Dissipative forces in classical electrodynamics and in ideal gases

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Effective theory of dissipative forces in harmonic systems:

- linearly coupled harmonic oscillators
- a point charge in CED
- particle in an ideal fermi gas
- equations of motion of the current in an ideal fermi gas

Harmonic oscillators

Equations of motion

System x; environment y:

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega_0^2}{2}x^2 + jx + \sum_n \left(\frac{m}{2}\dot{y}_n^2 - \frac{m\omega_n^2}{2}y_n^2 - g_n y_n x\right)$$

Stability: $\sum_n \frac{g_n^2}{\omega_n^2} < m^2\omega_0^2$

In. cond.: $x(t_i) = \dot{x}(t_i) = y_n(t_i) = \dot{y}_n(t_i) = 0,\ t_i \to -\infty$

EoM y_n : $m\ddot{y}_n = -m\omega_n y_n - g_n x \Longrightarrow y_n(\omega) = \frac{g_n x(\omega)}{m[(\omega+i\epsilon)^2-\omega_n^2]}$

x : $-j(\omega) = [m(\omega^2 - \omega_0^2) - \Sigma^r(\omega)]x(\omega)$

Self energy: $\Sigma^r(\omega) = \sum_n \frac{g_n^2}{m}\frac{1}{(\omega+i\epsilon)^2-\omega_n^2}$

Solution: $x(t) = \int dt' D^r(t-t')j(t')$

Green function: $D^r(t) = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{m[(\omega+i\epsilon)^2-\omega_0^2]-\Sigma^r(\omega)}$

Harmonic oscillators

Equations of motion

Spectral function: $\rho(\Omega) = \sum_n \frac{g_n^2}{2m\omega_n}\delta(\omega_n - \Omega)$

Self energy: $\Sigma^r(\omega) = \sum_n \frac{g_n^2}{m}\frac{1}{(\omega+i\epsilon)^2-\omega_n^2} = \int d\Omega \frac{2\rho(\Omega)\Omega}{(\omega+i\epsilon)^2-\Omega^2}$

Ohmic form: $\rho(\Omega) = \frac{\Theta(\Omega)g^2\Omega}{m\Omega_D(\Omega_D^2+\Omega^2)} \Longrightarrow \Sigma^r(\omega) = -\frac{g^2\pi}{m\Omega_D}\frac{\Omega_D+i\omega}{\omega^2+\Omega_D^2}$

Green function: $D^r(t) = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{m[(\omega+i\epsilon)^2-\omega_0^2]-\Sigma^r(\omega)}$

EoM: $$0 = \left[m\omega^2 + \frac{g^2\pi}{m\Omega_D}\frac{\Omega_D+i\omega}{\omega^2+\Omega_D^2} - m\omega_0^2\right]x(\omega)$$

↗

$\mathcal{O}(\omega)$: Newton's friction force
$\mathcal{O}(\omega^2)$: acausality

Irreversibility, dissipation and acausality

Normal mode mixing

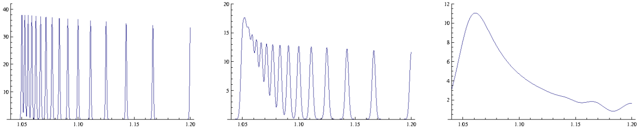
Harmonic model: $L = \frac{m}{2}\dot{x}^2 - \frac{m\omega_0^2}{2}x^2 + \sum_n \left(\frac{m}{2}\dot{y}_n^2 - \frac{m\omega_n^2}{2}y_n^2 - g_n y_n x\right)$

(Rubin 1960,...,Caldeira, Leggett 1983,..., Hu, Paz, Zhang (1991),...)

Energy injected into x:
- distributed over infinitely many normal modes
- observation time t_o: $\mathcal{O}\left(\frac{1}{t_o}\right)$ line spread
- no gap: ∞ many unresolved modes by finite time observations: **dissipation**

Effective spectral function: $\rho(\frac{\omega}{\omega_0})$:

$m = 1$
$\omega_n = \frac{\omega_0}{n}$
$n = 1, \ldots, 20$



$T = 2000$ $\quad$ $T = 700$ $\quad$ $T = 100$

Irreversibility: $[\lim_{T\to\infty}, \lim_{N\to\infty}] \neq 0$, phase transition

Acausality:

- integration of Newton's equation $\Longrightarrow$finite systems are causal
- infinite systems: $[\lim_{N\to\infty}, \lim_{\Delta t\to 0}] \neq 0$

Extended action principle

Problem 1.: $\not{T}$ and irreversibility of an effective theory

System: x, environment: y

Dynamics: time translation invariant, T-invariant $\Longrightarrow S[x,y]$
EoM: $\frac{\delta S[x,y]}{\delta x} = \frac{\delta S[x,y]}{\delta y} = 0$
In. cond. : $x(t_i), \dot{x}(t_i), y(t_i), \dot{y}(t_i)$ are given

Effective theory: $\frac{\delta S[x,y]}{\delta y} = 0 \Longrightarrow y = y[x]$, $S_{eff}[x] = S[x, y[x]]$

$$\frac{\delta S[x,y[x]]}{\delta x} = \frac{\delta S[x,y[x]]}{\delta x} + \frac{\delta S[x,y[x]]}{\delta y}\frac{\delta y[x]}{\delta x} = 0$$

Explicite **symmetry breaking by the environment** in. cond.:
- time translation invariance
- T: environment is not seen but
  effective EoM depends on the environment in. cond.
  $y(t_f) \neq y(t_i)$, $\dot{y}(t_f) \neq \dot{y}(t_i)$

Irreversibility: $t_i \to -\infty$ - time translation invariance recovered
- $\not{T}$ remains

Extended action principle

Problem 2.: initial conditions

1. Dissipative forces: final coordinates are disordered/unknown
   - in. cond. should be used by specifying $x(t_i)$ and $\dot{x}(t_i)$
   - EoM is needed at t_f but $\frac{\delta S}{\delta x(t_f)} = p_f \neq 0$ is unacceptable
2. Missing environment initial conditions:
   $x, y_1, \dots, y_N$, linearly coupled harmonic oscillators
   $\boldsymbol{y}(t_i) = \dot{\boldsymbol{y}}(t_i) = 0$, $t_i \to -\infty$
   Effective EoM: $(c_0 + c_2\partial_t^2 + \cdots + c_{2(N+1)}\partial_t^{2(N+1)})x = 0$
   - higher order derivatives in time
   - needs the environment in. cond. to solve
   - $\not{T}$ encoded by in. cond., rather than EoM

Q: How to solve EoM without knowing all in. cond.?

A: Variational method remains well defined

Goal: Retarded Green function by functional manipulations

Extended action principle

Problem 3.: non-conservative (semiholonomic) forces

d'Alembert principle: Virtual work $= (F - m\ddot{x})\delta x = 0$

Holonomic force: $F(x,\dot{x})\delta x = -\delta x \partial_x U(x,\dot{x}) - \delta\dot{x}\partial_{\dot{x}} U(x,\dot{x})$

Integrating d'Alembert principle in time:

$$0 = \delta \int_{t_i}^{t_f} dt \left[\frac{m}{2}\dot{x}^2 - U(x,\dot{x})\right] - \delta x(m\dot{x} + \partial_{\dot{x}} U)\Big|_{t_i}^{t_f}$$

Semiholonomic force: $x \to \hat{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$ - active: x^+

- passive: x^- (environment)

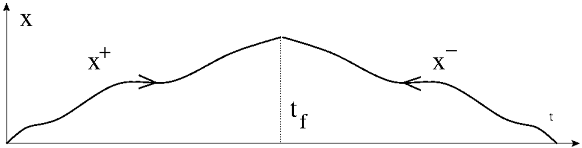
Virtual work:

$$F(x,\dot{x})\delta x = -[\delta x \partial_{x^+} U(\hat{x},\dot{\hat{x}}) + \delta\dot{x}\partial_{\dot{x}^+} U(\hat{x},\dot{\hat{x}})]_{|x^+ = x^- = x}$$

Is this extension enough to cover effective theories?

Extended action principle

Solution: replay the motion backward in time



↑
flipping of the time arrow

$$\tilde{x}(\tilde{t}) = \begin{cases} x(\tilde{t}) & t_i < \tilde{t} < t_f, \\ x(2t_f - \tilde{t}) & t_f < \tilde{t} < 2t_f - t_i, \end{cases}$$

$$\hat{x}(t) = \begin{pmatrix} x^+(t) \\ x^-(t) \end{pmatrix} = \begin{pmatrix} \tilde{x}(t) \\ \tilde{x}(2t_f - t) \end{pmatrix}$$

↖
doublers

$x^-(t)$: $T^2 = 1 \Longrightarrow$ identical time arrows for $x^+(t)$ and $x^-(t)$

Extended action principle

Lagrangian

both doublers are dynamical $\Longrightarrow L(\hat{x}, \dot{\hat{x}})$

A. Both follow the same motion: $x^+(t) = x^-(t)$ not in QM!

$$\Longrightarrow \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \tau \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} x^- \\ x^+ \end{pmatrix} \Longrightarrow L(\hat{x}, \dot{\hat{x}}) = \pm L(\tau\hat{x}, \tau\dot{\hat{x}})$$

B. In. cond.: $L(\hat{x}, \dot{\hat{x}}) = -L(\tau\hat{x}, \tau\dot{\hat{x}})$

$$x^+(t_f) = x^-(t_f) \Longrightarrow \frac{\delta S}{\delta x(t_f)} = p_f^+ - p_f^- = 0 \leftarrow \text{CTP}$$

$\nearrow$ time reversal

C. Degenerate action for $x^+(t) = x^-(t)$

$$\Longrightarrow \quad L(\hat{x}, \dot{\hat{x}}) = L(x^+, \dot{x}^+) - L(x^-, \dot{x}^-) + L_{spl}(\hat{x}, \dot{\hat{x}})$$
$$L_{spl}(\hat{x}, \dot{\hat{x}}) = i\frac{\epsilon}{2}(x^{+2} + x^{-2})$$

Extended action principle

Green function for harmonic systems

Action: $S[\hat{x}] = \frac{1}{2}\hat{x}\hat{K}\hat{x}$

Green function: $\hat{D} = \hat{K}^{-1}$

Trajectory: $x(t) = -\sum_{\sigma'}\int dt' D_0^{\sigma\sigma'}(t,t')\sigma' j(t')$

CTP symmetry (σ-independence): $D^{++} + D^{--} = D^{+-} + D^{-+}$

$$\hat{D} = \begin{pmatrix} D^n & -D^f \\ D^f & -D^n \end{pmatrix} + iD^i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\hat{K} = \hat{D}^{-1} = \begin{pmatrix} K^n & K^f \\ -K^f & -K^n \end{pmatrix} + iK^i \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Retarded, advanced components: $K^{\overset{r}{a}} \equiv K^n \pm K^f$, $D^{\overset{r}{a}} \equiv D^n \pm D^f$

Inversion: $K^{\overset{r}{a}} = \left(D^{\overset{r}{a}}\right)^{-1}$, $K^i = -(D^a)^{-1}D^i(D^r)^{-1}$

H. O.: $L = \frac{m}{2}\dot{x}^2 - \frac{m\Omega^2}{2}x^2$

$$\hat{D}(\omega) = \frac{1}{m}\begin{pmatrix} \frac{1}{\omega^2-\Omega^2+i\epsilon} & -2\pi i\Theta(-\omega)\delta(\omega^2-\omega_0^2) \\ -2\pi i\Theta(\omega)\delta(\omega^2-\omega_0^2) & -\frac{1}{\omega^2-\Omega^2-i\epsilon} \end{pmatrix}$$

Extended action principle

Green function for interacting system

Legendre transformation:

$$W[\hat{j}] = S[\hat{x}] + \hat{j}\hat{x}, \quad \frac{\delta S[\hat{x}]}{\delta \hat{x}} = -\hat{j}$$

Inverse transformation:

$$S[\hat{x}] = W[\hat{j}] - \hat{x}, \quad \frac{\delta W[\hat{j}]}{\delta \hat{j}} = \hat{x}$$

Green functions:

$$W[\hat{j}] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_n} \int dt_1 \cdots dt_n D^{\sigma_1,\dots,\sigma_n}(t_1,\dots,t_n) j^{\sigma_1}(t_1) \cdots j^{\sigma_n}(t_n)$$

Solution:

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma,\sigma_1,\dots,\sigma_n} \int dt_1 \cdots dt_n D^{\sigma,\sigma_1,\dots,\sigma_n}(t,t_1,\dots,t_n) j^{\sigma_1}(t_1) \cdots j^{\sigma_n}(t_n)$$

Residuum theorem: - **no runaway solutions**

- possible acausality

Extended action principle

Action

D. To recover time translation invariance: $t_i \to -\infty$, $t_f \to \infty$

Harmonic system: $S_0[\hat{x}] = \frac{1}{2}\hat{x}\hat{D}^{-1}\hat{x}$

$(D^{-1})^n = m(\omega^2 - \Omega^2)$, $(D^{-1})^f = i\text{sign}(\omega)\epsilon$, $(D^{-1})^i = \epsilon$

$$\begin{aligned} S &= \int dt \left[\frac{m}{2}\dot{x}^{+2}(t) - \frac{m\Omega^2}{2}x^{+2} - \frac{m}{2}\dot{x}^{-2}(t) + \frac{m\Omega^2}{2}x^{-2}\right] + S_{bc} \\ S_{bc} &= \frac{\epsilon}{\pi}\int_{-\infty}^{\infty} dtdt' \frac{x^+(t)x^-(t')}{t - t' + i\epsilon} + \frac{i\epsilon}{2}\int_{-\infty}^{\infty} dt[x^{+2}(t) + x^{-2}(t)] \\ &= \underbrace{\frac{i\epsilon}{2}\int_{-\infty}^{\infty} dt[x^+(t) - x^-(t)]^2}_{inf.\ decoherence} + \underbrace{\frac{\epsilon}{\pi}P\int_{-\infty}^{\infty} dtdt' \frac{x^+(t)x^-(t')}{t - t'}}_{inf.\ entanglement} \end{aligned}$$

Non-harmonic systems: $S[\hat{x}] = S[x^+] - S[x^-] + S_{bc}[\hat{x}]$

Extended action principle

Effective action

System: x, environment: y

$$
\begin{aligned}
S[x,y] &= S_s[x] + S_e[x,y] \\
S[\hat{x},\hat{y}] &= S_s[x^+] + S_e[x^+,y^+] - S_s[x^-] - S_e[x^-,y^-] + S_{bc}[\hat{x},\hat{y}]
\end{aligned}
$$

Effective action: $\frac{\delta S[\hat{x},\hat{y}]}{\delta \hat{y}} = 0 \Longrightarrow \hat{y} = \hat{y}[\hat{x}]$

$$
\begin{aligned}
S_{eff}[\hat{x}] &= S_s[x^+] + S_e[x^+,y^+[\hat{x}]] - S_s[x^-] - S_e[x^-,y^-[\hat{x}]] + S_{bc}[\hat{x},\hat{y}[\hat{y}]] \\
&= S_s[x^+] - S_s[x^-] + S_{infl}[\hat{x}] + S_{bc}[\hat{x},\hat{y}[\hat{y}]]
\end{aligned}
$$

Influence functional *(Feynman, Vernon 1963)*:

$$
S_{infl}[\hat{x}] = S_e[x^+,y^+[\hat{x}]] - S_e[x^-,y^-[\hat{x}]]
$$

Better parametrization:

$$
S_{eff}[\hat{x}] = S_1[x^+] - S_1[x^-] + S_2[\hat{x}] + S_{bc}[\hat{x},\hat{y}[\hat{y}]], \quad (S_2[0,x^-] = S_2[x^+,0] = 0)
$$

Extended action principle

Effective action

Keldysh parametrization: $x^{\pm} = x \pm \frac{x^d}{2}$ *(Keldysh 1964)*
Advantage: $x^+(s) = x^-(s)$, $x^d(s) = 0 \Longrightarrow S = \mathcal{O}\left(x^d\right)$ is sufficient

EoM for x^d:

$$
\begin{aligned}
0 &= \frac{\delta}{\delta x^d}\left\{S_1\left[x + \frac{x^d}{2}\right] - S_1\left[x - \frac{x^d}{2}\right] + S_2\left[x + \frac{x^d}{2}, x - \frac{x^d}{2}\right]\right\}_{|x^d=0} \\
&= \frac{\delta S_1[x]}{\delta x} + \frac{\delta S_2[x^+, x^-]}{\delta x^+}_{|x^+=x^-=x}
\end{aligned}
$$

$\uparrow$
semiholonomic forces

- S_1: Holonomic forces (Noether theorem available)
- S_2: Semiholonomic forces, environment excitations
- $L = \frac{m}{2}\dot{x}^{+2} - \frac{m\omega^2}{2}x^{+2} - \frac{m}{2}\dot{x}^{-2} + \frac{m\omega^2}{2}x^{-2} + \frac{k}{2}(x^-\dot{x}^+ - x^+\dot{x}^-)$
  EoM: $m\ddot{x}^{\pm} = -m\omega^2 x^{\pm} - k\dot{x}^{\mp}$ *(Bateman 1931)*

Extended action principle

Effective action

- New symplectic structure
- Semiholonomic forces are sufficient: CTP symmetry is preserved during the elimination of degrees of freedom
- System-environment interactions mapped into system^{+}-system^{-} interactions

Harmonic oscillators

Effective action

Action:

$$S[\hat{x},\hat{y}] = \frac{1}{2}\hat{x}\hat{D}_0^{-1}\hat{x} + \frac{1}{2}\sum_n \hat{y}_n \hat{G}_n^{-1}\hat{y}_n + \hat{x}\left(\hat{j} - \hat{\sigma}\sum_n g\hat{y}_n\right)$$

Green function: $\hat{D} = \frac{1}{\hat{D}_0^{-1} - \hat{\sigma}\hat{\Sigma}\hat{\sigma}}, \quad \hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Self energy: $\hat{\Sigma} = \sum_n g_n^2 \hat{G}_n = \begin{pmatrix} \Sigma^n & -\Sigma^f \\ \Sigma^f & -\Sigma^n \end{pmatrix} + i\Sigma^i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Effective action: $S_{eff}[\hat{x}] = \frac{1}{2}\hat{x}\hat{D}^{-1}\hat{x} + \hat{j}\hat{x} = x_d(D^{r-1}x + j)$

Retarded Geen function:

$$D^r = \frac{1}{D_0^{-1} - \Sigma^r}$$

$$\Sigma^r = \Sigma^n + \Sigma^f$$

$$D_0^{-1} = -\partial_t^2 - \omega_0^2$$

EoM: $x = -D^r j$

A point charge

- Crossoever at the classical electron radius, $r_0 = \frac{e^2}{mc^2}$
- UV divergent Coulomb self-force
- Regulated Green function:
  - Smoothed action-at-a-distance on the light cone: $\delta(x^2) \to \delta_\ell(x^2)$ (off shell EMF)
  - Retardation: $\mathrm{sign}(x^0)\delta(x^2) = 0 \Longrightarrow \delta_\ell(0) = 0$ (only CTP!)
    - $\mathrm{sign}(x^0)$ is Lorentz invariant but its smeared version not
    - avoid the coincidences of singularities in $\delta(x^2)\mathrm{sign}(x^0)$
  - $\ell \gg r_0$: cutoff independent traditional electrodynamics
  - $\ell \ll r_0$: cutoff dependent (acausal) self interaction
- No runaway solution
- QED: Remains valid for non-relativistic charges

A point charge

Action

$$
\begin{aligned}
S_{ch}[\hat{x}] &= -m_B c \sum_{\sigma=\pm} \sigma \int ds \sqrt{\dot{x}^{\sigma 2}(s)} \\
S_{EMF}[\hat{A}] &= -\frac{1}{8\pi c} \sum_{\sigma=\pm} \sigma \int \frac{d^4p}{(2\pi)^4} Q(p^2) p^2 A^\sigma_\mu(-p) T^{\mu\nu}(p) A^\sigma_\nu(p) \\
S_i[\hat{x}, \hat{A}] &= -\frac{e}{c} \sum_{\sigma=\pm} \sigma \int ds \dot{x}^{\sigma\mu}(s) A^\sigma_\mu(x^\sigma(s))
\end{aligned}
$$

Q: UV regulator; $T^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$

A point charge

Effective action

The elimination of the EMF:

$$\begin{aligned}\Re S_{eff}[\hat{x}] &= -m_B c\int ds\left[\sqrt{\dot{x}^{+2}(s)}-\sqrt{\dot{x}^{-2}(s)}\right]\\ &\quad +\frac{2\pi e^2}{c}\int ds ds'\dot{\hat{x}}(s)\Re\hat{D}(x(s)-x(s'))\dot{\hat{x}}(s')\end{aligned}$$

(Schwarzschild 1903; Tetrode 1922; Fokker 1929) + far field

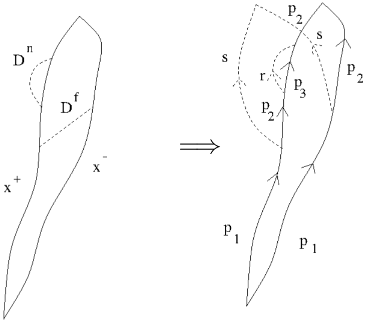
Regulated Green function: $\Re\hat{D}(x)=\frac{\delta_\ell(x^2)}{4\pi}\begin{pmatrix}-1 & -\mathrm{sign}(x^0)\\ \mathrm{sign}(x^0) & 1\end{pmatrix}$

Regulated Dirac-delta: $\delta_\ell(z)=\delta(z-\ell^2)$ or $\delta_\ell(z)=\frac{\Theta(z)}{12\ell^4}ze^{-\frac{\sqrt{z}}{\ell}}$

A point charge

Effective action

$S_s \leftrightarrow$ semiholonomic forces $\leftrightarrow$ environment excitations



Free motion between vertices:

$p_1 = p_2 + s$, $p_2 = p_3 + r$

D^n: conservative self interactions

D^f: radiation

A point charge

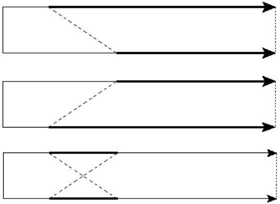
Influence functional

$$
\begin{aligned}
S_{infl} \quad = \quad & \frac{e^2}{2c}\int dsds'[\dot{x}^+\delta((x^+ - x'^+)^2)\dot{x}'^+ - \dot{x}^-\delta((x^- - x'^-)^2)\dot{x}'^-] \\
& + \frac{e^2}{c}\int dsds'\dot{x}^+ \mathrm{sign}(x^{+0} - x'^{-0})\delta((x^+ - x'^-)^2)\dot{x}'^-
\end{aligned}
$$

with $x = x(s)$, $x' = x(s')$

Far field interaction through the "end of time"?

Like photon emission and absorption:



(a): $x^{+0} < x'^{-0}$

(b): $x^{+0} > x'^{-0}$

(a) + (b)

A point charge

Equation of motion

$u \ll \ell$: expanding $\delta_\ell((x(s) - x(s+u))^2)$:

Quadratic influence Lagrangian:

$$L_{infl}(s) = x^d(s)\frac{4e^2}{c}\int_{-\infty}^{0} du\delta'_\ell(u^2)[x(s+u) - u\dot{x}(s+u) - x]$$

EoM: $$m_B c\ddot{x}^\mu = -\frac{e^2}{c\ell}\ddot{x}^\mu + (g^{\mu\nu} - \dot{x}^\mu\dot{x}^\nu)\left[K_\nu + \mathcal{O}\left(x^2\ell\right)\right]$$

$\nearrow$ $\ddot{x}\dot{x} = 0$

$$K(s) = \frac{2e^2}{3c}\left\{\dddot{x}(s) \quad \leftarrow \quad \text{Abraham-Lorentz}\right.$$

$$\left. \underbrace{-6\int_{-\infty}^{0} du\delta'_\ell(u^2)\left[x(s+u) - u\dot{x}(s+u) - x(s) + \frac{u^2}{2}\ddot{x}(s) + \frac{u^3}{3}\dddot{x}(s)\right]}_{\mathcal{O}(\ell)}\right\}$$

A point charge

Light-cone anomaly

- Abraham-Lorentz force is $\mathcal{O}\left(\ell^0\right)$ due to the non-uniform convergence of the loop-integral

- EMF dynamics remains off-shell sensitive:
  $D_r(x,y) = D_{r\ell}(x,y)\left[1 + \frac{(x-y)^2}{s_0^2}\right] \Longrightarrow m_B \to m_B + a_1 \frac{e^2}{2c^2 s_0}$

- classical analogy of chiral (light-cone propagating fermion) anomaly

A point charge

Renormalization

World line: $x(s) = x_0 e^{-i\omega s}$, $x_d(s) = x_d$

$$
\begin{aligned}
L_{eff} &= -x_d x_0 m_B c\omega^2 \left[1 + \frac{\lambda_B}{6}\frac{1+i\ell\omega}{(1-i\ell\omega)^3}\right] \\
&= -x_d x_0 mc \left[\omega^2 + i\omega^3 \lambda(\omega)\frac{2e^2}{3c}\right]
\end{aligned}
$$

- mass: $m = m_B(1 + \frac{g_B}{6})$
- coupling constant: $\lambda_B = \frac{e^2}{m_B c^2 \ell} \Longrightarrow \lambda = \frac{e^2}{mc^2\ell} = \frac{r_0}{\ell}$
  $\lambda(\omega) = \lambda \frac{1-3i\ell\omega-(\ell\omega)^2}{(1-i\ell\omega)^3}$
- Landau pole: $\lambda_B = \frac{\lambda}{1-\frac{\lambda}{6}} = \frac{r_0}{\ell - \frac{r_0}{6}}$

Quantum CTP formalism

Expectation value rather than transition amplitude (Schwinger vs. Feynman)

- Reduplication is inherent: bra $\leftrightarrow x^-$ and ket $\leftrightarrow x^+$
  $\bar{A}(t) = \langle\psi(0)|U^\dagger(t)AU(t)|\psi(0)\rangle$ *(Schwinger 1961)*
  $\uparrow \qquad \uparrow$
  independent building up of interactions (graphs)
- Naive quantization of open systems (with semiholonomic forces):
  $x \to (x^+, x^-) \Longrightarrow \psi(x^+, x^-) \sim \rho(x^+, x^-)$
  (new light on Gleason theorem)
- Generating functional

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{j}]} &= \mathrm{Tr}T[e^{-\frac{i}{\hbar}\int dt(H(t)-j^+(t)x(t))}]\rho_i T^*[e^{\frac{i}{\hbar}\int dt(H(t)+j^-(t)x(t))}] \\
&= \int D[\hat{x}]e^{\frac{i}{\hbar}S_0[x^+]-\frac{i}{\hbar}S_0[x^-]+\frac{i}{\hbar}\int dt\hat{j}(t)\hat{x}(t)} \\
&= \int_{x^+(t_f)=x^-(t_f)} D[\hat{x}]e^{\frac{i}{\hbar}S[\hat{x}]+\frac{i}{\hbar}\int dt\hat{j}(t)\hat{x}(t)}
\end{aligned}
$$

Convergence of the path integral: $S \to S + S_{bc}$

Quantum CTP formalism

Effective theory

System $\phi(x)$, environment $\psi(x)$, action $S[\phi, \psi] = S_s[\phi] + S_e[\phi, \psi]$

(Wilsonian) Effective action:

$$e^{\frac{i}{\hbar}S_{eff}[\hat{\phi}]} = e^{\frac{i}{\hbar}S_s[\phi^+]-\frac{i}{\hbar}S_s[\phi^-]}\int D[\hat{\psi}]e^{\frac{i}{\hbar}S_e[\phi^+,\psi^+]-\frac{i}{\hbar}S_e[\phi^-,\psi^-]+\frac{i}{\hbar}S_{bc}[\hat{\psi}]}$$

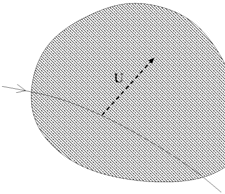
$$\begin{aligned} S_{eff}[\hat{\phi}] &= S_s[\phi^+] - S_s[\phi^-] + S_{infl}[\hat{\phi}] \\ &= S_1[\phi^+] - S_1[\phi^-] + S_2[\phi^+, \phi^-],\ S_2[0, \phi] = S_2[\phi, 0] = 0 \end{aligned}$$

- Semiholonomic forces $\leftrightarrow S_2[\phi^+, \phi^-] \leftrightarrow$ Entanglement
- Feynman graphs representation of entanglement

Ideal gas of fermions

The model

Particle interacting with the ideal gas by the potential $U(\boldsymbol{x})$:



$$
\begin{aligned}
S[\boldsymbol{x},\psi^\dagger,\psi] &= S_p[\boldsymbol{x}] + S_g[\psi^\dagger,\psi] + S_i[\boldsymbol{x},\psi^\dagger,\psi] \\
S_p[\boldsymbol{x}] &= \int dt \left[\frac{M}{2}\dot{\boldsymbol{x}}^2(t) - V(\boldsymbol{x}(t))\right] \\
S_g[\psi^\dagger,\psi] &= \int dt d^3y \psi^\dagger(t,\boldsymbol{y}) \left[i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta + \mu\right]\psi(t,\boldsymbol{y}) \\
S_i[\boldsymbol{x},\psi^\dagger,\psi] &= \int dt d^3y U(\boldsymbol{y}-\boldsymbol{x}(t))\psi^\dagger(t,\boldsymbol{y})\psi(t,\boldsymbol{y}) = \psi^\dagger\Gamma[\boldsymbol{x}]\psi
\end{aligned}
$$

$V(\boldsymbol{x})$ is steep enough to justify $L_{eff} = \mathcal{O}\left(x^2\right)$

Ideal gas of fermions

Influence functional

$$
\begin{aligned}
e^{\frac{i}{\hbar}S_{infl}[\hat{\boldsymbol{x}}]} &= \int D[\hat{\psi}]D[\hat{\psi}^\dagger]e^{\frac{i}{\hbar}\hat{\psi}^\dagger(\hat{F}^{-1}+\hat{\Gamma}[\hat{\boldsymbol{x}}])\hat{\psi}} \\
S_{infl}[\hat{\boldsymbol{x}}] &= -i\hbar \mathrm{Tr}\ln[\hat{F}^{-1}+\hat{\Gamma}] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'=\pm}\sigma\sigma' j^\sigma G^{\sigma\sigma'} j^{\sigma'} + \mathcal{O}\left(\hat{j}^3\right) \\
j^\sigma(t,\boldsymbol{y}) &= U(\boldsymbol{y}-\boldsymbol{x}^\sigma(t))
\end{aligned}
$$

$G^{\sigma_1\sigma_2}(x_1,x_2) =$  $= -i\hbar n_s \hat{F}^{\sigma_1\sigma_2}(x_1,x_2)\hat{F}^{\sigma_2\sigma_1}(x_2,x_1)$

Ideal gas of fermions

Density-density two-point function

$$
\begin{aligned}
\hat{G} &= \begin{pmatrix} G^f + iG^i & -G^f + iG^i \\ G^f + iG^i & -G^n + iG^i \end{pmatrix} \\
G^n_{\omega\boldsymbol{k}} &= G^+_{\omega\boldsymbol{k}} + G^+_{-\omega\boldsymbol{k}} \quad \text{(near field)} \\
G^f_{\omega\boldsymbol{k}} &= G^-_{\omega\boldsymbol{k}} - G^-_{-\omega\boldsymbol{k}} \quad \text{(far field)} \\
iG^i_{\omega\boldsymbol{k}} &= G^-_{\omega\boldsymbol{k}} + G^-_{-\omega\boldsymbol{k}} \\
G^+_{\omega\boldsymbol{k}} &= \frac{n_s}{\hbar^2}\Re\int\frac{d^3k}{(2\pi)^3}n_{\boldsymbol{k}}\frac{1}{\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}+i\epsilon} \qquad \text{(Lindhard fn.)} \\
G^-_{\omega\boldsymbol{k}} &= i\pi n_s\int\frac{d^3k}{(2\pi)^3}n_{\boldsymbol{k}}\delta(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})(n_{\boldsymbol{k}+\boldsymbol{q}}-1)
\end{aligned}
$$

n_s: spin degeneracy

$n_{\boldsymbol{k}}$: occupation number

Ideal gas of fermions

$L^{(2)}_{infl}$

$$
\begin{aligned}
S_{infl}[\hat{\boldsymbol{x}}] &= -\frac{1}{2}\sum_{\sigma\sigma'=\pm}\sigma\sigma' j^{\sigma} G^{\sigma\sigma'} j^{\sigma'}, \quad j^{\sigma}(t,\boldsymbol{y}) = U(\boldsymbol{y}-\boldsymbol{x}^{\sigma}(t)) \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\sigma\sigma'\int_{\omega\boldsymbol{k}} dt dt' U^2_{\boldsymbol{k}} e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}^{\sigma}(t)-\boldsymbol{x}^{\sigma'}(t'))} G^{\sigma,\sigma'}_{\omega\boldsymbol{k}}
\end{aligned}
$$

Change of variables: $t \to t + \frac{u}{2}$, $t' \to t - \frac{u}{2}$

$$
\begin{aligned}
L_{infl}(t) &= -\frac{1}{2}\sum_{\sigma\sigma'}\sigma\sigma'\int_{\omega\boldsymbol{k}} du U^2_{\boldsymbol{k}} e^{-i\omega u+i\boldsymbol{k}[\boldsymbol{x}^{\sigma}(t+\frac{u}{2})-\boldsymbol{x}^{\sigma'}(t-\frac{u}{2})]} G^{\sigma,\sigma'}_{\omega\boldsymbol{k}} \\
&= \frac{1}{2}\int_{\boldsymbol{k}} U^2_{\boldsymbol{k}}\Bigg[(\boldsymbol{k}\vec{\Delta}_1\boldsymbol{x})(\boldsymbol{k}\vec{\Delta}_1\boldsymbol{x}^d)G^n_{0\boldsymbol{k}} - (\boldsymbol{k}\vec{\Delta}_1\boldsymbol{x})(\boldsymbol{k}\vec{\Delta}_0\boldsymbol{x}^d)G^f_{0\boldsymbol{k}} \\
&\quad + \frac{(\boldsymbol{k}\vec{\Delta}_1\boldsymbol{x}^d)^2 - (\boldsymbol{k}\vec{\Delta}_0\boldsymbol{x}^d)^2}{4} iG^i_{0\boldsymbol{k}}\Bigg]
\end{aligned}
$$

u dependence: $\vec{\Delta}_j\boldsymbol{x} = \sum_{n=0}^{\infty}\frac{(-i)^{2n+j}}{2^{2n+j}(2n+j)!}\boldsymbol{x}^{(2n+j)}\partial_{\omega}^{2n+j}$

Ideal gas of fermions

$\Re L^{(2)}_{infl}$: equation of motion

$$\begin{aligned}
L_{infl} &= \boldsymbol{x}^d \left(-k\dot{\boldsymbol{x}} - \delta M \ddot{\boldsymbol{x}} + \alpha \dddot{\boldsymbol{x}} + i d_0 \boldsymbol{x}^d - i d_2 \ddot{\boldsymbol{x}}^d + \mathcal{O}\left(\frac{d^4}{dt^4}\right)\right) \\
&\quad + \mathcal{O}\left(x^d x^3\right) \\
k &= \frac{1}{24\pi^2 v_F} \int dk k U_k^2 \partial_{ix} G^f_{0k}(x,y) \qquad \leftarrow \quad \text{non}-\text{rel. gas} \\
\delta M &= \frac{1}{48\pi^2 v_F^2} \int dk U_k^2 \partial^2_{ix} G^n_{0k}(x,y) \\
\alpha &= -\frac{1}{144\pi^2 v_F^3} \int \frac{dk}{k} U_k^2 \partial^3_{ix} G^f_{0k}(x,y)
\end{aligned}$$

$\Re L_{infl}$:

$$M_R \langle \ddot{\boldsymbol{x}} \rangle = -\langle \boldsymbol{\nabla} U(\boldsymbol{x}) \rangle - k \langle \dot{\boldsymbol{x}} \rangle + \alpha \langle \dddot{\boldsymbol{x}} \rangle + \mathcal{O}\left(\frac{d^4}{dt^4}\right) + \mathcal{O}\left(\langle x^3 \rangle\right)$$

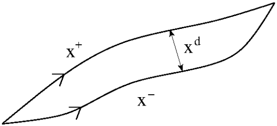
mass renormalization: $M_R = M + \delta M$

Ideal gas of fermions

$\Im L_{infl}^{(2)}$: decoherence

$$
\begin{aligned}
L_{infl} &= \boldsymbol{x}^d \left(-k\dot{\boldsymbol{x}} - \delta M \ddot{\boldsymbol{x}} + \alpha \dddot{\boldsymbol{x}} + i d_0 \boldsymbol{x}^d - i d_2 \ddot{\boldsymbol{x}}^d + \mathcal{O}\left(\frac{d^4}{dt^4}\right)\right) \\
&\quad + \mathcal{O}\left(x^d x^3\right) \\
d_0 &= -\frac{1}{48\pi^2} \int dk k^2 U_k^2 G_{0k}^i(x,y) \\
d_2 &= \frac{1}{96\pi^2 v_F^2} \int dk U_k^2 \partial_{ix}^2 G_{0k}^i(x,y)
\end{aligned}
$$

$e^{-\frac{1}{\hbar}\Im S_{infl}[\boldsymbol{x}^d]}$: suppression



Decoherence and consistence in the coordinate diagonal basis

Ideal gas of fermions

Density-density two-point function, $T = 0$

$$
\begin{aligned}
x &= \frac{\omega}{|\boldsymbol{k}|v_F}, \quad y = \frac{|\boldsymbol{k}|}{k_F}, \quad v_F = \frac{\hbar k_F}{m} \\
G^{\pm}_{\omega,\boldsymbol{k}} &= \frac{n_s k_F m}{2\pi^2\hbar^2} g^{\pm}(x,y) \\
g^{+}(x,y) &= -\frac{1}{4} + \frac{1}{4y}\left[1 - \left(x - \frac{y}{2}\right)^2\right] \ln\left|\frac{1 + x - \frac{y}{2}}{1 - x + \frac{y}{2}}\right| \\
g^{-}(x,y) &= -\frac{\pi}{4y}\begin{cases} 1 - (\frac{y}{2} - x)^2 & y > 2, \quad -1 < \frac{y}{2} - x < 1 \\ 1 - (\frac{y}{2} - x)^2 & y < 2, \quad -1 < \frac{y}{2} - x < -1 + y \\ 2xy & y < 2, \quad 0 < x < 1 - \frac{y}{2} \end{cases}
\end{aligned}
$$

Ideal gas of fermions

$L^{(2)}_{infl}$: $T = 0$

$x = \frac{\omega}{|\boldsymbol{k}|v_F} = \frac{v_{ph}}{v_F}, y = \frac{|\boldsymbol{k}|}{k_F} \sim 0,\ v_F = \frac{\hbar k_F}{m}$: $g^f_{\omega\boldsymbol{k}} = -i\frac{\pi}{2}x,\ g^i_{\omega\boldsymbol{k}} = -i\frac{\pi}{2}|x|$

$$\begin{aligned}
\Re L_{infl} &= \boldsymbol{x}^d(-k\dot{\boldsymbol{x}} - \delta M\ddot{\boldsymbol{x}}) + \mathcal{O}\left(\boldsymbol{x}^4\right), \quad k = \frac{k_F m}{48\pi^3\hbar^2 v_F}\int_0^\infty dk k^3 U_k^2 \\
\Im L_{infl} &= \frac{1}{8}\int_{\boldsymbol{k}} U^2_{\boldsymbol{k}}[(\boldsymbol{k}\vec{\Delta}_1\boldsymbol{x}^d)^2 - (\boldsymbol{k}\vec{\Delta}_0\boldsymbol{x}^d)^2]G^i_{\boldsymbol{k}\dot{\boldsymbol{x}},\boldsymbol{k}} \\
&= i\lambda\boldsymbol{x}^{d2}|\dot{\boldsymbol{x}}|f\left(\frac{\boldsymbol{x}^d\dot{\boldsymbol{x}}}{|\boldsymbol{x}^d||\dot{\boldsymbol{x}}|}\right) \ \leftarrow \ \text{no decoherence for } \dot{\boldsymbol{x}} = 0
\end{aligned}$$

$$\begin{aligned}
\lambda &= \frac{k_F m}{48\pi^3\hbar^2 v_F}\int_0^\infty dk k^4 U_k^2 \\
f(\boldsymbol{u}\boldsymbol{v}) &= \frac{1}{\boldsymbol{u}^2|\boldsymbol{v}|}\int d\boldsymbol{n}(\boldsymbol{n}\boldsymbol{u})^2|\boldsymbol{n}\boldsymbol{v}|
\end{aligned}$$

Loop integral is dominated by $x, y \sim 0$:

- $k^4 U_k^2 \sim 0$ for $y = \frac{k}{k_F} \gg 1$: the potential can not resolve the gas particles separately
- $k^4 U_k^2 \sim 0$ for $|x| = \frac{m|\boldsymbol{k}\dot{\boldsymbol{x}}|}{k\hbar k_F} \gg 1$: particle motion slower than the average velocity of the gas particles

Ideal gas of fermions

$L^{(2)}_{infl}$: $T=0$, screened Coulomb interaction

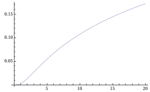
$$U_k = \frac{4\pi e^2}{k^2 + r_s^{-2}}, \quad r_s^2 = \frac{4e^2 m k_F}{\pi^2 \hbar^2} \quad \leftarrow \text{ IR regulator for } k$$

Not to resolve the gas particles: extended particle form factor $\rho(\boldsymbol{x}) \Longrightarrow U_{\boldsymbol{k}} \to \rho_{\boldsymbol{k}} U_{\boldsymbol{k}}$

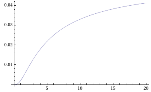
$$\rho(\boldsymbol{x}) = \frac{e^{-r/r_a}}{8\pi r_a^3} \ (\sim\text{H atom})$$

$$k = \frac{\hbar}{a_0^2} f_k\left(\frac{r_s}{r_a}\right), \quad \lambda = \frac{\hbar}{r_a a_0^2} f_\lambda\left(\frac{r_s}{r_a}\right), \quad a_0 = \frac{\hbar^2}{me^2}$$

$$f_k(z) = \frac{z^4[1-9z^2-9z^4+17z^6-6z^4(3+z^2)\ln z^2]}{36\pi(1-z^2)^5}$$



$$f_\lambda(z) = \frac{z^4(1+5z)}{96(1+z)^5}$$



Density independent k and λ!

Ideal gas of fermions

Quadratic influence functional at $T=0$, screened Coulomb interaction

Asymptotic:

$$k \sim \frac{\hbar}{a_0^2}\begin{cases}\frac{6\ln(\frac{r_s}{r_a})^2-17}{36\pi} & \frac{r_s}{r_a}\to\infty\\ \frac{(\frac{r_s}{r_a})^4}{36\pi} & \frac{r_s}{r_a}\to 0\end{cases}, \qquad \lambda \sim \frac{\hbar}{r_a a_0^2}\begin{cases}\frac{5}{96} & \frac{r_s}{r_a}\to\infty\\ \frac{(\frac{r_s}{r_a})^4}{96} & \frac{r_s}{r_a}\to 0\end{cases}$$

↑

linear UV divergence of
$\lambda = \frac{k_F m}{48\pi^3\hbar^2 v_F}\int_0^\infty dk k^4 U_k^2$

$(\frac{r_s}{r_a})^4$ for $\frac{r_s}{r_a}\to 0$: $U_{\boldsymbol{k}}^2$ in

$$S_{infl}[\hat{\boldsymbol{x}}] = -\frac{1}{2}\sum_{\sigma\sigma'}\sigma\sigma'\int_{\omega\boldsymbol{k}} dtdt' U_{\boldsymbol{k}}^2 e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}^{\sigma}(t)-\boldsymbol{x}^{\sigma'}(t'))} G^{\sigma,\sigma'}_{\omega\boldsymbol{k}}$$

Scales: $v(t) = v_0 e^{-\frac{t}{\tau}}$, $\tau = \dfrac{m a_0^2}{\hbar f_k(\frac{r_s}{r_a})}$,

$$e^{-\Im S_{eff}} = e^{-Dtx^{d2}},\ D = \frac{|\dot{\boldsymbol{x}}| f_\lambda(\frac{r_s}{r_a})}{r_a a_0^2}$$

Effective theory of the current in an ideal gas

Inertial forces

Nonlinear change of coordinates - $S = \mathcal{O}\left(x^2\right) \Longrightarrow S = \mathcal{O}\left(x^3\right)$

- inertial forces $\Longrightarrow$ **interaction**

Harmonic oscillator: $y = \frac{x^2}{2}$

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \quad \rightarrow \quad \frac{m}{4y}\dot{y}^2 - m\omega^2 y$$

Nontrivial dynamics of $j^\mu(x) = \langle\bar{\psi}(x)\gamma^\mu\psi(x)\rangle$ in an ideal electron gas

Effective theory of the current in an ideal gas

Equation of motion

Generator functional:

$$\begin{aligned} e^{\frac{i}{\hbar}W[\hat{a}]} &= \int D[\hat{\psi}]D[\hat{\bar{\psi}}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}(\hat{G}^{-1}+\not{\hat{a}})\hat{\psi}} \\ W[\hat{a}] &= \frac{\hbar}{2}\hat{a}\hat{G}\hat{a}+\mathcal{O}\left(\hat{a}^3\right) \end{aligned}$$

Effective action:

$$\Gamma[\hat{j}] = -\frac{\hbar}{2}\hat{j}\hat{G}\hat{j}+\mathcal{O}\left(\hat{j}^3\right)$$

Linearized equation of motion:

$$\frac{1}{\hbar}\hat{G}^{-1}\hat{a} = \hat{j}$$

Effective theory of the current in an ideal gas

Equation of motion

$$j^\mu = (n_0 + n, \boldsymbol{j}),\; a^\mu = (\phi, \boldsymbol{a})$$

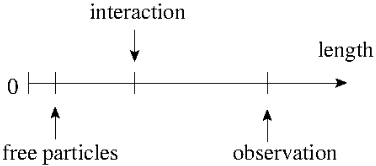
$$-\frac{v_F}{\pi^2\hbar}\phi = \left[\sum_{j,k=0}^{\infty} b_{\ell,j,k}\left(\frac{i\omega}{v_F|\boldsymbol{q}|}\right)^j\left(\frac{\boldsymbol{q}^2}{k_F^2}\right)^k\right]n$$

$$\frac{T^2}{v_F\hbar}\boldsymbol{a} - \frac{b_{t,2,0}}{b_{\ell,2,0}}\frac{v_F}{\pi^2\hbar}\frac{\omega\boldsymbol{q}}{\boldsymbol{q}^2}\phi = \left[\sum_{j,k=0}^{\infty} b_{t,j,k}\left(\frac{i\omega}{v_F|\boldsymbol{q}|}\right)^j\left(\frac{\boldsymbol{q}^2}{k_F^2}\right)^k + \frac{\boldsymbol{q}\otimes\boldsymbol{q}}{\boldsymbol{q}^2}\sum_{j,k=0}^{\infty} b_{j,k}\left(\frac{i\omega}{v_F|\boldsymbol{q}|}\right)^j\left(\frac{\boldsymbol{q}^2}{k_F^2}\right)^k\right]\boldsymbol{j}$$

- Odd and even powers of ω mixed: dissipation
- Deviation from the phenomenological Navier-Stokes eq.:
  - Charge conservation: $\frac{\omega}{|\boldsymbol{q}|} \leftrightarrow \omega$ (phenomenology did not foresee $\frac{1}{|\boldsymbol{q}|}$)
  - $\mathcal{O}\left(\omega^2\right)$ needed for the IR normal modes

Effective theory of the current in an ideal gas

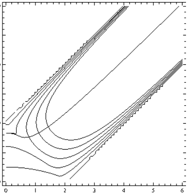
Interaction?



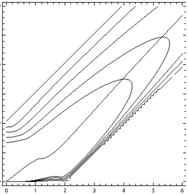
Effective theory of the current in an ideal gas

Normal modes

$x = \frac{\omega}{|\boldsymbol{k}| v_F}$, $y = \frac{|\boldsymbol{k}|}{k_F}$



longitudinal

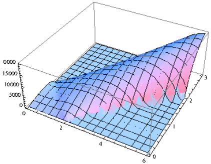


transverse

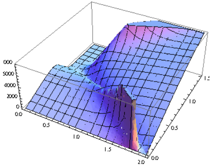
Effective theory of the current in an ideal gas

Decoherence

Quadratic decoherence (consistency) strength: $c(x,y) = \frac{\delta^2 \Im S_{eff}}{\delta a^+(-q)\delta a^-(q)}$



$T = 0$



$T \neq 0$

Summary

1. Extended action principle for open systems
   - Redoubling of the degrees of freedom
   - Existence of doubler revealed by quantum fluctuations only
2. Effective theories
   - influence functional = closed + open effective interactions
   - dissipative forces $\leftrightarrow$ entanglement
3. Examples:
   - Radiation back-reaction in CED
   - $T = 0$, extended particle and screened Coulomb potential density independence, $n \to 0$?
   - Dissipative effective current dynamics in an ideal fermion gas
     - Short distance single-particle core
     - Screened by interactions