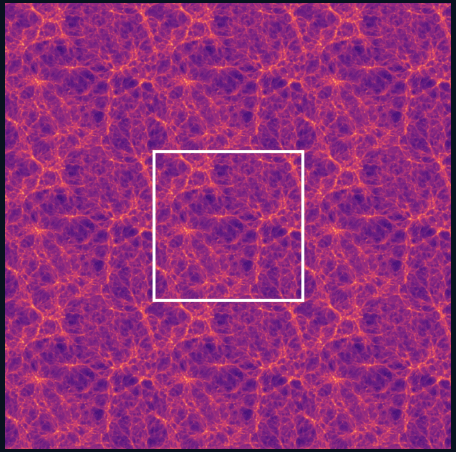
Escaping the Torus of Numerical Cosmology

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FIKUT NEXTGEN 2026, Budapest

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Virtually all simulations you have seen live on a torus



Particles leave the box through one face and re-enter through the opposite one. That gives space the topology of a **3-torus**:

$$\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

The choice is rarely stated, but it sits in the equation of motion:

$$m_i \ddot{\boldsymbol{x}}_i = \underbrace{\sum_{j \neq i} \frac{m_i m_j \boldsymbol{F}_{ij}}{a^3}}_{\text{gravity}} \underbrace{- 2 m_i H \dot{\boldsymbol{x}}_i}_{\text{expansion}} + \underbrace{\text{boundary term}}_{\text{topology}}$$

Simulation studies vary the first two terms all the time, but hardly ever the third.

Why did the torus win?

Good reasons

- The Poisson equation is fast to solve with FFTs on a periodic mesh.
- Without edges, every point is equivalent.
- Linear momentum is conserved.
- Resolution is uniform and the mean density is fixed.

Four decades of infrastructure

1985 Efstathiou et al. compare force solvers "in the fundamental cube of an infinite periodic universe".

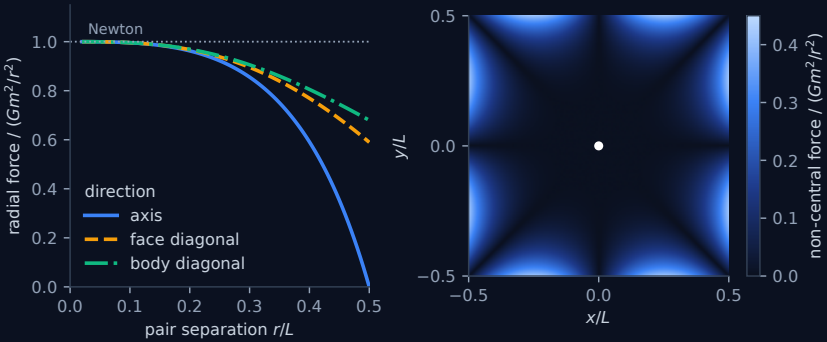
2005 gadget-2 and the Millennium run.

Now Most codes, initial condition generators and analysis tools assume the cube.

A practical choice

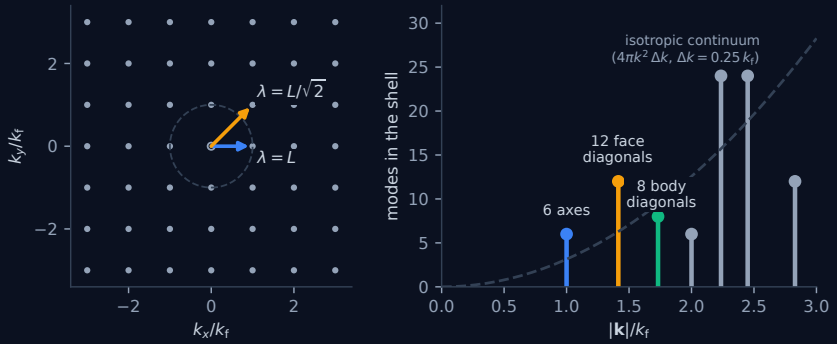
The torus was a sensible choice, and much good science has been done with it. It suits the computer, but no observation points to it. So what does it cost?

Cost 1: gravity stops being Newtonian



- Summing over the periodic images weakens the radial force by different amounts in different directions. At $r = 0.4\,L$ it is 59% of Newton's value along an axis and 81% along a body diagonal.
- The non-central force component acts as a torque, so **angular momentum is not conserved**.

Cost 2: only some wavelengths are allowed



- Waves must fit the box, so large-scale power is cut off differently by direction: L along an axis, $L/\sqrt{2}$ and $L/\sqrt{3}$ along the diagonals. The mean-density mode $k = 0$ is missing.
- Rácz et al. (2021) find an overall **negative, anisotropic bias** in periodic simulations.

When do the artefacts matter?

Usually safe

- Scales well below the box size, since at $r = 0.1\,L$ the force error is under 1%.
- Angle-averaged statistics in large boxes.

Expect trouble

- In small boxes, such as high-resolution hydro runs, the largest mildly non-linear scales approach L (Rácz et al. 2021). High-z ΛCDM needs larger boxes than usual (Bagla & Ray 2005).
- The box fixes the mean density and has no modes beyond its size, so σ_8 is underestimated (Sirko 2005) and super-sample effects are missing (Takada & Hu 2013).
- We expect tests of isotropy, spin alignment or rotation to be most exposed, since the artefact itself has a direction.
- Lightcones built by replicating the box repeat structures. The resulting bias is mostly within the errors, except for high-order statistics on small sky areas (Chen & Yu 2024).

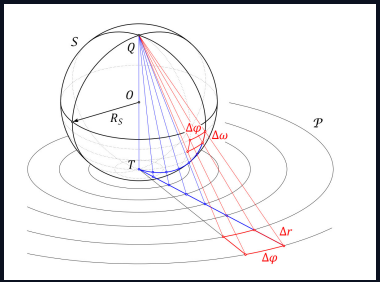
What should a better topology provide?

Boundaries without an artificial preferred direction, a force law that is Newtonian locally and consistent with the topology globally, initial conditions with the statistics of that topology, and a cost we can afford.

	$\mathbb{T}^3$ **cube**	$\mathbb{R}^3$ **sphere**	$\mathbb{S}^1 \times \mathbb{R}^2$ **cylinder**	$\mathbb{S}^3/I^*$ **dodecahedron**
Symmetry	translations, $\mathrm{O_h}$	$\mathrm{SO}(3)$ about the centre	axial translations, $\mathrm{SO}(2)$ about the axis	homogeneous: every point equivalent
Periodic images	infinite lattice	none	infinite chain	119, summed exactly
Boundary	periodic	open, homogeneous exterior	periodic axis, open radially	closed, faces glued with a twist
Longest modes	box sides	enclosing box	L_z axial, box transverse	discrete modes of $\mathbb{S}^3/I^*$
Mean density	fixed	free	free	fixed
Mass resolution	uniform	falls with radius	falls away from the axis	uniform on $\mathbb{S}^3$
Spatial curvature	none	none	none	positive

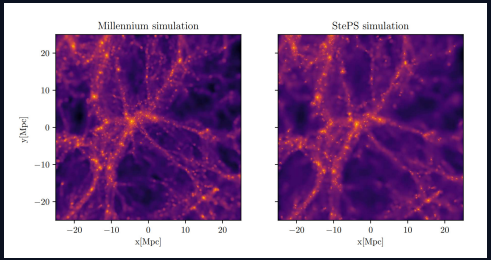
Every topology gives something up, so the choice should depend on the question.

Example 1: the sphere and the stereographic projection



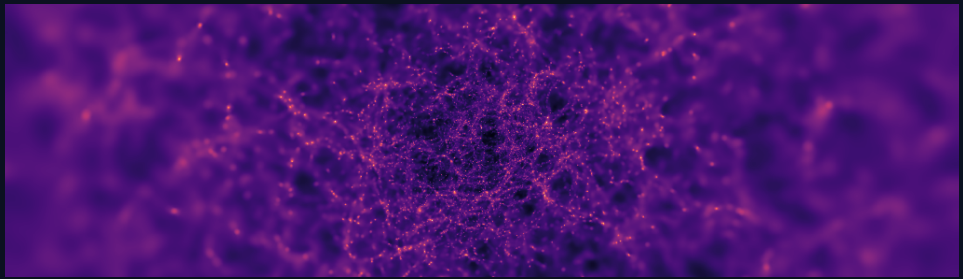
$$\omega = 2\arctan\left(\frac{r}{2R_\mathrm{S}}\right)$$

Inverse **stereographic projection** maps all of $\mathbb{R}^3$ onto a finite hypersphere. This is the basis of StePS, *STEreographically Projected cosmological Simulations*.



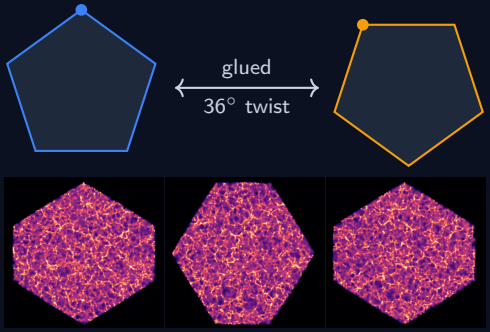
- There are no images and the space is rotationally symmetric about the centre, but resolution falls with radius.
- Rerunning Millennium with the same parameters and seed took 106 h on 12 GPUs, against 683 h on 512 processors. The structures match at about 1.6× coarser central resolution (Rácz et al. 2018, 2019).

Example 2: the cylinder, $\mathbb{S}^1 \times \mathbb{R}^2$



- The space is periodic along one axis and open in the radial direction, which is compactified stereographically. Rotations about the axis and translations along it remain symmetries (Rácz et al. 2026).
- Voids, filaments and halos form in the high-resolution region around the axis, and resolution declines toward the edge (Pál et al. 2026).

Example 3: the Poincaré dodecahedron, $\mathbb{S}^3/I^*$



- Gluing opposite faces of a spherical dodecahedron gives the Poincaré dodecahedral space, which is closed and positively curved.
- Every particle has exactly 119 images, so the force sum is finite and exact and needs no Ewald table.
- The direction-dependent image force starts at $(r/R)^5$, against $(r/L)^3$ in $\mathbb{T}^3$ (Roukema & Różański 2009). Curvature generally matters more than topology, though (Vigneron & Roukema 2023).

Test configuration

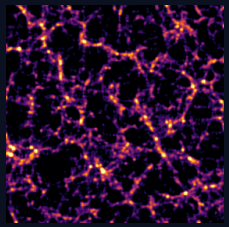
The run uses $\mathbb{S}^3/I^*$ forces on a flat Friedmann background. It is a numerical experiment, not a model of our Universe, and to our knowledge the first ΛCDM N-body run in this topology.

The cosmic web forms there too

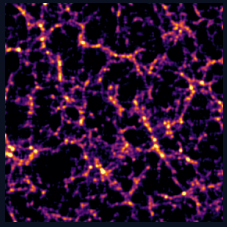
1.2 Gpc

same seed, $z = 0$:

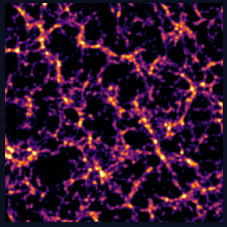
the same web forms



$\mathbb{T}^3$, 256^3



$\mathbb{S}^3/I^*$, 256^3

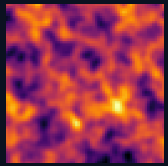


$\mathbb{S}^3/I^*$, 1024^3 glass

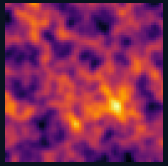
50 Mpc

same seed:

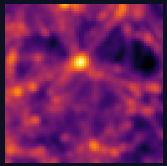
the runs start alike and diverge by $z = 0$



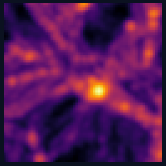
$\mathbb{T}^3$, $z = 30$



$\mathbb{S}^3/I^*$, $z = 30$



$\mathbb{T}^3$, $z = 0$



$\mathbb{S}^3/I^*$, $z = 0$

Takeaways

Why the torus?

We adopted it forty years ago because computers needed it. It is an ingenious convenience, but importantly, not a law of nature!

When does it matter?

Above all in small boxes, and many widely used high-resolution runs are small, from TNG50 to the thousands of CAMELS boxes. There the box distorts the clustering of matter in which their galaxies form. Anything that measures a direction is probably exposed as well.

What else is there?

The cosmic web grows in spheres, cylinders, and dodecahedra just as well. Every boundary demands a sacrifice, so let your question choose the shape instead of blindly following the conventions.

The periodic cube is an excellent tool, but it is a choice we should state.

Thank you for your attention!

Backup slides

Nothing observational asks for this

What the CMB says

No matched circle pairs in WMAP or Planck, so the shortest loop through us is longer than 98.5% of the last-scattering diameter (Petersen et al. 2023).

For a cubic torus, Planck finds a similar limit: inscribed radius $R_{\rm i} > 0.97\,\chi_{\rm rec}$ at 99% CL.

That diameter is about 28 Gpc, far larger than a typical simulation box.

The fair objection

"Nobody thinks the Universe is a torus. It is only a boundary condition. Just use a bigger box."

This often works, because errors at a fixed scale shrink as L grows.

At fixed mass resolution, however, $N \propto L^3$, and scales close to L stay affected whatever L is.

So the useful question is when the torus starts to matter.

Ewald summation in a periodic cube

One particle represents an infinite lattice of copies, with the mean density removed. For $L = G = m = 1$:

$$\boldsymbol{a}(\boldsymbol{r}) = -\sum_{\boldsymbol{n}} \frac{\boldsymbol{r}-\boldsymbol{n}}{|\boldsymbol{r}-\boldsymbol{n}|^3}\left[\operatorname{erfc}(\alpha\rho_{\boldsymbol{n}}) + \frac{2\alpha\rho_{\boldsymbol{n}}}{\sqrt{\pi}}\, e^{-\alpha^2\rho_{\boldsymbol{n}}^2}\right] - 2\sum_{\boldsymbol{h}\neq 0} \frac{\boldsymbol{h}}{h^2}\, e^{-\pi^2 h^2/\alpha^2} \sin(2\pi\, \boldsymbol{h}\cdot\boldsymbol{r}), \quad \rho_{\boldsymbol{n}} = |\boldsymbol{r}-\boldsymbol{n}|$$

Near the particle

$\boldsymbol{a} \simeq -\boldsymbol{r}/r^3 + \frac{4\pi}{3}\boldsymbol{r} + \mathcal{O}(r^3)$. The linear term is isotropic, so the direction dependence only starts at r^3.

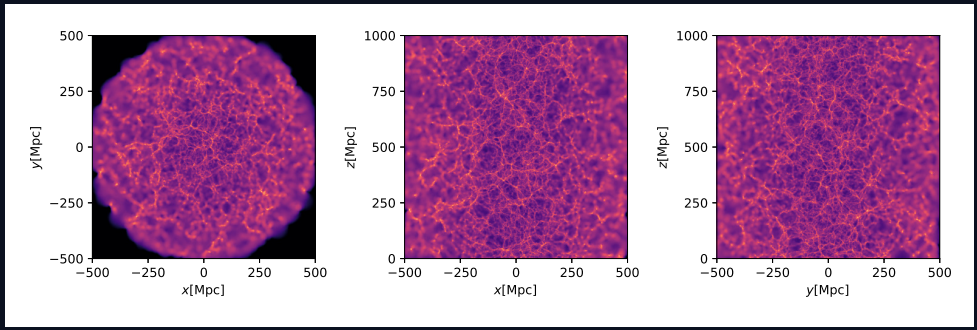
Checks behind the figure

The result is independent of α to 10^{-13}, vanishes at the three symmetry points, and matches a direct lattice sum with a neutralising background to five decimals.

Conservation laws by topology

	Lin. mom.	Ang. mom.	Symmetry	Periodic correction
$\mathbb{T}^3$ (periodic cube)	✓ all 3	×	$\mathrm{O_h}$	3D Ewald
$\mathbb{R}^3$ (spherical StePS)	×	✓ full $\mathrm{SO}(3)$	$\mathrm{SO}(3)$	none
$\mathbb{S}^1 \times \mathbb{R}^2$ (cylindrical)	✓ z-axis	✓ z-comp.	$\mathrm{SO}(2) \times$ transl.	1D Ewald

The first cylindrical run: parameters



Planck 2018 ΛCDM · $R_{\rm sim} = 500$ Mpc · $L_z = 1$ Gpc · 2.4×10^7 particles · $z_{\rm init} = 49$ (2LPT) · 640 CPU cores · Rácz et al. 2026

A topology test on the cylinder, and a null result

Shamir (2025) reported a galaxy-spin asymmetry in JWST/JADES data. Since gravity on the cylinder is anisotropic by construction, could the topology produce such a signal?

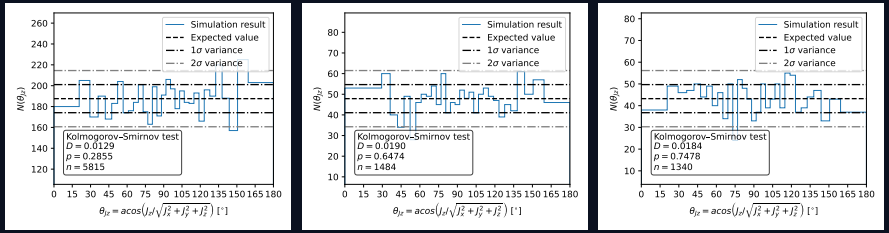


Figure 1: Halo spin orientation θ_{J_z} relative to the periodic axis, three mass bins at $z = 0.5$.

Null result

The cylinder topology does not produce the asymmetry reported from JWST.

The $\mathbb{S}^3/I^*$ test configuration

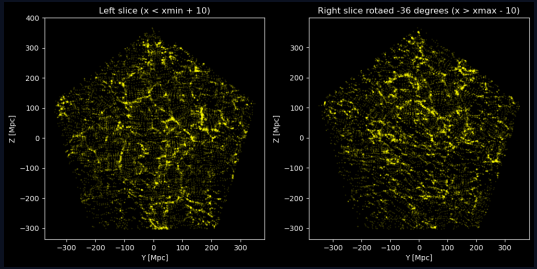
Setup

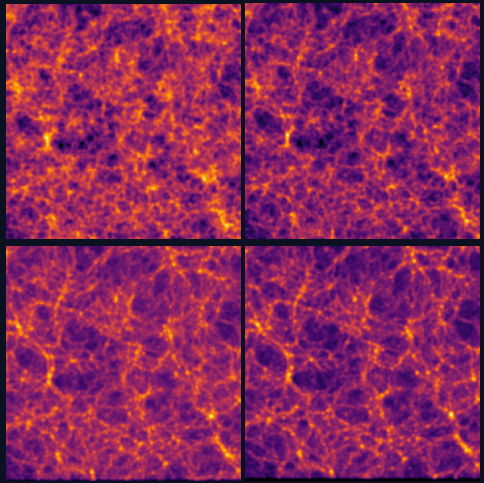
- Curvature radius $R = 3100$ Mpc, used here as a test configuration (129 Mpc for the 50 Mpc run).
- Geodesic inradius $\frac{\pi}{10}R \approx 970$ Mpc, or $R \tan \frac{\pi}{20} \approx 490$ Mpc in the stereographic chart. Slices are shown in the chart.
- Same seed as the Gadget-4 $\mathbb{T}^3$ reference, 256^3 and 1024^3 in a 1.2 Gpc cube, clipped to the domain.

Caveats

- Expansion is flat ($\Omega_\mathrm{m} + \Omega_\Lambda = 1$), so R is not tied to the background. This is not a self-consistent closed cosmology.
- The default integrator drifts particles in the flat chart, an approximation of order $(r/R)^2$.
- The code is experimental and the results are unpublished.

$\mathbb{S}^3/I^*$: opposite faces and evolution





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