .$$



Hermann Bondi [2]

C_{AB} bondi shear

N^A angular momentum aspect

M mass aspect

Radial charges

From the null constraint equation one can derive conservation laws for quantities

1. With an arbitrary Y^A conformal Killing vector of the unit 2-sphere:

$$\overset{[1]}{Q}(Y^A) := \int d^2\Omega \; Y^A \left[r^4 e^{-2\beta} \gamma_{AB} (\partial_r U^B) + 2r^2 D_A \beta \right]$$

2. If the function $\lambda(x^A)$ is an arbitrary linear combinantion of constants and $\ell \leq 1$ spherical harmonics, we have:

$$\overset{[2]}{Q}(\lambda) := \int d^2\Omega \; \lambda \left(V e^{-2\beta} - \frac{r}{2} e^{-2\beta} D^A \partial_r (r^2 U_A) - r D^2 \beta \right)$$

Along any u = *const.* hypersurface

$$\partial_r \overset{[1]}{Q} = 0 \qquad\qquad \partial_r \overset{[2]}{Q} = 0$$

Radial charges

Relevance to the gluing problem: they are gauge-idependent obstructions of the 4-dimensional C^2 gluing when $m = 0$.

The gluing is only possible when the two charges evaluated on S_1 and S_2 equipped with the sphere data x_1 and x_2 satisfy:

$$\overset{[1]}{Q}[x_1](Y^A) = \overset{[1]}{Q}[x_2](Y^A) \qquad \text{and} \qquad \overset{[2]}{Q}[x_1](\lambda) = \overset{[2]}{Q}[x_2](\lambda)$$

3. ADM charges

ADM charges

Arnowitt, **D**eser and **M**isner: 3+1 foliation with $t = const$ surfaces, Hamiltonian dynamics of GR

1. Energy

Total energy (matter + gravitational field) of an asymptotically flat spacetime

$$E_{ADM} := -\lim_{r\to\infty} \frac{1}{8\pi} \int_{S^2} (R_{ab} - Rh_{ab}) X^b n^a dA \qquad X = x^a \partial_a$$

2. Angular momentum

The ADM angular momentum is defined as

$$(L_{ADM})_a = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (K_{bc} - \mathrm{tr}K h_{bc})(\phi_{(a)})^c n^b \, dA$$

where the conformal Killing vector $(\phi_{(a)})_b = \varepsilon_{acb} x^c$ is a rotation field in $\mathbb{R}^3$.

Comparison between ADM and radial charges

1. ADM energy and $\overset{[2]}{Q}(1)$

The ADM energy is

$$E_{ADM} = \lim_{r\to\infty}\left(M + \mathcal{O}(r^{-1})\right) = M$$

assuming M = *const*. Whereas the 2nd radial charge after an asymptotic series expansion

$$\lim_{r\to\infty}\overset{[2]}{Q}(1) = \lim_{r\to\infty}\int d^2\Omega\left(-2M + \mathcal{O}(r^{-1})\right) = -8\pi M$$

Therefore it is clear that

$$\lim_{r\to\infty}\overset{[2]}{Q}(1) \propto E_{ADM}$$

Comparison between ADM and radial charges

2. ADM angular momentum and $\overset{[1]}{Q}(H^A_{(1,m)})$

Magnetic spherical harmonics:

$$H_A^{(\ell,m_a)} = \frac{1}{\sqrt{\ell(\ell+1)}} * D_A Y^{(\ell,m_a)}$$

$*$: Hodge dual

The non-linear forms do not match, so let's try

- Linearizing around Minkowski $g_{\mu\nu} \to g_{\mu\nu} + \epsilon h_{\mu\nu}$

- Choosing the gauge where $h_{ur} = 0 \;\Rightarrow\; \delta\beta = 0$

Comparison between ADM and radial charges

2. ADM angular momentum and $\overset{[1]}{Q}(H^A_{(1,m)})$

Thus the ADM angular momentum is

$$L_{ADM} = \sqrt{\frac{2\pi}{3}} \int d^2\Omega \left(r^2 \partial_r h_{uA} - 2r h_{uA} \right) H^A_{(1,m_a)}$$

And the first radial charge in the $\delta\beta = 0$ gauge is

$$\overset{[1]}{Q}(H^A_{(1,m_a)}) = - \int d^2\Omega \left(r^2 \partial_r h_{uA} - 2r h_{uA} \right) H^A_{(1,m_a)}$$

Thus

$$\overset{[1]}{Q}(H^A_{(1,m_a)}) = -\sqrt{\frac{3}{2\pi}} L_{ADM}$$

4. BMS charges

BMS charges

Bondi-van der **B**urg-**M**etzner-**S**achs group: symmetry group of asymptotically flat spacetimes at $\mathcal{I}$

- includes the isometry group of Minkowski spacetime, the Poincaré group
- and an infinite number of supertranslations (generalization of the 4 Minkowski translations) shifts u independently at every angle on $\mathcal{I}$

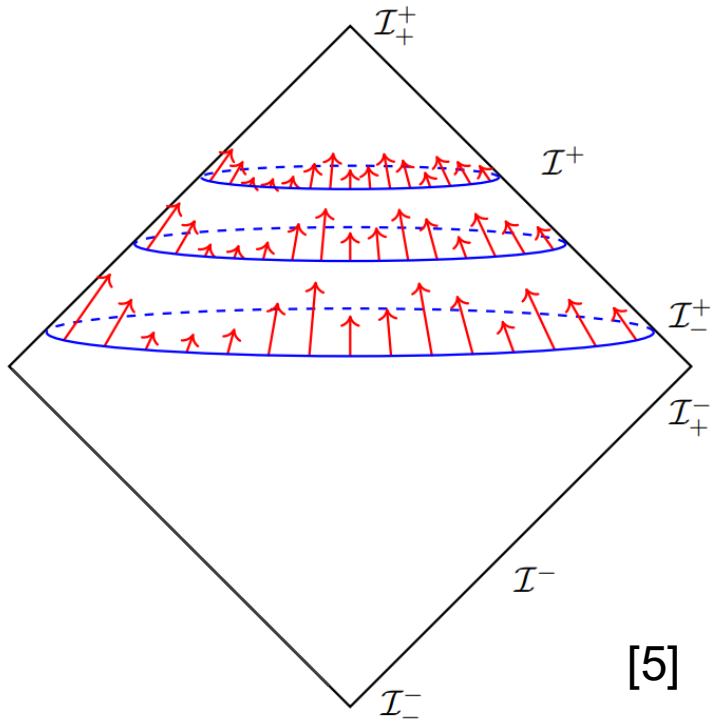
There exists a charge for every generator and each cross section of $\mathcal{I}^+$

On a u = *const* cut the charge associated with the generator

is

$$\vec{\xi_1} = \frac{1}{2}(u-u_0)D_A Y^A\,\partial_u + Y^A\partial_A - \frac{1}{2}rD_A Y^A\partial_r$$

$$\mathcal{Q}(\vec{\xi_1}) = \frac{1}{16\pi}\int d^2\Omega\,\left(\frac{1}{2}r^2 D_A Y^A + r^3 D_A Y^A \partial_r\beta - r^4\,e^{-2\beta}\,\gamma_{AB}\,Y^A\,\partial_r U^B\right)$$



[5]

Comparison of BMS and radial charges

Let's compare the BMS charge with $\overset{[1]}{Q}(Y^A)$

After linearization in the $\delta\beta = 0$ gauge

$$\mathcal{Q}(\vec{\xi}_1) = \frac{1}{16\pi}\int d^2\Omega\Big(r^2\partial_r h_{uA} - 2rh_{uA}\Big)Y^A$$

$$\overset{[1]}{Q}(Y^A) = -\int d^2\Omega\Big(r^2\partial_r h_{uA} - 2rh_{uA}\Big)Y^A$$

match

Summary

1. The charge $\overset{[2]}{Q}(1)$ is proportional to the ADM energy in the asymptotic limit.

2. The linear charge $\overset{[1]}{Q}(H^A_{(1,m_a)})$ is proportional to the linear ADM angular momentum.

3. The linear charge $\overset{[1]}{Q}(Y^A)$ is proportional to the $\mathcal{Q}$ BMS charge.

Thank you for your attention!

References

[1] Robert Sansom, The null gluing problem for the Einstein equations, Seminar on Mathematical General Relativity Sorbonne Université, 19th December, 2024

[2] Humanist Heritage webpage, https://heritage.humanists.uk/hermann-bondi-1919-2005/, 2026.

[3] Techincal University of Munich, Department of Mathematics webpage, https://www.math.cit.tum.de/en/math/department/history/, 2026.

[4] University of Vienna Memorial Book webpage, https://gedenkbuch.univie.ac.at/en/person/hans-thirring, 2026.

[5] Andrew Strominger. Lectures on the infrared structure of gravity and gauge theory, p. 67, 2018.

[6] Éanna É. Flanagan and David A. Nichols. Conserved charges of the extended Bondi-Metzner-Sachs algebra. Physical Review D, 95(4), 2017.

[7] Piotr T. Chruściel and Wan Cong. Characteristic gluing with λ I. Linearised Einstein equations on four-dimensional spacetimes, 2024.

Appendix

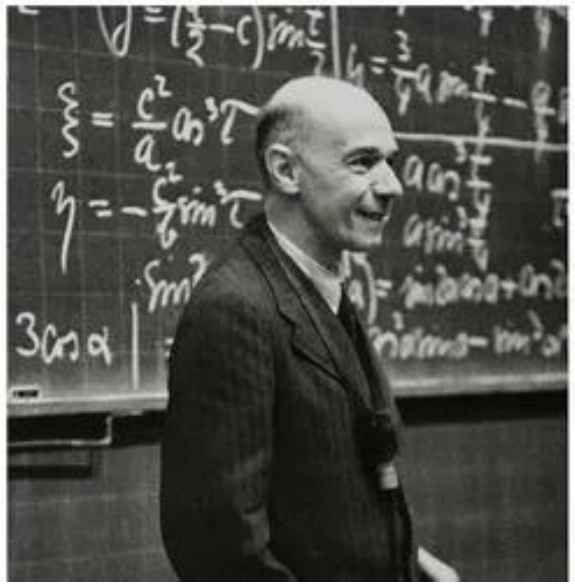
A. Charges in slowly rotating Kerr spacetime

Charges in slowly rotating Kerr spacetime

Before the exact stationary, axis-symmetric solution by Kerr, Lense and Thirring gave an appoximate solution

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + r^2\sin^2\theta\left(d\phi - \frac{2Ma}{r^3}dt\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2$$

It can be regarded as the small a limit of the Kerr metric as $g_{\mu\nu} \approx g_{\mu\nu}(a) + \mathcal{O}(a^2)$





Josef Lense and
Hans Thirring [3,4]

Charges in slowly rotating Kerr spacetime

The ADM charges of Lense—Thirring spacetime:

$$E_{ADM} = \lim_{r\to\infty} M\,\frac{8M^3a^2(4M-r)+3r^3(2M+r)(4M+r)}{3r^{5/2}(2M+r)^{5/2}} = M$$

$$(P_{ADM})_a = 0$$

$$(C_{ADM})_a = 0$$

$$(L_{ADM})_r = 0\ , \qquad (L_{ADM})_\theta = 0\ , \qquad (L_{ADM})_\varphi = \lim_{r\to\infty} a\left(M + \frac{8M^3}{3(2M+r)^2}\right) = aM$$

And the radial charges:

$$\overset{[2]}{Q}(1) = -8\pi M$$

$$\overset{[1]}{Q}(H^A_{(1,0)}) = -\sqrt{\frac{3}{2\pi}}aM$$

B. Transport equations

Further transport equations

From additional null constraint equations one can find more gauge-dependent obstructions

Gauge-dependent: can be removed with suitably chosen sphere perturbations

Goal: find the obstruction of the continuous gluing of $\partial_u h_{uA}$

Further transport equations

From additional null constraint equations one can find more gauge-dependent obstructions

Gauge-dependent: can be removed with suitably chosen sphere perturbations

Goal: find the obstruction of the continuous gluing of $\partial_u h_{uA}$

From the transport equation $\partial_r \overset{(1)}{H}_{uA} = \frac{3}{4r^5}\mathring{D}^B\Big(P + 2\varepsilon\Big)h_{AB}$, where $P(h)_{AB} = \Big(\frac{1}{2}\mathring{D}^2 - \varepsilon\Big)h_{AB}$ and

$$\overset{(1)}{H}_{uA} = 3\partial_u \check{h}_{uA} + r\partial_r\partial_u\check{h}_{uA} - \frac{1}{2r^3}\partial_u\mathring{D}^B h_{AB} + \frac{3}{8}r^{-8/3}\partial_r\Big(r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A(\check{h}_{uB})]\Big)$$
$$- \frac{V}{4r^4}\mathring{D}^B\partial_r h_{AB} - \frac{V}{4r^5}\mathring{D}^B h_{AB} - \frac{3}{8r^4}\mathring{D}^B P(h)_{AB}$$

we obtain the charge [6]

$$\mathcal{Q}_1(\mu^A) = \int_{S^2} d^2\Omega\ \mu^A \overset{(1)}{H}_{uA} \qquad \mu^A : \ell = 1, 2 \text{ SH}$$

Further transport equations

Let's find the non-linear $\overset{(1)}{H}_{uA}$. From the derivation of the appropriate transport equation we find

$$\overset{(1)}{H}_{uA} := -\partial_u \overset{(0)}{H}_{uA} + \frac{1}{2r} D^B \partial_u \gamma_{AB} + \frac{3}{8} r^{-8/3} \partial_r (r^{8/3} D^B \mathrm{TS}[D_A U_B]) + \frac{1}{4r^2} D^B \partial_r (V \partial_r \gamma_{AB}) - \frac{3e^{2\beta}}{4r^2} D^B \mathrm{TS}\Big[D_A D_B \beta\Big]$$

where

$$\overset{(0)}{H}_{uA} := -\frac{4}{r} D_A \beta + 3e^{-2\beta} U_A + r e^{-2\beta} \gamma_{AB} \partial_r U^B$$

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$$+ \frac{1}{4r^2} D^B \partial_r (V \partial_r \gamma_{AB}) - \frac{3e^{2\beta}}{4r^2} D^B \mathrm{TS}\Big[D_A D_B \beta\Big]$$

where

$$\overset{(0)}{H}_{uA} := -\frac{4}{r} D_A \beta + 3e^{-2\beta} U_A + r e^{-2\beta} \gamma_{AB} \partial_r U^B$$

After asymptotic series expansion, the first non-zero order is

$$\overset{(1)}{H}_{uA} = \left[\frac{1}{2} N^{BC} \mathring{D}_B C_{AC} - \frac{1}{16} \underbrace{\left(\mathring{D}^C \mathring{D}_C + 5\right)}_{\text{annihilates } \ell = 1,2 \text{ SH-s}} \mathring{D}^B C_{AB}\right] r^{-3} + \mathcal{O}(r^{-4})$$

news tensor:

$$N_{AB} = \partial_u C_{AB}$$

recall:

$$\partial_r \overset{(1)}{H}_{uA} = \frac{3}{4r^5} \mathring{D}^B \Big(P + 2\varepsilon\Big) h_{AB}$$

The residual charge associated with the C^3 gluing problem is

$$\mathcal{Q}_1(\mu^A) = \frac{1}{2r^3} \int d^2\Omega \; \mu^A \Big(N^{BC} \mathring{D}_B C_{AC}\Big) + \mathcal{O}(r^{-4})$$