The degrees of freedom in general relativity

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ELMO Seminar
Budapest, 14 November 2014

Plan

Outline

Janus-faced GR: No clear distinction in general relativity

The arena:

All the pre-GR physical theories provide a distinction between the **arena** in which physical phenomena take place and the **phenomena** themselves.

	arena:	phenomena:
classical mechanics	phase space	dynamical trajectories
electrodynamics	Minkowski spacetime	dynamics of the Maxwell field
general relativity	curved spacetime	the geometry of the spacetime

Such a clear distinction between the arena and the phenomenon is simply not available in general relativity

the metric plays both roles.

- GR is more than merely a field theoretic description of gravity.
  It is a certain body of universal rules:
  - modeling the space of events by a four-dimensional differentiable manifold
  - the use of tensor fields and tensor equations to describe physical phenomena
  - use of the (otherwise dynamical) metric in measuring of distances, areas, volumes, angles ...

The degrees of freedom in GR:

What are the degrees of freedom?

- in a theory possessing an initial value formulation: "degrees of freedom" is a synonym of "how many" distinct solutions of the equations exist
- in ordinary particle mechanics: the number of degrees of freedom is the number of quantities that must be specified as initial data divided by two

The degrees of freedom in the linearized theory:

Einstein (1916, 1918): the field equations involve **two degrees of freedom per spacetime point** when studying linearized theory

Is the full nonlinear theory characterized by two degrees of freedom?

Darmois (1927): probably the earliest **answer in the confirmatory** based on consideration of the Cauchy (or initial value) problem

The degrees of freedom in GR:

What are the main issues?

- "... **no way singles** out precisely which functions (i.e., which of the 12 metric or extrinsic curvature components or functions of them) can be freely specified, which functions are determined by the constraints, and which functions correspond to gauge transformations. Indeed, **one of the major obstacles to developing a quantum theory of gravity** is the inability to single out the physical degrees of freedom of the theory. "

  R.M. Wald: General Relativity, Univ. Chicago Press, (1984)
- How these two degrees of freedom may be expressed in terms of the components of the metric tensor and its derivatives (or such combinations of these as, e.g. the Riemann tensor)?
  - Notably, there may be **many possible representations**.
- The main issue is **not to find the only legitimate quantities** representing the gravitational degrees of freedom.
- Rather, **finding a particularly convenient embodiment** of this information which could be used in solving various problems of physical interest.

The outline:

Based on three recent papers

1. I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
2. I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, to appear in Class. Quantum Grav.
3. I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, submitted to Class. Quantum Grav.

The main message:

1. smooth Einsteinian spaces of **Euclidean and Lorentzian signature**
   - smoothly foliated by a **two-parameter family of codimension-two-surfaces** which are orientable compact without boundary in M
2. the **Bianchi identity** and **a pair of nested** $1+n$ **and** $1+[n-1]$ **decompositions**
   - **explore the relations** of the various projections of the field equations
   - indicate: mixed hyperbolic–hyperbolic initial value problem can be introduced
3. solving the constraints:
   - since the early works by A. Lichnerowicz (1944) and J.W. York (1972) the constraints are solved by transforming them into a semilinear elliptic system by applying the "conformal method" ... as opposed to this ...
   - a **new gauge fixing** and some **geometrically distinguished variables regardless** whether the primary space is Riemannian or Lorentzian
   - the $1+n$ **momentum constraint** as a **first order symmetric hyperbolic syst.**.
   - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
4. the true degrees of freedom to gravity?

Assumptions:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dim. ($n \geq 3$), smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian or Riemannian metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$G_{ab} - \mathscr{G}_{ab} = 0$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence, $\nabla^a \mathscr{G}_{ab} = 0$.

  - or, more conventionally,

$$[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}$$

matter fields satisfying their field equations with energy-momentum tensor T_{ab} and with cosmological constant Λ

  -

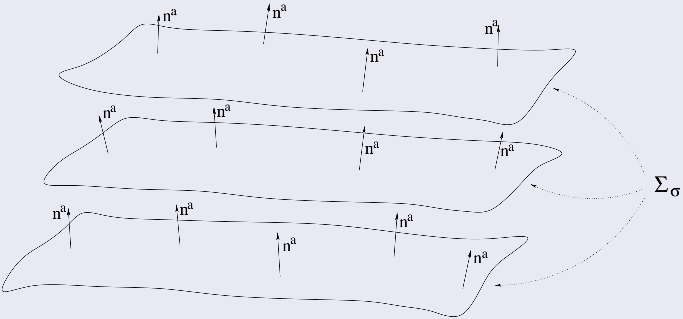
$$\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}$$

Outline

The primary $1+n$ splitting:

No restriction on the topology by Einstein's equations! (local PDEs)

- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some codimension one manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.



Projections:

The projection operator:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$\boxed{n^a n_a = \epsilon}$$

  - the sign of the norm of n^a is not fixed. ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, respectively.

- **the projection operator**

$$\boxed{h^a{}_b = g^a{}_b - \epsilon\, n^a n_b} \qquad [g^a{}_b \text{ is the identity operator}]$$

  to the level surfaces of $\sigma : M \to \mathbb{R}$.

- **the induced metric** on the $\sigma = const$ level surfaces

$$\boxed{h_{ab} = h^e{}_a h^f{}_b\, g_{ef}}$$

  while D_a will denote the covariant derivative operator associated with h_{ab}.

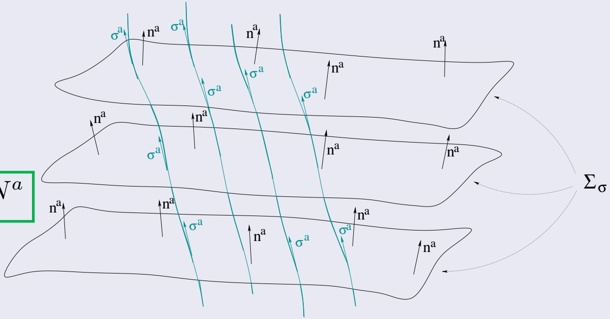
Decompositions of various fields:

Examples:

- a form field: $$L_a = \delta^e{}_a\, L_e = (h^e{}_a + \epsilon\, n^e n_a)\, L_e = \boldsymbol{\lambda}\, n_a + \mathbf{L}_a$$
  - where $\boldsymbol{\lambda} = \epsilon\, n^e\, L_e$ and $\mathbf{L}_a = h^e{}_a\, L_e$
- "time evolution vector field"

$$\sigma^a : \quad \sigma^e \nabla_e \sigma = 1$$

$$\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a$$



- where N and N^a denotes the 'laps' and 'shift' of $\sigma^a = (\partial_\sigma)^a$:

$$N = \epsilon\, (\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e$$

Decompositions of various fields:

Any symmetric tensor field P_{ab} can be decomposed

in terms of n^a and fields living on the $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$

It is also rewarding to inspect the decomposition of the contraction $\nabla^a P_{ab}$:

$$\epsilon\,(\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\,(K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\,\epsilon\, \dot{n}^e \mathbf{p}_e]$$
$$(\nabla^a P_{ae})\, h^e{}_b = \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}]$$

$$\dot{n}_a := n^e \nabla_e n_a = -\epsilon\, D_a \ln N$$

back

Decompositions of various fields:

Examples:

- the metric

$$g_{ab} = \epsilon\, n_a n_b + h_{ab}$$

- the "source term"

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

where

$$\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$$

- the r.h.s. of our basic field equation $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

Relations between various parts of the basic equations:

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot n^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

back

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}) + [\,(K^e{}_e)\, E^{(\mathcal{M})}_b + E^{(\mathcal{H})}\, \dot n_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})\,\dot n^a\,] = 0$$

a first order symmetric hyperbolic linear homogeneous system for $(E^{(\mathcal{H})}, E^{(\mathcal{M})}_i)^T$ fosh

Theorem

Let (M, g_{ab}) be as specified above. Assume that the metric h_{ab} induced on the $\sigma = const$ level surfaces is Riemannian. Then, **regardless whether g_{ab} is of Lorentzian or Euclidean signature**, *any solution to the reduced equations $E^{(\mathcal{EVOL})}_{ab} = 0$ is also a solution to the full set of field equations $G_{ab} - \mathscr{G}_{ab} = 0$ provided that the constraint expressions $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_a$ vanish on one of the $\sigma = const$ level surfaces.* back

Relations between various parts of the basic equations:

Corollary

If the constraint expressions $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_a$ *vanish on all the* $\sigma = const$ *level surfaces then the relations*

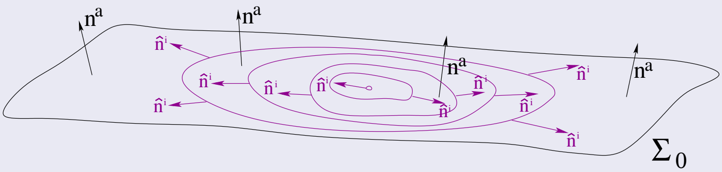
$$\begin{aligned} K^{ab}\, E^{(\mathcal{EVOL})}_{ab} &= 0 \\ D^a E^{(\mathcal{EVOL})}_{ab} - \epsilon\, \dot{n}^a\, E^{(\mathcal{EVOL})}_{ab} &= 0 \end{aligned}$$

hold for the evolutionary expression $E^{(\mathcal{EVOL})}_{ab}$.

The secondary $1 + [n-1]$ splitting:

Assume now that on one of the $\sigma = const$ level surfaces—say on Σ_0—there exists a smooth function $\rho : \Sigma_0 \to \mathbb{R}$, with nowhere vanishing gradient such that:

- the $\rho = const$ level surfaces $\mathscr{S}_\rho$ are homologous to each other and such that they are orientable compact without boundary in M.



- The metric h_{ij} on Σ_0 can be decomposed as

$$h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j$$

in terms of the positive definite metric $\hat{\gamma}_{ij}$, induced on the $\mathscr{S}_\rho$ hypersurfaces,

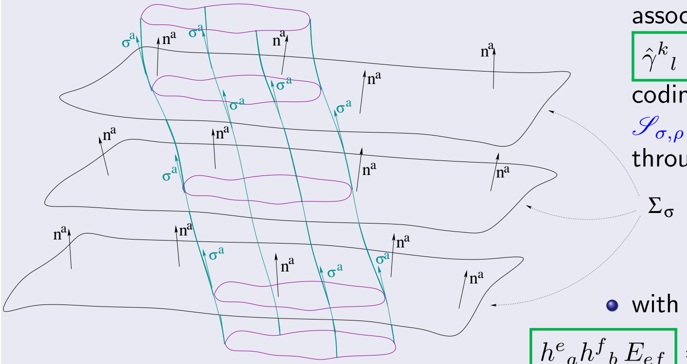
- and the unit norm field

$$\hat{n}^i = \hat{N}^{-1} \left[(\partial_\rho)^i - \hat{N}^i \right]$$

normal to the $\mathscr{S}_\rho$ hypersurfaces on Σ_0, where $\hat{N}$ and $\hat{N}^i$ denotes the 'laps' and 'shift' of an 'evolution' vector field $\rho^i = (\partial_\rho)^i$ on Σ_0.

Secondary projections:

The Lie transport of this foliation of Σ_0 along the integral curves of the vector field σ^a yields then a two-parameter foliation $\mathscr{S}_{\sigma,\rho}$:



- the fields $\hat{n}^i$, $\hat{\gamma}_{ij}$ and the associated projection op. $\boxed{\hat{\gamma}^k{}_l = h^k{}_l - \hat{n}^k \hat{n}_l}$ to the codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$ get to be well-defined throughout M.

- with some algebra

$$\boxed{h^e{}_a h^f{}_b \, E_{ef}} = E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})}$$

- can be put into the form

$$h^e{}_a h^f{}_b \, E_{ef} = \boxed{{}^{(n)}E_{ij} = {}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij}}$$

The integrability condition for ${}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij} = 0$

$$ {}^{(n)}E_{ab} = \hat{E}^{(\mathcal{H})}\hat{n}_a\hat{n}_b + [\hat{n}_a\hat{E}^{(\mathcal{M})}_b + \hat{n}_b\hat{E}^{(\mathcal{M})}_a] + (\hat{E}^{(\mathcal{EVOL})}_{ab} + \hat{\gamma}_{ab}\hat{E}^{(\mathcal{H})}) $$

$$ \hat{E}^{(\mathcal{H})} = \hat{n}^e\hat{n}^f\,{}^{(n)}E_{ef}, \quad \hat{E}^{(\mathcal{M})}_a = \hat{\gamma}^e{}_a\hat{n}^f\,{}^{(n)}E_{ef}, \quad \hat{E}^{(\mathcal{EVOL})}_{ab} = \hat{\gamma}^e{}_a\hat{\gamma}^f{}_b\,{}^{(n)}E_{ef} - \hat{\gamma}_{ab}\hat{E}^{(\mathcal{H})} $$

Lemma

The integrability condition $D^a[{}^{(n)}\mathscr{G}_{ab}] = 0$ holds on Σ_σ if the momentum constraint expression $E^{(\mathcal{M})}_b$, along with its Lie derivative $\mathscr{L}_n E^{(\mathcal{M})}_b$, vanishes there.

Lemma

$\hat{E}^{(\mathcal{H})}$ and $\hat{E}^{(\mathcal{M})}_a$ vanish identically along a worldline representing a regular origin.

Corollary

Assume that $E^{(\mathcal{M})}_b = 0$ on all the Σ_σ level surfaces and a regular origin exist in M. Then any solution to the secondary reduced equations $\hat{E}^{(\mathcal{EVOL})}_{ij} = 0$ is also a solution to the full set of secondary equations ${}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij} = 0$. Theorem

Set up for a mixed initial value problem:

We have seen ...

1. $E_b^{(\mathcal{M})} \equiv 0 \Longrightarrow D^a[{}^{(n)}\mathscr{G}_{ab}] = 0 \Longrightarrow$ it suffices to solve the reduced equations $\hat{E}_{ij}^{(\mathcal{EVOL})} = 0$ & a regular origin exist $\mathscr{W}_{\mathscr{S}_{\rho_*}}$ in $M \Longrightarrow$ to get solution to the full set of equations ${}^{(n)}E_{ij} = {}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij} = 0$.
2. ${}^{(n)}E_{ij} = h^e{}_a h^f{}_b\, E_{ef} = E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})} = 0$ & $E_b^{(\mathcal{M})} \equiv 0$ ◂ $\nabla^a E_{ab} = 0$

$\Longrightarrow \boxed{\mathscr{L}_n\, E^{(\mathcal{H})} + E^{(\mathcal{H})}\,(K^e{}_e) = 0} \Longrightarrow \boxed{E^{(\mathcal{H})} \equiv 0 \ : \ E^{(\mathcal{H})} = 0 \ \text{ on } \ \Sigma_0}$

Theorem

To get solution to the full set of the primary Einstein's equations $G_{ab} - \mathscr{G}_{ab} = 0$ it suffices—regardless whether the primary metric g_{ab} is Riemannian or Lorentzian—to solve the secondary reduced equations $\hat{E}_{ij}^{(\mathcal{EVOL})} = 0$, along with $E_b^{(\mathcal{M})} = 0$ on all the Σ_σ hypersurfaces, provided that $E^{(\mathcal{H})} = 0$ holds on Σ_0 and there exists a regular origin $\mathscr{W}_{\mathscr{S}_{\rho_*}}$ in M.

The explicit forms:

Expressions in the $1+n$ decomposition:

$$
\begin{aligned}
E^{(\mathcal{H})} &= n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} \\
E^{(\mathcal{M})}_a &= h^e{}_a n^f E_{ef} = D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a \\
E^{(\mathcal{EVOL})}_{ab} &= {}^{(n)}\!R_{ab} + \epsilon\,\{-\mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon N^{-1}\,D_a D_b N\} \\
&\quad + \tfrac{1+\epsilon}{(n-1)}\,h_{ab}\,E^{(\mathcal{H})} - \left(\mathfrak{S}_{ab} - \tfrac{1}{n-1}\,h_{ab}\,[\mathfrak{S}_{ef}\,h^{ef} + \epsilon\,\mathfrak{e}]\right)
\end{aligned}
$$

where

$$
\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\,h^e{}_a n^f \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b \mathscr{G}_{ef}
$$

and the extrinsic curvature K_{ab} which is defined as

$$
K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\,\mathscr{L}_n h_{ab}
$$

where $\mathscr{L}_n$ stands for the Lie derivative with respect to n^a

The explicit forms:

Expressions in the $1+[n-1]$ decomposition:

$$\begin{aligned}
\hat{E}^{(\mathcal{H})} &= \tfrac{1}{2}\,\{-\hat{R} + (\hat{K}^l{}_l)^2 - \hat{K}_{kl}\hat{K}^{kl} - 2\,\hat{\mathfrak{e}}\}\,,\\
\hat{E}^{(\mathcal{M})}_i &= \hat{D}^l\hat{K}_{li} - \hat{D}_i\hat{K}^l{}_l - \hat{\mathfrak{p}}_i\,,\\
\hat{E}^{(\mathcal{EVOL})}_{ij} &= \hat{R}_{ij} - \mathscr{L}_{\hat{n}}\hat{K}_{ij} - (\hat{K}^l{}_l)\hat{K}_{ij} + 2\,\hat{K}_{il}\hat{K}^l{}_j - \hat{N}^{-1}\,\hat{D}_i\hat{D}_j\hat{N}\\
&\quad + \hat{\gamma}_{ij}\{\mathscr{L}_{\hat{n}}\hat{K}^l{}_l + \hat{K}_{kl}\hat{K}^{kl} + \hat{N}^{-1}\,\hat{D}^l\hat{D}_l\hat{N}\} - [\hat{\mathfrak{S}}_{ij} - \hat{\mathfrak{e}}\,\hat{\gamma}_{ij}]
\end{aligned}$$

where $\hat{D}_i, \hat{R}_{ij}$ and $\hat{R}$ denote the covariant derivative operator, the Ricci tensor and the scalar curvature of $\hat{\gamma}_{ij}$, respectively. The 'hatted' source terms $\hat{\mathfrak{e}}$, $\hat{\mathfrak{p}}_i$ and $\hat{\mathfrak{S}}_{ij}$ and the extrinsic curvature $\hat{K}_{ij}$ are defined as

$$\hat{\mathfrak{e}} = \hat{n}^k\hat{n}^l\,{}^{(n)}\mathscr{G}_{kl}\,, \quad \hat{\mathfrak{p}}_i = \hat{\gamma}^k{}_i\,\hat{n}^l\,{}^{(n)}\mathscr{G}_{kl} \quad \text{and} \quad \hat{\mathfrak{S}}_{ij} = \hat{\gamma}^k{}_i\hat{\gamma}^l{}_j\,{}^{(n)}\mathscr{G}_{kl}$$

and

$$\hat{K}_{ij} = \hat{\gamma}^l{}_i\,D_l\,\hat{n}_j = \tfrac{1}{2}\,\mathscr{L}_{\hat{n}}\hat{\gamma}_{ij}$$

Outline

So far only completely generic decompositions were used.

There is an obvious need for a suitable gauge choice:

in local coordinates x^α the Ricci tensor reads as

$$R_{\mu\nu} = -\tfrac{1}{2}\, g^{\varepsilon\sigma}\,\{\partial_\varepsilon\partial_\sigma g_{\mu\nu} + \partial_\mu\partial_\nu g_{\varepsilon\sigma} - \partial_\varepsilon\partial_\nu g_{\mu\sigma} - \partial_\mu\partial_\sigma g_{\varepsilon\nu}\} + F_{\mu\nu}(g_{\lambda\kappa}, \partial_\gamma g_{\lambda\kappa})$$

$F_{\mu\nu}$ is quadratic in the Christoffel symbols:

- $\Gamma^\gamma{}_{\alpha\beta} = \tfrac{1}{2} g^{\gamma\varepsilon}\,\{\partial_\alpha g_{\varepsilon\beta} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\alpha\beta}\}$
- $\Gamma^\gamma{}_{\alpha\beta}$ involves the inverse metric $g^{\mu\nu}$ which can be given as $g'^{\mu\nu}/\det(g)$, where $g'^{\mu\nu}$ can at best be determined as a polynomial of degree n of the components $g_{\mu\nu}$ in an $n+1$ dimensional space.
- $F_{\mu\nu} = F'_{\mu\nu}/[\det(g)]^2$, where $F'_{\mu\nu}$ and $[\det(g)]^2$ are separately polynomials of degree $2(n+1)$ in $g_{\varepsilon\sigma}$ and $\partial_\gamma g_{\varepsilon\sigma}$

The complexity can be reduced

by the following gauge fixing:

There exist a smooth function $\Omega : M \to \mathbb{R}$—which does not vanish except at an origin where the foliation $\mathscr{S}_\rho$ smoothly reduces to a point on the Σ_0 level surfaces—such that the induced metric $\hat{\gamma}_{ij}$ can be decomposed as

$$\hat{\gamma}_{ij} = \Omega^2 \, \gamma_{ij}$$

where γ_{ij} is such that

$$\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij}) = 0$$

on each of the $\mathscr{S}_\rho$ surfaces.

What does the second relation mean?

In virtue of

$$\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij}) = \mathscr{L}_\rho \, \ln[\det(\gamma_{ij})]$$

the determinant is independent of ρ but may depend on the ‘angular’ coordinates.

Verifications

The conformal structure:

- Do the desired smooth function $\Omega : M \to \mathbb{R}$ and the metric γ_{ij} exist?
-
$$\boxed{\hat\gamma^{ij}(\mathscr{L}_\rho \hat\gamma_{ij}) = \cancel{\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij})} + (n-1)\,\mathscr{L}_\rho(\ln \Omega^2)}$$
- for any smooth distribution of the induced $[n-1]$-metric $\hat\gamma_{ij}$ on the $\mathscr{S}_\rho$ surfaces one may integrate the above relation along the integral curves of ρ^a on Σ_0, starting at $\mathscr{S}_0$.
- one gets $\Omega^2 = \Omega^2(\rho, x^3, \dots, x^{n+1})$ as

$$\boxed{\Omega^2 = \Omega_0^2 \cdot \exp\left[\frac{1}{n-1}\int_0^\rho \left(\hat\gamma^{ij}(\mathscr{L}_\rho \hat\gamma_{ij})\right)\, d\tilde\rho\right]}$$

where $\Omega_0 = \Omega_0(x^3, \dots, x^{n+1})$ denotes the conformal factor at $\mathscr{S}_0$.

The decomposition of the extrinsic curvature:

The Σ_σ hypersurfaces in both cases are spacelike:

$$K_{ij} = \boldsymbol{\kappa}\, \hat{n}_i \hat{n}_j + [\hat{n}_i\, \mathbf{k}_j + \hat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

$$\begin{aligned}
\boldsymbol{\kappa} &= \hat{n}^k \hat{n}^l\, K_{kl} = \hat{n}_k (\mathscr{L}_n \hat{n}^k) \\
\mathbf{k}_i &= \hat{\gamma}^k{}_i \hat{n}^l\, K_{kl} = \tfrac{1}{2} \hat{\gamma}^k{}_i\, (\mathscr{L}_n \hat{n}_k) - \tfrac{1}{2} \hat{\gamma}_{ki}\, (\mathscr{L}_n \hat{n}^k) \\
\mathbf{K}_{ij} &= \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j\, K_{kl} = \tfrac{1}{2}\, \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j\, (\mathscr{L}_n \hat{\gamma}_{kl})
\end{aligned}$$

$$\mathbf{K}^l{}_l = \hat{\gamma}^{kl}\, \mathbf{K}_{kl} = \tfrac{1}{2}\, \hat{\gamma}^{ij} (\mathscr{L}_n \hat{\gamma}_{ij}) = \tfrac{n-1}{2}\, \mathcal{L}_n \ln \Omega^2$$

(conformal invariant) projection taking the trace free parts on the $\mathscr{S}_{\sigma,\rho}$ surfaces:

$$\Pi^{kl}{}_{ij} = \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j - \tfrac{1}{n-1}\, \hat{\gamma}_{ij} \hat{\gamma}^{kl} = \gamma^k{}_i \gamma^l{}_j - \tfrac{1}{n-1}\, \gamma_{ij} \gamma^{kl}$$

$$\mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\, \gamma_{ij} (\gamma^{ef} \mathbf{K}_{ef}) \quad \text{and} \quad \mathring{\hat{K}}_{ij} = \hat{K}_{ij} - \tfrac{1}{n-1}\, \gamma_{ij} (\gamma^{ef} \hat{K}_{ef})$$

The $1+n$ constraints

The momentum constraint:

$$\boxed{E^{(\mathcal{M})}_a = h^e{}_a n^f E_{ef} = D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a = 0}$$

◂ div

$$(\hat K^l{}_l)\,\mathbf{k}_i + \hat D^l \mathring{\mathbf{K}}_{li} + \boldsymbol{\kappa}\,\dot{\hat n}_i + \mathscr{L}_{\hat n}\mathbf{k}_i - \dot{\hat n}^l\,\mathbf{K}_{li} - \hat D_i\boldsymbol{\kappa} - \tfrac{n-2}{n-1}\,\hat D_i(\mathbf{K}^l{}_l) - \epsilon\,\mathfrak{p}_l\,\hat\gamma^l{}_i = 0$$

$$\boldsymbol{\kappa}\,(\hat K^l{}_l) + \hat D^l\mathbf{k}_l - \mathbf{K}_{kl}\hat K^{kl} - 2\,\dot{\hat n}^l\,\mathbf{k}_l - \mathscr{L}_{\hat n}(\mathbf{K}^l{}_l) - \epsilon\,\mathfrak{p}_l\,\hat n^l = 0$$

where $\boxed{\dot{\hat n}_k = \hat n^l D_l \hat n_k = -\hat D_k(\ln\hat N)}$

After some algebra in coordinates $(\rho, x^3, \dots, x^{n+1})$ adopted to the foliation $\mathscr{S}_{\sigma,\rho}$:

$$\left\{\begin{pmatrix} \frac{n-1}{(n-2)\,\hat N}\,\hat\gamma^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -\frac{(n-1)\,\hat N^K}{(n-2)\,\hat N}\,\hat\gamma^{AB} & -\hat\gamma^{AK} \\ -\hat\gamma^{BK} & -\hat N^K \end{pmatrix}\partial_K\right\}\begin{pmatrix}\mathbf{k}_B \\ \mathbf{K}^E{}_E\end{pmatrix} + \begin{pmatrix}\mathscr{B}^A_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})}\end{pmatrix} = 0$$

Is a first order **symmetric hyperbolic** system for the vector valued variable

$$\boxed{(\mathbf{k}_B, \mathbf{K}^E{}_E)^T}$$

where **the 'radial coordinate' ρ plays the role of 'time'**. ... with characteristic cone (apart from the surfaces $\mathscr{S}_\rho$ with $\hat n^i\xi_i = 0$) $\boxed{[\hat\gamma^{ij} - (n-1)\,\hat n^i\hat n^j]\,\xi_i\xi_j = 0}$

The $1+n$ constraints

The Hamiltonian constraint:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\, {}^{(n)}R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

using $${}^{(n)}R = \hat R - \left\{2\,\mathscr{L}_{\hat n}(\hat K^l{}_l) + (\hat K^l{}_l)^2 + \hat K_{kl}\hat K^{kl} + 2\,\hat N^{-1}\,\hat D^l\hat D_l\hat N\right\}$$

$$-\epsilon\,\hat R + \epsilon\left\{2\,\mathscr{L}_{\hat n}(\hat K^l{}_l) + (\hat K^l{}_l)^2 + \hat K_{kl}\,\hat K^{kl} + 2\,\hat N^{-1}\,\hat D^l\hat D_l\hat N\right\}$$
$$+ 2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + (\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathbf{K}_{kl}\,\mathbf{K}^{kl} - 2\,\mathfrak{e} = 0$$

- $\not\epsilon = \pm 1$ elliptic equation for Ω: using $\hat K^l{}_l = \frac{n-1}{2}\,\mathscr{L}_{\hat n}\ln\Omega^2 - \hat N^{-1}\,\mathbb{D}_k\hat N^k$ and

$$\hat\gamma_{ij} = \Omega^2\,\gamma_{ij} \Longrightarrow \hat R = \Omega^{-2}\left[{}^{(\gamma)}R - (n-2)\left\{\mathbb{D}^l\mathbb{D}_l\ln\Omega^2 + \tfrac{(n-3)}{4}(\mathbb{D}^l\ln\Omega^2)(\mathbb{D}_l\ln\Omega^2)\right\}\right]$$

- parabolic equation for $\hat N$: $\hat K^l{}_l = \hat N^{-1}\,[\frac{n-1}{2}\,\mathscr{L}_\rho\ln\Omega^2 - \hat D_k\hat N^k]$ & $\hat N^{-1}\,\hat D^l\hat D_l\hat N$

  R. Bartnik (1993), R. Weinstein & B. Smith (2004)

- algebraic equation for $\boldsymbol{\kappa}$ provided that $\mathbf{K}^E{}_E$ does not vanish

Solving the constraints:

- The momentum constraint (satisfying a hyperbolic system) can always be solved as an initial value problem with initial data specified at some $\mathscr{S}_\rho \subset \Sigma_\sigma$ for the variables $\mathbf{k}_B, \mathbf{K}^E{}_E = \frac{n-1}{2}\,\mathcal{L}_n \ln \Omega^2$.
- In solving the coupled initial value problem the dependent variables, i.e. $\Omega, \hat{N}^A, \gamma_{AB}, \boldsymbol{\kappa} = \hat{n}_k(\mathscr{L}_n \hat{n}^k), \mathscr{L}_n\,\gamma_{AB} = 2\,\Omega^{-2}\,\mathring{\mathbf{K}}_{ij}$ are all freely specifiable on Σ_σ whereas $\hat{N}$ is determined by the Hamiltonian constraint.

Theorem

In terms of the geometrically distinguished variables the Hamiltonian and momentum constraints can be given as a parabolic–hyperbolic system. This coupled parabolic–hyperbolic system can be solved on the hypersurfaces Σ_0 for

$$\boxed{\hat{N}, \mathbf{k}_B, \mathscr{L}_n\Omega}$$

once a sufficiently regular choice for the variables

$$\boxed{\Omega, \gamma_{AB}, \hat{N}^A, \boldsymbol{\kappa} = \hat{n}_k(\mathscr{L}_n \hat{n}^k), \mathscr{L}_n\,\gamma_{AB}}$$

had been made throughout Σ_0. We also have the freedom to choose initial data to the parabolic–hyperbolic system on one of the two-surfaces $\mathscr{S}_\rho$ foliating Σ_0.

Outline

Summary:

1. smooth Einsteinian spaces of **Euclidean and Lorentzian signature** [$n+1$-dim. $(n \geq 3)$]
   - smoothly foliated by a **two-parameter family of codimension-two-surfaces**:
2. the **Bianchi identity** and **a pair of nested** $1+n$ **and** $1+[n-1]$ **decompositions**
   - **explored relations** of the various projections of the field equations $\Longrightarrow$
   - mixed hyperbolic–hyperbolic initial value problem can be introduced
3. solving the constraints:
   - a **new gauge fixing** and some **geometrically distinguished variables regardless** whether the primary space is Riemannian or Lorentzian
   - the $1+n$ **momentum constraint** as a **first order symmetric hyperbolic system**.
   - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
4. the **conformal structure** γ_{ij}, defined on the foliating codimension-two surfaces $\mathscr{S}_\rho$, appears to provide a **convenient embodiment** of the true degrees of freedom to various metric theories of gravity

Thanks for your attention

First order symmetric hyperbolic linear homogeneous system for $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E_e^{(\mathcal{M})} + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot n^e\, E_e^{(\mathcal{M})}) - \epsilon\, K^{ae}\,(E_{ae}^{(\mathcal{EVOL})} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

$$\mathscr{L}_n E_b^{(\mathcal{M})} + D^a (E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})}) + [\,(K^e{}_e)\, E_b^{(\mathcal{M})} + E^{(\mathcal{H})}\, \dot n_b - \epsilon\,(E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})})\, \dot n^a\,] = 0$$

- When writing them out explicitly in some local coordinates $(\sigma, x^1, \ldots, x^n)$ adopted to the vector field $\sigma^a = N\, n^a + N^a\!: \quad \sigma^e \nabla_e \sigma = 1$ and the foliation $\{\Sigma_\sigma\}$, read as

$$\left\{ \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N}\, h^{ij} \end{pmatrix} \partial_\sigma + \begin{pmatrix} -\frac{1}{N}\, N^k & h^{ik} \\ h^{jk} & -\frac{1}{N}\, N^k\, h^{ij} \end{pmatrix} \partial_k \right\} \begin{pmatrix} E^{(\mathcal{H})} \\ E_i^{(\mathcal{M})} \end{pmatrix} = \begin{pmatrix} \mathscr{E} \\ \mathscr{E}^j \end{pmatrix}$$

  where the source terms $\mathscr{E}$ and $\mathscr{E}^j$ are linear and homogeneous in $E^{(\mathcal{H})}$ and $E_i^{(\mathcal{M})}$. ◂ back

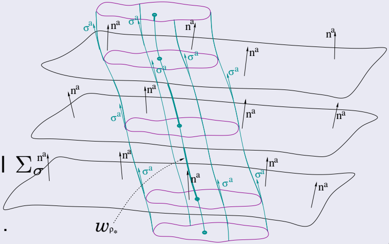
- It is also informative to inspect the characteristic cone associated with the above equation which—apart from the hypersurfaces Σ_σ with $n^i \xi_i = 0$—can be given as

$$(h^{ij} - n^i n^j)\, \xi_i \xi_j = 0$$

Having an origin

A world-line $\mathscr{W}_{\rho_*}$ represents an origin in M:

- If the foliating codimension-two-surfaces smoothly reduce to a point on the Σ_σ level surfaces at the location $\rho = \rho_*$. back
  - Note that then Ω vanishes at $\rho = \rho_*$. $\Longrightarrow$
  - The existence of an origin on the individual Σ_σ level surfaces is signified by the limiting behavior $\hat{\gamma}^{ij}(\mathscr{L}_\rho \hat{\gamma}_{ij}) \to \pm\infty$ while $\rho \to \rho_*^{\pm}$.



To have a regular origin in M:

- One needs to impose further conditions excluding the occurrence of various defects such as the existence of a conical singularity.
  - An origin $\mathscr{W}_{\rho_*}$ will be referred as being **regular** if there exist smooth functions $\hat{N}_{(2)}, \Omega_{(3)}$ and $\hat{N}^A_{(1)}$ such that, in a neighborhood of the location $\rho = \rho_*$ on the Σ_σ level surfaces, the basic variables $\hat{N}, \Omega$ and $\hat{N}^A$ can be given as

$$\hat{N} = 1 + (\rho - \rho_*)^2\, \hat{N}_{(2)},\ \ \Omega = (\rho - \rho_*) + (\rho - \rho_*)^3\, \Omega_{(3)},\ \ \hat{N}^A = (\rho - \rho_*)\, \hat{N}^A_{(1)}$$