Eötvös Loránd University, Doctoral School of Physics
HUN-REN Wigner Research Centre for Physics

Volume Concepts
in the Thermodynamics of Black Holes

Réka Somogyfoki
somogyfoki.reka@wigner.hu
joint work with Péter Ván

2026-11-06

Introduction
Why do black holes need a volume?

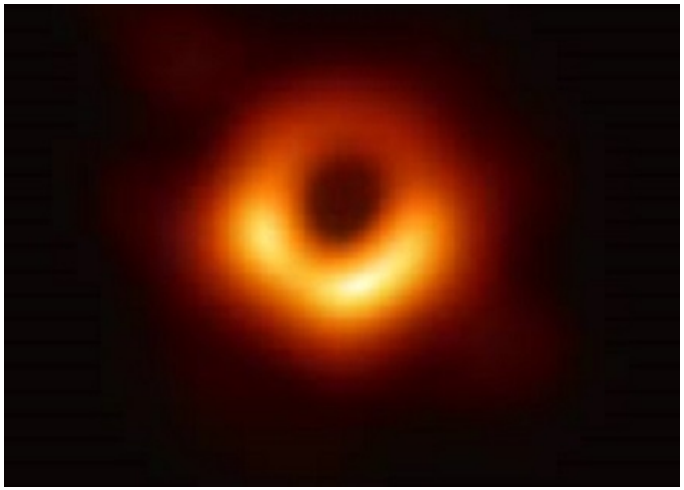


Image from Event Horizon Telescope project

Three problems, one origin

- **Geometry:** volume in GR is slicing dependent; the horizon only defines an *area*
- **Extensivity:** $S \propto M^2$ is not first-order homogeneous
- **Stability:** negative heat capacity [1]

$$S = 4\pi U^2 \Rightarrow -\frac{1}{C_V T^2} = 8\pi \Rightarrow C_V < 0$$

Can a thermodynamic volume restore extensivity and stability?

Volume concepts

Many roads to a black hole volume

Approach	**Volume**	Λ	**Extensivity / stability**
Hawking–Page [2]	none	fixed	non-extensive; $C < 0$ for small black holes
Extended, geometric [3, 4]	$V = \frac{4\pi}{3} r_S^3$	pressure $P = -\Lambda/8\pi$	S and V degenerate; radiation law violated
Thermodynamic [1, 5]	$V \propto r_S^5$ (Christodoulou–Rovelli)	generalized pressure	Euler-homogeneous, concave S
Holographic, boundary [6]	area of a York boundary	fixed	extensive for large black holes
Restricted phase space [7]	no P–V pair (central charge C)	fixed	first-order homogeneous mass

Volume concepts
No volume vs. geometric volume

No volume: $M = U$ [2]

$\Lambda = -3/l^2$ fixed, $\quad S = \pi R^2$

$$U = \frac{R}{2}\left(1 + \frac{R^2}{l^2}\right)$$

$$F = 0 \;\Rightarrow\; T_{HP} = \frac{1}{\pi l}$$

Geometric V: $M = H$ [3, 4]

$\Lambda = -8\pi P, \quad V = \frac{\partial H}{\partial P}\big|_S = \frac{4\pi}{3} R^3$

$$H = \frac{R}{2} + \frac{4\pi}{3} P R^3$$

$$G = 0 \;\Rightarrow\; T_{HP} = \sqrt{\frac{8P}{3\pi}}$$

Same $T(R) = \frac{1}{4\pi R}\left(1 + 8\pi P R^2\right)$, same stability limit and Hawking–Page temperature.

Volume concepts
No volume vs. geometric volume

No volume: $M = U$ [2]

$\Lambda = -3/l^2$ fixed, $S = \pi R^2$

$$U = \frac{R}{2}\left(1 + \frac{R^2}{l^2}\right)$$

$$F = 0 \;\Rightarrow\; T_{HP} = \frac{1}{\pi l}$$

Geometric V: $M = H$ [3, 4]

$\Lambda = -8\pi P$, $V = \frac{\partial H}{\partial P}\big|_S = \frac{4\pi}{3} R^3$

$$H = \frac{R}{2} + \frac{4\pi}{3} P R^3$$

$$G = 0 \;\Rightarrow\; T_{HP} = \sqrt{\frac{8P}{3\pi}}$$

Same $T(R) = \frac{1}{4\pi R}\left(1 + 8\pi P R^2\right)$, same stability limit and Hawking–Page temperature.

The problem

S and V are both functions of R alone: the state space is **degenerate**, and $E = H - PV = R/2$.

Thermodynamic volume

Extensivity selects the volume

Euler-homogeneous entropy - Biró et al., 2018 [1]

$$S(U,V) = \xi\, U^{\alpha} V^{\beta}, \qquad \alpha + \beta = 1, \qquad p = \frac{\beta}{\alpha}\frac{U}{V}$$

Thermodynamic volume

Extensivity selects the volume

Euler-homogeneous entropy - Biró et al., 2018 [1]

$$S(U,V) = \xi\, U^{\alpha} V^{\beta}, \qquad \alpha + \beta = 1, \qquad p = \frac{\beta}{\alpha}\frac{U}{V}$$

Two conditions fix the volume

$$S = \pi r_S^2, \quad U \propto r_S, \quad V \propto r_S^{\gamma} \implies \alpha + \gamma\beta = 2$$

Stefan–Boltzmann: $\dfrac{\beta}{\alpha} = \dfrac{1}{3} \implies \alpha = \dfrac{3}{4}, \quad \beta = \dfrac{1}{4}, \quad \gamma = 5$

Thermodynamic volume

Extensivity selects the volume

Euler-homogeneous entropy - Biró et al., 2018 [1]

$$S(U,V) = \xi\, U^{\alpha} V^{\beta}, \qquad \alpha + \beta = 1, \qquad p = \frac{\beta}{\alpha}\frac{U}{V}$$

Two conditions fix the volume

$$S = \pi r_S^2, \quad U \propto r_S, \quad V \propto r_S^{\gamma} \implies \alpha + \gamma\beta = 2$$

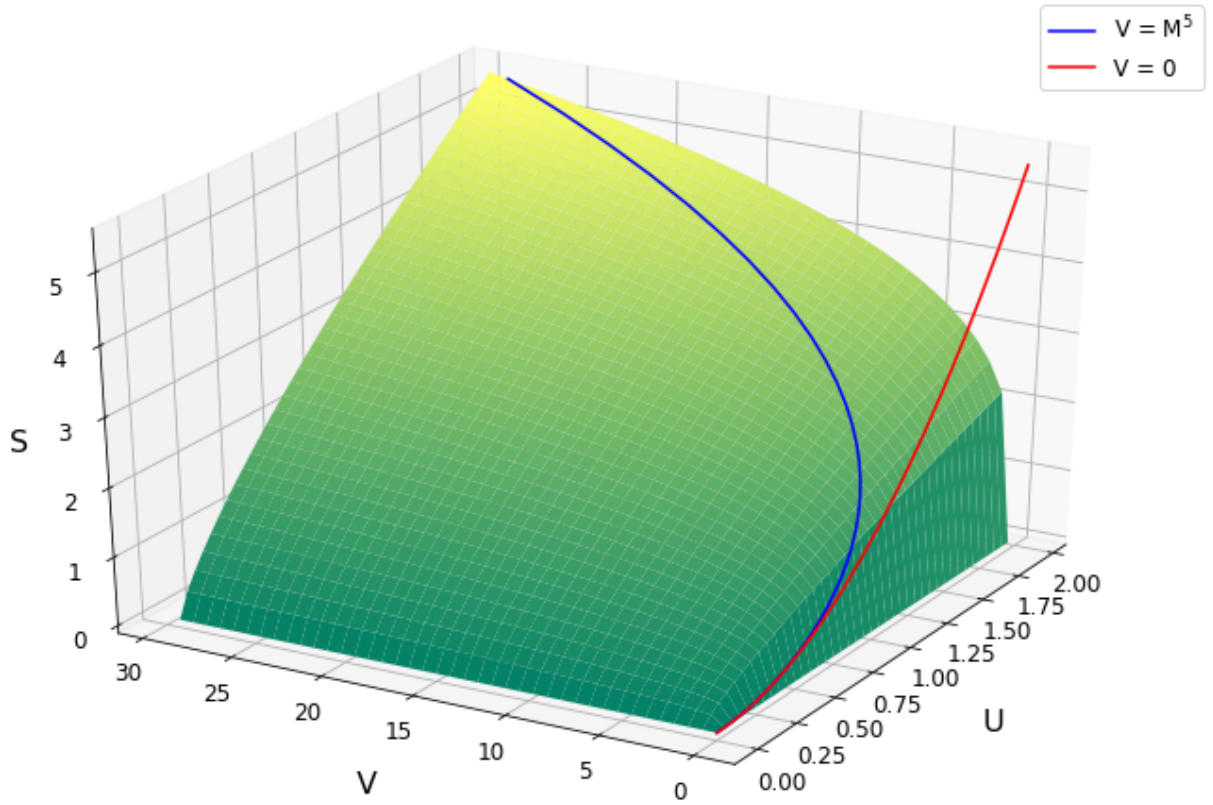
Stefan–Boltzmann: $\dfrac{\beta}{\alpha} = \dfrac{1}{3} \implies \alpha = \dfrac{3}{4}, \quad \beta = \dfrac{1}{4}, \quad \gamma = 5$

Same scaling from geometry

Christodoulou–Rovelli: $V(v) = 3\sqrt{3}\,\pi M^2 v$ [8]; over the evaporation time $v \sim M^3$, so $V \sim r_S^5$ [9].

Thermodynamic volume

Concavity of the entropy



With volume, $S(U, V)$ is concave (stable). The $V \sim M^5$ path stays on it; its projection to $V = 0$ is the convex, unstable curve [1, 10].

Anti-de Sitter spacetime

Christodoulou–Rovelli volume in AdS [12]

Generalized pressure

$$\Lambda = -\frac{6PS}{\pi} \quad \text{(previously: } \Lambda = -8\pi P\text{)} \quad \Longrightarrow \quad V = \frac{\partial H}{\partial P}\Big|_S = r_S^5$$

Schwarzschild–AdS = Kiselev, $w = -5/3$ [11, 5]

$$H(R,P) = \frac{R}{2} + PR^5, \qquad T(R,P) = \frac{1}{4\pi R} + \frac{5P}{2\pi}R^3$$

$$G(R,P) = H - TS = \frac{R}{4} - \frac{3}{2}PR^5$$

Anti-de Sitter spacetime

Christodoulou–Rovelli volume in AdS [12]

Generalized pressure

$$\Lambda = -\frac{6PS}{\pi} \quad \text{(previously: } \Lambda = -8\pi P\text{)} \quad \Longrightarrow \quad V = \left.\frac{\partial H}{\partial P}\right|_S = r_S^5$$

Schwarzschild–AdS = Kiselev, $w = -5/3$ [11, 5]

$$H(R,P) = \frac{R}{2} + PR^5, \qquad T(R,P) = \frac{1}{4\pi R} + \frac{5P}{2\pi}R^3$$

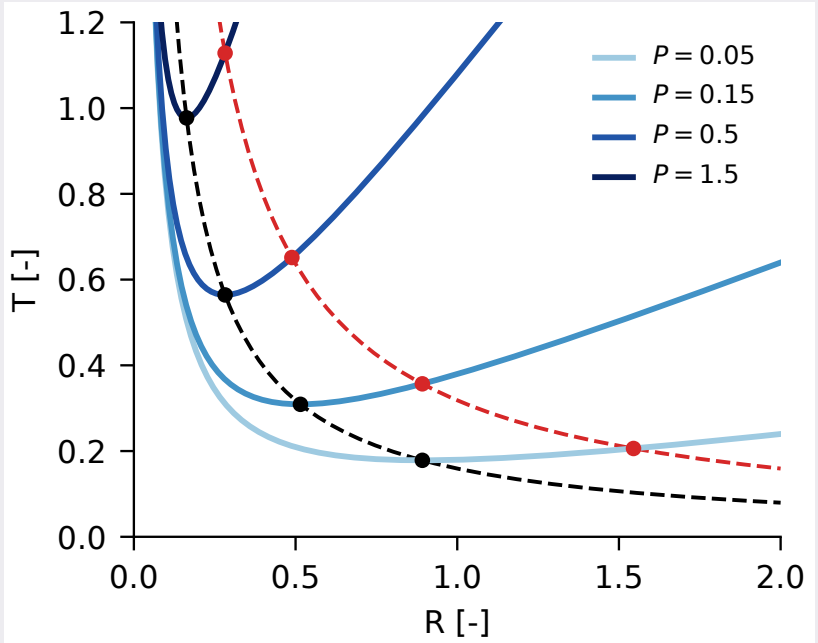
$$G(R,P) = H - TS = \frac{R}{4} - \frac{3}{2}PR^5$$

Hawking–Page temperature from $G = 0$

$$T_{HP} = \sqrt{\frac{8P}{3\pi}} \quad \longrightarrow \quad T_{HP} = \frac{2}{3\pi}(6P)^{1/4}$$

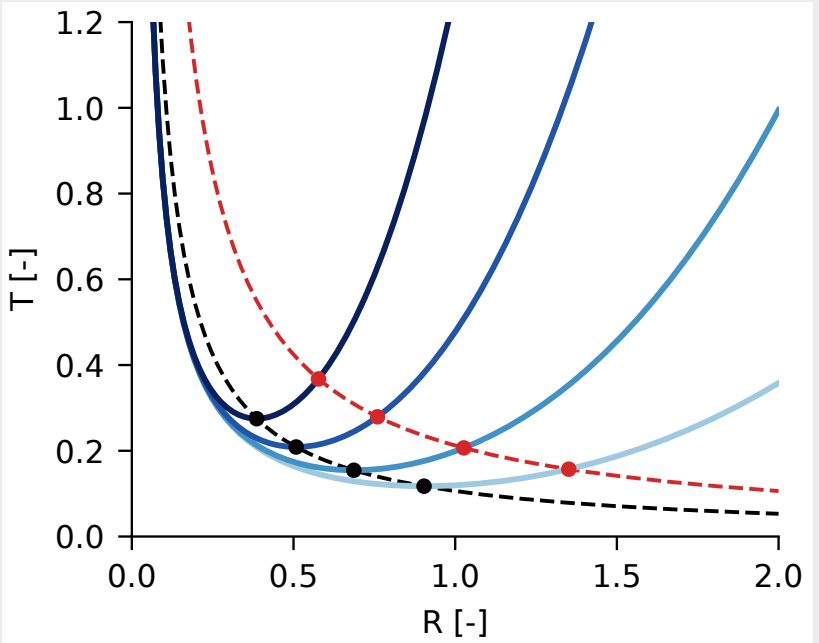
Anti-de Sitter spacetime

Stability and Hawking–Page boundaries [2]



Geometric volume

$R \geq \frac{1}{2\pi T}$ (local), $R \geq \frac{1}{\pi T}$ $(G < 0)$



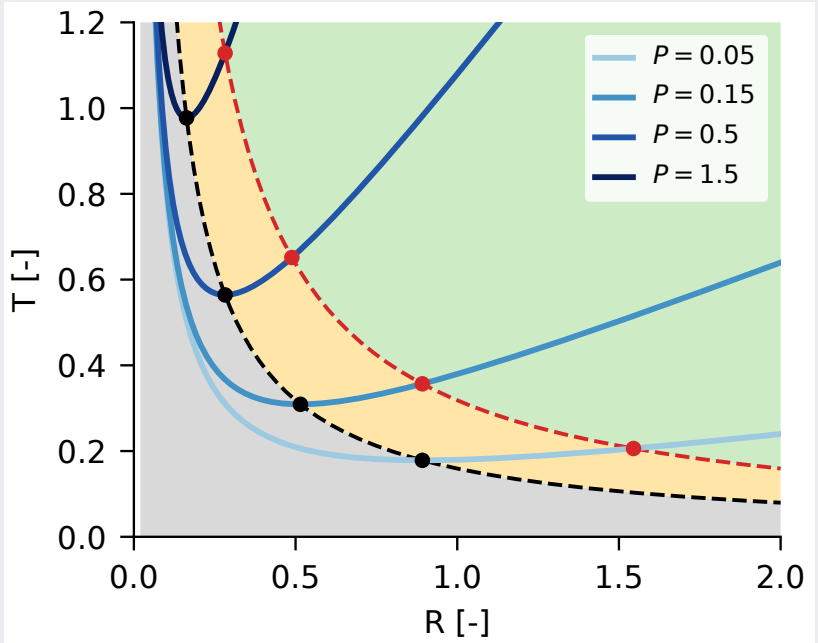
Christodoulou–Rovelli volume

$R \geq \frac{1}{3\pi T}$ (local), $R \geq \frac{2}{3\pi T}$ $(G < 0)$

Isobars $T(R)$ for $P = 0.05,\ 0.15,\ 0.5,\ 1.5$. Black dashed: local stability limit; red dashed: $G = 0$. Dots: minima (black) and Hawking–Page points (red); the boundaries do not depend on P.

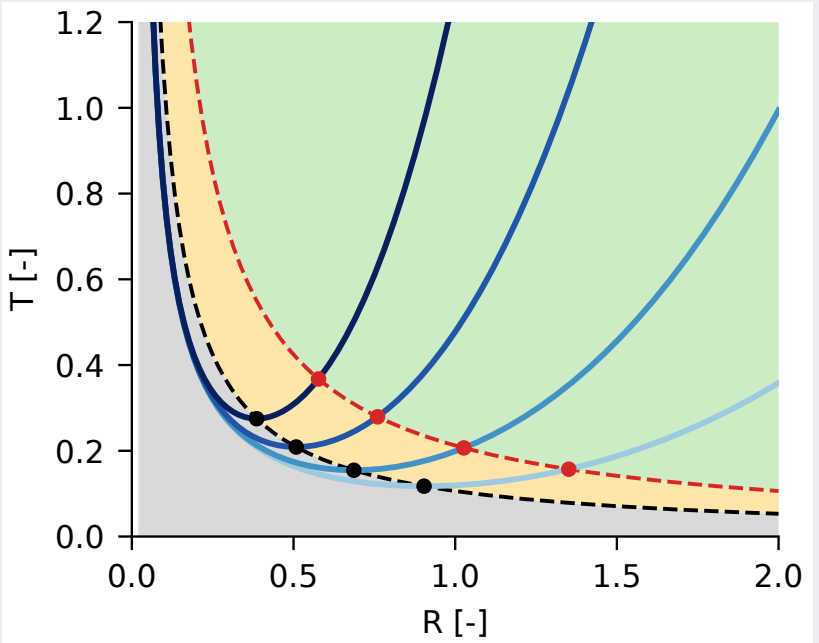
Anti-de Sitter spacetime

Stability and Hawking–Page boundaries [2]



Geometric volume

$R \geq \frac{1}{2\pi T}$ (local), $R \geq \frac{1}{\pi T}$ $(G < 0)$



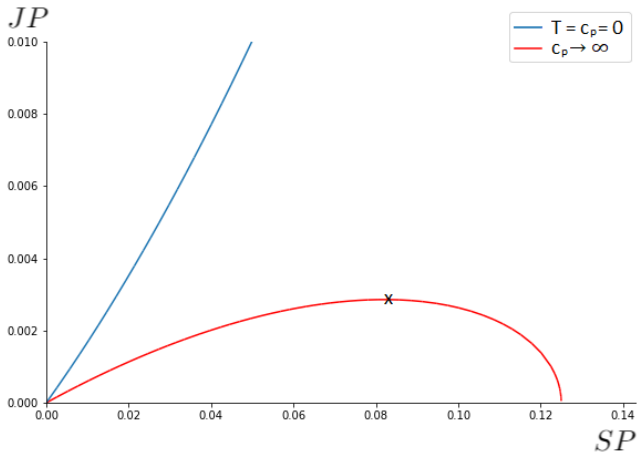
Christodoulou–Rovelli volume

$R \geq \frac{1}{3\pi T}$ (local), $R \geq \frac{2}{3\pi T}$ $(G < 0)$

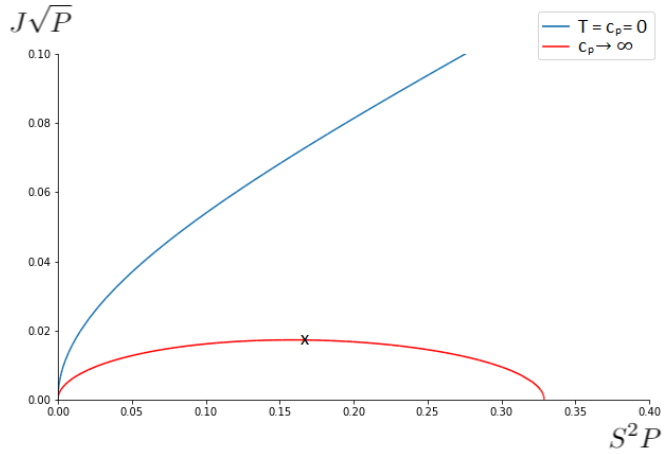
Isobars $T(R)$ for $P = 0.05,\ 0.15,\ 0.5,\ 1.5$. Black dashed: local stability limit; red dashed: $G = 0$. Grey: unstable, yellow: metastable $(G > 0)$, green: stable $(G < 0)$.

Enthalpy with the new volume

$$H(S,P,J)=\frac{1}{2\sqrt{\pi S}}\sqrt{\left(S+\frac{2PS^3}{\pi^2}\right)^2+4\pi^2J^2\left(1+\frac{2PS^2}{\pi^2}\right)}$$



Geometric volume [3]
$(SP)_{crit} \approx 0.08204, \quad (JP)_{crit} \approx 0.002857$



Christodoulou–Rovelli volume
$(S^2P)_{crit} \approx 0.1584, \quad (J\sqrt{P})_{crit} \approx 0.01729$

$c_P \to \infty$ in red, $T = 0$ in blue. Critical point at $(\partial T/\partial S)_{P,J} = 0$: phase structure preserved, critical point stays physical ($T > 0$).

Open problems
Challenges of the volume concepts

- **Equilibrium meaning of the Christodoulou–Rovelli volume:** it grows in v; the r_S^5 scaling appears only over the evaporation time [9]. Boundary volumes avoid this [6].
- Λ **as a state variable:** varying Λ may change the theory rather than the state [13]; restricted phase space keeps Λ fixed [7, 14].
- **Rotating black holes:** critical points in different reduced coordinates are not yet comparable; the Hawking–Page transition in the new Kerr–AdS phase space is not yet located. Holographic volume was recently extended to Kerr–AdS [15].
- **Environment:** the generalized pressure describes a Kiselev-type black hole in an anisotropic fluid [11, 16]: physical meaning?

Summary

- No volume: **degenerate**, unstable, extensivity?
- Extensivity $+$ radiation law $\Rightarrow V \propto r_S^5$
- AdS: **stability restored**, new T_{HP}
- Kerr–AdS: critical point preserved

Bibliography I

[1] T.S. Biró, V.G. Czinner, H. Iguchi, and P. Ván.
Black hole horizons can hide positive heat capacity.
Physics Letters B, 782:228–231, 2018.

[2] S.W. Hawking and D.N. Page.
Thermodynamics of black holes in Anti-de Sitter space.
Communications in Mathematical Physics, 87(4):577–588, 1983.

[3] B.P. Dolan.
Where is the PdV in the first law of black hole thermodynamics?
Open Questions in Cosmology, 2012.

[4] E. Spallucci and A. Smailagic.
Maxwell's equal area law and the Hawking-Page phase transition.
Journal of Gravity, 2013, 2013.

[5] Réka Somogyfoki and Péter Ván.
Volume in the extensive thermodynamics of black holes: AdS and Kiselev spacetimes.
Acta Physica Polonica B Proceedings Supplement, 18:6–A12, 2025.

[6] Silvester G. A. Borsboom and Manus R. Visser.
Holographic pressure and volume for black holes.
Classical and Quantum Gravity, 2026.
arXiv:2602.05130.

[7] Zeyuan Gao and Liu Zhao.
Restricted phase space thermodynamics for AdS black holes via holography.
Classical and Quantum Gravity, 39(7):075019, 2022.

[8] M. Christodoulou and C. Rovelli.
How big is a black hole?
Physical Review D, 91(6):064046, 2015.

[9] Carlo Rovelli.
Black holes have more states than those giving the Bekenstein-Hawking entropy: a simple argument, 2017.
arXiv:1710.00218.

Bibliography II

[10] T.S. Biró, V.G. Czinner, H. Iguchi, and P. Ván.
Volume dependent extension of Kerr-Newman black hole thermodynamics.
Physics Letters B, 803:135344, 2020.

[11] V. V. Kiselev.
Quintessence and black holes.
Classical and Quantum Gravity, 20:1187, 2003.

[12] Réka Somogyfoki and Péter Ván.
Volume in the extensive thermodynamics of black holes, 2025.
arXiv:2503.22482.

[13] Robinson Mancilla.
Generalized euler equation from effective action: Implications for the Smarr formula in AdS black holes, 2024.
arXiv:2410.06605.

[14] Xuan-Rui Chen, Bin Wu, and Zhen-Ming Xu.
Black hole thermodynamics from Iyer-Wald formalism with variable Newton's constant, 2025.
arXiv:2508.04446.

[15] Van Quyet Hoang.
Holographic pressure and extensivity of rotating black holes at finite cutoff, 2026.
arXiv:2602.18526.

[16] Viktor G. Czinner and Hideo Iguchi.
Hawking-Rényi black hole thermodynamics, Kiselev solution, and cosmic censorship.
European Physical Journal C, 85:443, 2025.

[17] H. Quevedo, M.N. Quevedo, and A. Sánchez.
Homogeneity and thermodynamic identities in geometrothermodynamics.
The European Physical Journal C, 77(3):1–4, 2017.

Backup
Kiselev interpretation [11, 5]

Kiselev horizon condition

$$f(r) = 1 - \frac{2M}{r} - \frac{K}{r^{3w+1}} \xrightarrow{w=-5/3} M = \frac{R}{2}\left(1 - KR^4\right)$$

Comparison with the Christodoulou–Rovelli enthalpy

$$H(R,P) = \frac{R}{2}\left(1 + 2PR^4\right) \implies K = -2P$$

Environment

Anisotropic fluid: $\rho = -p_r$, $p_t \neq p_r$ [16]. Isotropic only for $w = -1$, the AdS case, which is not extensive in this scaling analysis [5].

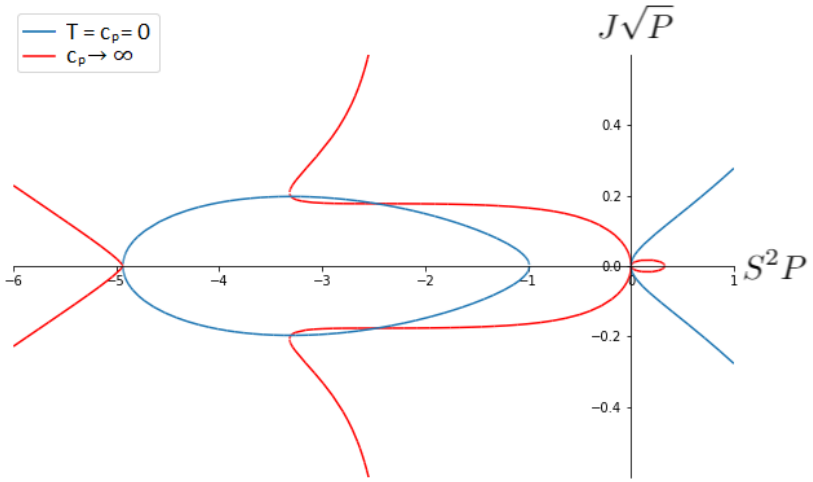
Backup

Full Kerr–AdS phase space with the new volume

Enthalpy and stability boundary

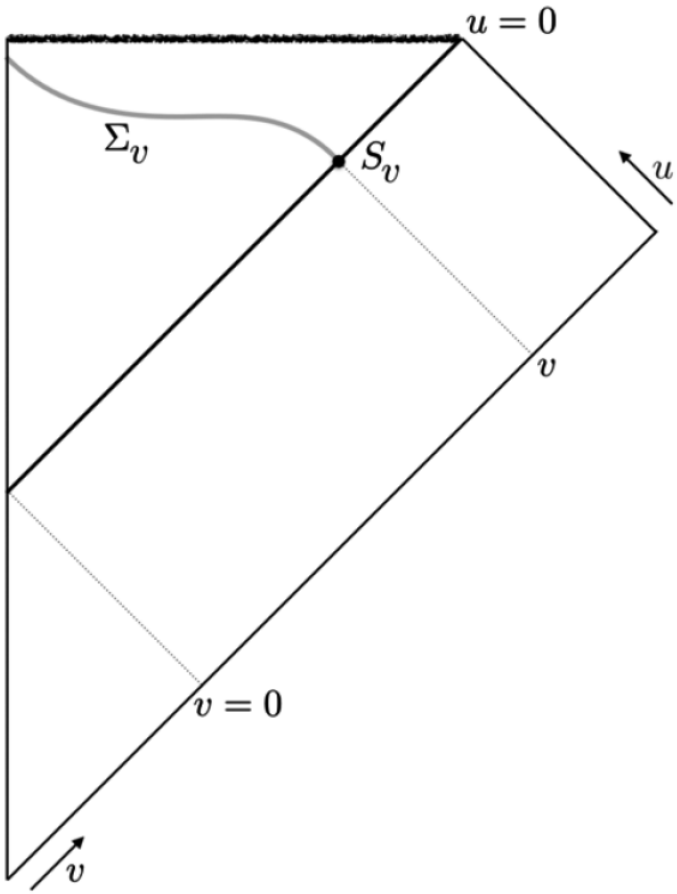
$$H(S,P,J) = \frac{1}{2\sqrt{\pi S}}\sqrt{\left(S + \frac{2PS^3}{\pi^2}\right)^2 + 4\pi^2 J^2\left(1 + \frac{2PS^2}{\pi^2}\right)}$$

$$c_P = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{P,J}}$$



Full reduced $J - S - P$ parameter space; $c_P \to \infty$ in red, $T = 0$ in blue.

Backup
The Christodoulou–Rovelli volume



Penrose diagram of a collapsing object [8]

Largest spacelike interior [8]

$$V(v) = 3\sqrt{3}\,\pi M^2 v$$

grows linearly in advanced time v

Over the evaporation time

$$v \sim M^3 \implies V \sim M^5 \sim r_S^5$$

Rovelli, 2017 [9]

Geometry and thermodynamics select the **same scaling**.



Extensivity

$$S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$$

Generalized extensivity - Quevedo et al., 2017 [17]:

$$\Psi(\lambda^{\beta_1} E^1, ..., \lambda^{\beta_n} E^n) = \lambda^{\beta_\psi} \Psi(E^1, ..., E^n)$$

Stability: concave entropy

$$\frac{\partial T(U,V)}{\partial U} = \frac{1}{C_V} > 0 \qquad \frac{\partial p(T,V)}{\partial V} \leq 0 \qquad \frac{1}{C_P} = \frac{\partial T(U,p)}{\partial U}\Big|_p > 0$$

Backup
Full phase space with geometric volume [3]

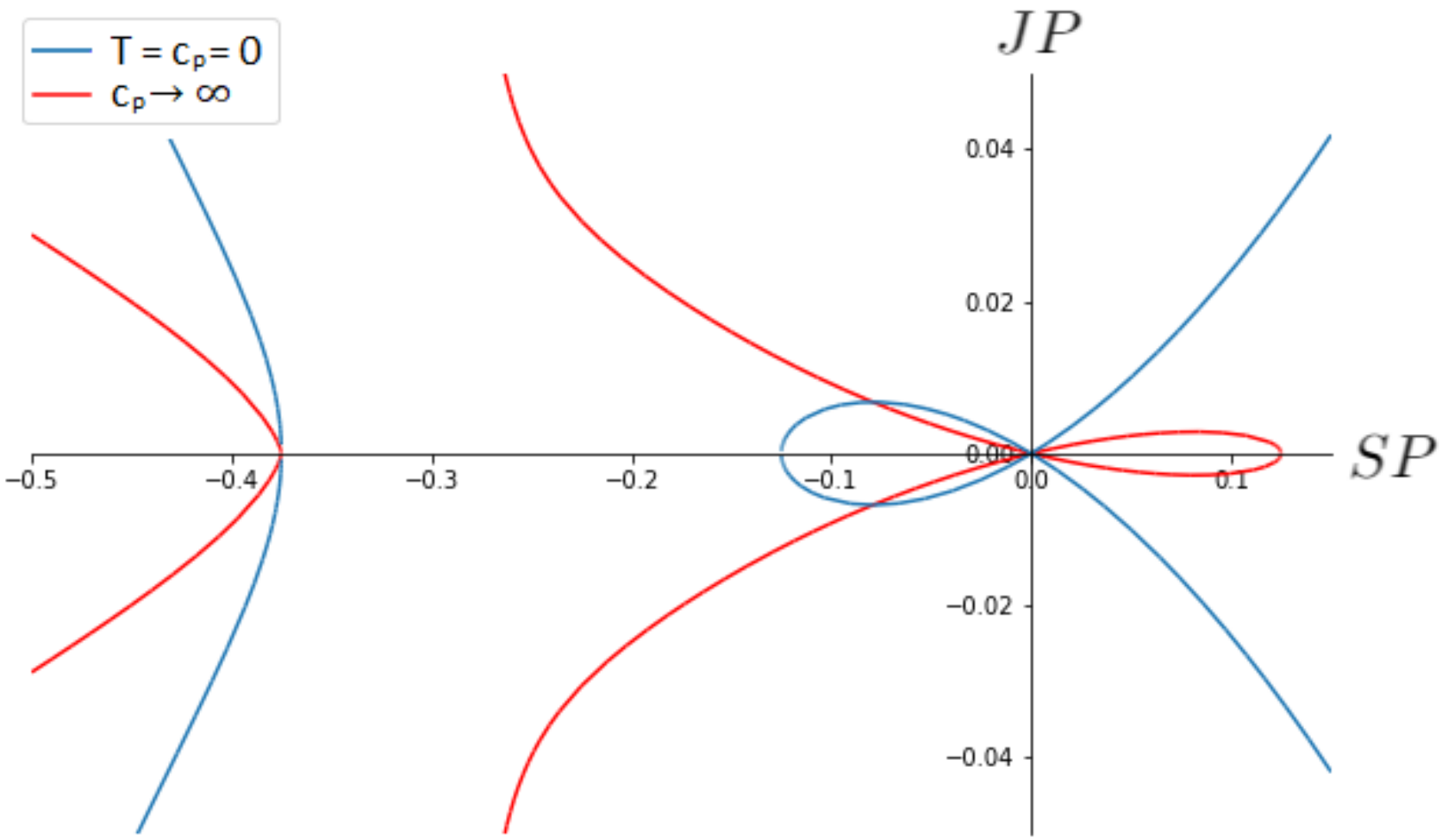


Figure: The full *JP*–*SP* phase space ($Q = 0$).

Equilibrium of pressures

$$p_{id} = \frac{nR_u T}{V} \qquad p_{rad} = \frac{\rho_r}{3} = \frac{\sigma}{3} T^4$$

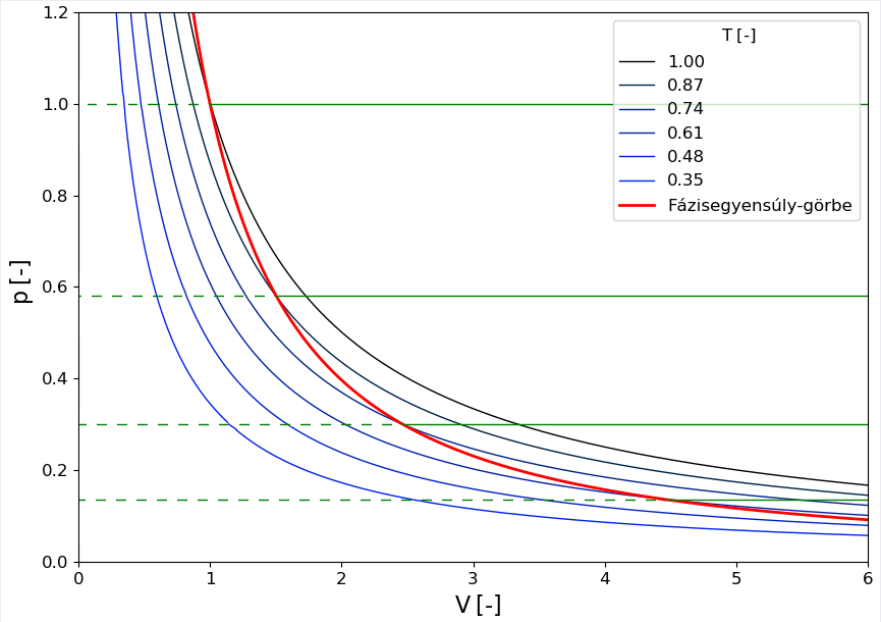


Figure: Phase equilibrium of radiation field and ideal gas in dimensionless coordinates. Equilibrium with radiation requires $G = 0$.

Schwarzschild

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

AdS–Kerr

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{1 - \frac{a^2}{L^2}}d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2$$
$$+ \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2 + a^2}{1 - \frac{a^2}{L^2}}d\phi\right)^2,$$

$$\Delta = \frac{(r^2 + a^2)(L^2 + r^2)}{L^2} - 2mr, \quad \Delta_\theta = 1 - \frac{a^2}{L^2}\cos^2\theta,$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Lambda = -\frac{3}{L^2} = -8\pi P$$