Analytic structure of nonperturbative quark propagators and meson processes

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Overview

Introduction

- Microscopic understanding of strongly interacting matter, in hadronic phase AND quark-gluon phase, is important for physics of heavy ion collisions & compact stars (hopefully insights into EoS & phase diagram).
- Lattice calculations, large majority of Dyson-Schwinger eq.'s calculations, and many other QFT studies are not done in the physical, Minkowski spacetime, but in 4-dim Euclidean space.
- $\Rightarrow$ What happens with Wick rotation (relating Minkowski with Euclid) must be under control, but this is highly nontrivial in the nonperturbative case.
- Thus, we need also nonperturbative Green's functions which permit Wick rotation.
- For solving Bethe-Salpeter equation and calculation of processes, extrapolation to complex momenta is necessary.
- $\Rightarrow$ Knowledge of the analytic behavior in the whole complex plane is needed.

Sketch of the Dyson-Schwinger approach

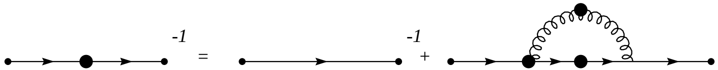
- Dyson-Schwinger (DS) approach: ranges from solving DS equations for Green's functions of non-perturbative QCD *ab initio*, to higher degrees of phenomenological modeling in applications including $T, \mu > 0$.

  e.g., [Alkofer, v.Smekal Phys. Rept. 353 (2001) 281], and [Roberts, Schmidt Prog.Part.Nucl.Phys. 45 (2000)S1]

- DS approach to quark-hadron physics = nonpertubative, covariant bound state approach with strong connections with QCD.

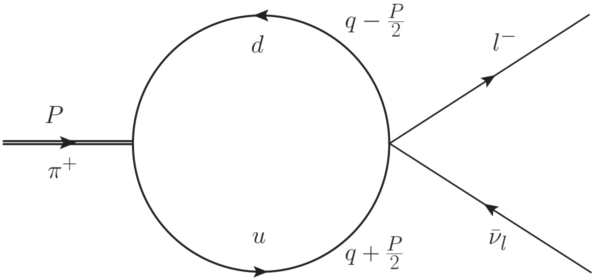
The Schwinger–Dyson equation for the quark propagator:

$$S^{-1}(p) = \not{p} - m - iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k,p) G_{\mu\nu}(p-k) \ . \qquad (1)$$



Already just $S(p)$ enables calculating an observable - pion decay constant

... since close to the chiral limit, $\Gamma_\pi^{BS} \approx -\frac{2B(q^2)}{f_\pi}\gamma_5$ is a good approximation



$$f_\pi = i\frac{N_c}{2}\frac{1}{M_\pi^2}\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\right)$$

$$\Gamma(\pi^+ \to e^+\nu_e) = \frac{1}{4\pi}G^2 f_\pi^2 \cos^2\theta_c(1-\frac{m_e^2}{M_\pi^2})^2 M_\pi m_e^2 \ .$$

Dynamically generated nonperturbative quark propagator

In principle, DS Eq. (1) yields the nonperturbatively dressed quark propagator

$$S(q) = \frac{A(q^2)\not{q} + B(q^2)}{A^2(q^2)q^2 - B^2(q^2)} = Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)} = -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2)$$

[$M(x)$ = dressed quark mass function, $Z(x)$ = wave-function renormalization]

- One usually gets just a model solution for the propagator $S(q)$, since one usually simplifies DS Eq. (1) by approximations & modeling! For example:
- 1.) rainbow(-ladder) for the dressed quark–gluon vertex: $\Gamma^\mu(k,p) \to \gamma^\mu$,
- 2.) $A(k^2) = 1$, 3.) various model Ansätze for the dressed gluon propagator, $g^2 G^{\mu\nu}(k) \propto \alpha_s^{\rm eff}(k^2)/k^2$, so that, e.g., Eq. (1) yields

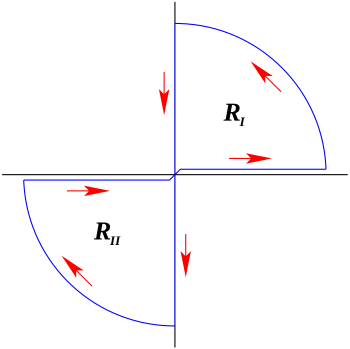
$$M(-p^2) = m + I(-p^2) = m + \int d^4k\, f(-k^2) g(-(k-p)^2)$$

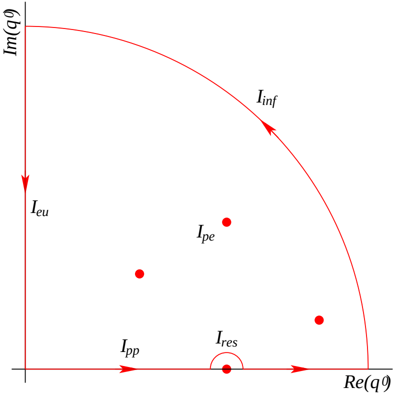
$$g(-k^2) = 4\pi C_F \frac{\alpha_s^{\rm eff}(-k^2)}{(-k^2)} \quad f(-k^2) = \frac{3i}{(2\pi)^4}\frac{M(k^2)}{k^2 - M^2(k^2)}$$

$$I(-p^2) = \int d^3k \int dk^0\, f(-(k^0)^2 + |\mathbf{k}|^2) g(-(k^0 - p^0)^2 + |\mathbf{k} - \mathbf{p}|^2)$$

Wick rotation

- Even with such approxim. & modeling, solving DS equations like Eq. (1), and related calculations (e.g., of f_π) with Green's functions like $S(q)$, are technically very hard to do in the physical, Minkowski space-time.
- Thus, additional simplification is sought by transforming to QFT in 4-dim. Euclidean space by the Wick rotation to the imaginary time component: $q^0 \to i\, q^0$.





The issue of singularities is more problematic than in the perturbative case!

Example: Separable approximation for low-E QCD

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(p-k) \approx g^{\mu\nu} \left[D_0 f_0(-p^2) f_0(-k^2) - D_1 (p\cdot k) f_1(-p^2) f_1(-k^2) \right]$$

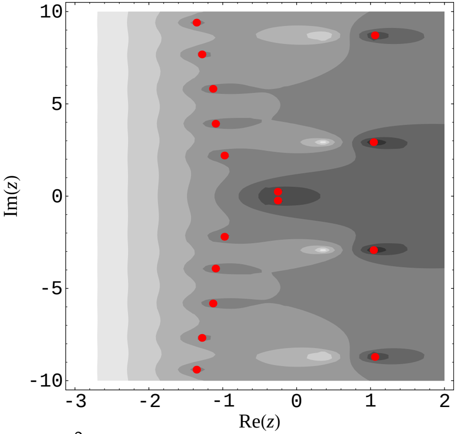
Schwinger–Dyson equation $\Rightarrow$

$$\begin{aligned} A(p^2) &= 1 + a f_1(-p^2) \\ B(p^2) &= m + b f_0(-p^2) \end{aligned}$$

Typical (phen. success.) Ansatz

$$\begin{aligned} f_0(x) &= e^{-x/\Lambda_0^2} \\ f_1(x) &= \frac{1 + e^{-x_0/\Lambda_1^2}}{1 + e^{-(x-x_0)/\Lambda_1^2}} \end{aligned}$$

produces **poles** which are **obstacles to Wick rotation!**



The contour plot of $z \mapsto \log |A(z)^2 z + B(z)^2|$ close to the origin. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$ and extend much further than the depicted region.

General properties a quark propagator should have:

- $S(q) \to S_{\rm free}(q)$ because of asymptotic freedom
  $\Rightarrow \sigma_{V,S}(-q^2) \to 0$ for $q^2 \in \mathbb{C}$ and $q^2 \to \infty$
- $\sigma_{V,S}(-q^2) \to 0$ cannot be analytic over the whole complex plane
- positivity violation $\leftrightarrow$ confinement $\leftrightarrow$ no Källén-Lehmann spectral representation exists

Try Ansaetze of the form (meromorphic parameterizations):

$$S(p) = \frac{1}{Z_2} \sum_{j=1}^{n_p} r_j \left(\frac{\not p + a_j + ib_j}{p^2 - (a_j + ib_j)^2} + \frac{\not p + a_j - ib_j}{p^2 - (a_j - ib_j)^2} \right)$$

$$\text{Thus,} \qquad \sigma_V(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j(x + a_j^2 - b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

$$\text{and} \qquad \sigma_S(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j a_j(x + a_j^2 + b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

Constraints:

$$\sum_{j=1}^{n_p} r_j = \frac{1}{2} \qquad\qquad \sum_{j=1}^{n_p} r_j a_j = 0$$

3R quark propagator - 3 poles on the real axis

Parameters: $n_p = 3,\ a_1 = 0.341,\ a_2 = -1.31,\ a_3 = -1.35919,\ b_1 = 0,\ b_2 = 0,$ $b_3 = 0,\ r_1 = 0.365,\ r_2 = 1.2,\ r_3 = -1.065,\ Z_2 = 0.982731.$

$\Rightarrow$ No obstacles to Wick rotation $\Rightarrow$ pion decay constant calculated equivalently

* in the Minkowski space:

$$f_\pi^2 = -i \frac{N_c}{4\pi^3 M_\pi^2} \int_0^\infty \xi^2\, d\xi \int_{-\infty}^{+\infty} dq^0\, B(q^2)\, \mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\gamma_5 S(q-\frac{P}{2})\right)$$

where $q^2 = (q^0)^2 - \xi^2$, $\xi = |\mathbf{q}|$, and $q \cdot P = M_\pi q^0$, OR

* in the Euclidean space:

$$f_\pi^2 = \frac{3}{8\pi^3 M_\pi^2} \int_0^\infty dx\, x \int_0^\pi d\beta\, \sin^2\beta\, B(q^2)\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\gamma_5 S(q-\frac{P}{2})\right)$$

where $q^2 = -x$ and $q \cdot P = -iM_\pi \sqrt{x} \cos\beta$.

3R quark propagator Ansatz (Alkofer & al.)

Numerically, for $\xi = 0.5$

$$|I_{\mathrm{pp}} + I_{\mathrm{res}} + I_{\mathrm{inf}} + I_{\mathrm{eu}} - I_{\mathrm{pe}}| \sim 10^{-8}$$

Particular integrals for $\xi = 0.5$

$$\begin{aligned} I_{\mathrm{pp}} &\approx 4 \cdot 10^{-12} \,, \\ I_{\mathrm{res}} &= -0.0221314i \\ I_{\mathrm{inf}} &\approx 4 \cdot 10^{-12} \,, \\ I_{\mathrm{eu}} &= 0.0221314i \\ I_{\mathrm{pe}} &= 0 \end{aligned}$$

$\Rightarrow f_\pi = 0.072\ \mathrm{GeV}.$

Wick rotation

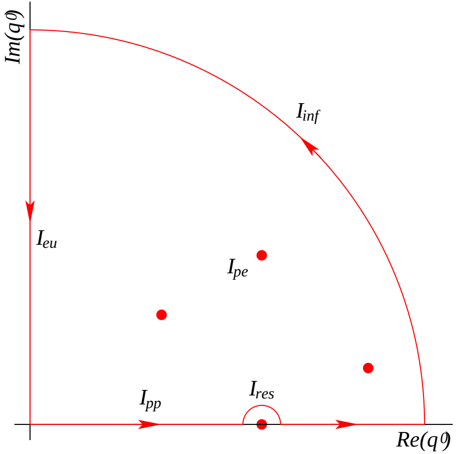


Figure : The curve of Wick rotation in p^0 complex plane (schematically).

2CC quark propagator - with 2 pairs of complex conj. poles

Parameters: $n_p = 2$, $a_1 = 0.351$, $a_2 = -0.903$, $b_1 = 0.08$, $b_2 = 0.463$, $r_1 = 0.360$, $r_2 = 0.140$, $Z_2 = 1$.

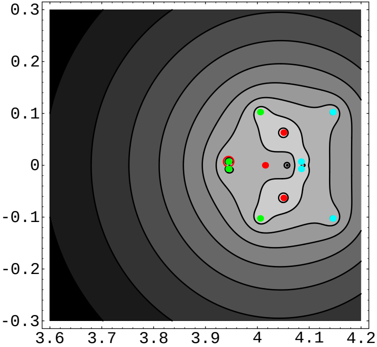


Figure : Contour plot of the complex function $q^0 \mapsto |\mathrm{trace}(q^0, \xi)|$ for $\xi = 4.0$. Dots are the poles of the function $q^0 \mapsto \mathrm{trace}(q^0, \xi)$.

2CC quark propagator Ansatz (Alkofer & al.)

For $\xi = 0.5$ the intergrals over q^0 are:

$$
\begin{aligned}
I_{\rm pp} &= 0.184102 \\
I_{\rm res} &= 0.0166534 i \\
I_{\rm inf} &\approx 0 \\
I_{\rm eu} &= 0.0230837 i \\
I_{\rm pe} &= 0.184102 + 0.0397371 i
\end{aligned}
$$

Cauchy's residue theorem OK: $|I_{\rm pp} + I_{\rm res} + I_{\rm eu} - I_{\rm pe}| = 1.4 \cdot 10^{-9}$

$\Rightarrow f_\pi = 0.071 \ \mathrm{GeV}.$

In spite of the popularity of quark propagators with CC poles, there are objections besides not alowing naive Wick rotation: possible problems for causality and unitarity of the theory.
Also, Benić *et al.*, PRD 86 (2012) 074002, have shown that CC poles of the quark propagator cause thermodynamical instabilities at nonvanishing temperature and density.

Presently favorite quark propagator

$$S(q) = \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} = Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)}$$

$$M(z) = \ln\left(\frac{(z+a_1)(z+a_2)(z+a_3)}{(z+b_1)(z+b_2)(b+b_3)}\right) \quad \& \quad Z(z) = 1$$

$$\sigma(z) = \frac{1}{z + M^2(z)} \ .$$

The favorite set of parameters:

a_1	a_2	a_3	b_1	b_2	b_3
3.00278	1.78718	0.554466	2.92927	2.01400	0.401162

Presently favorite quark propagator

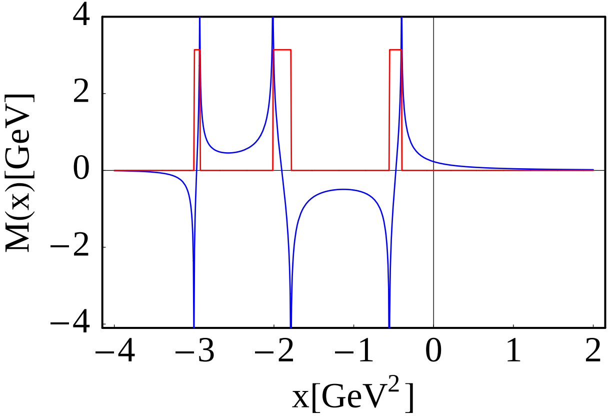


Figure : Real (blue) and imaginary (red) part of the function $x \mapsto M(x)$.

Presently favorite quark propagator: f_π calculation

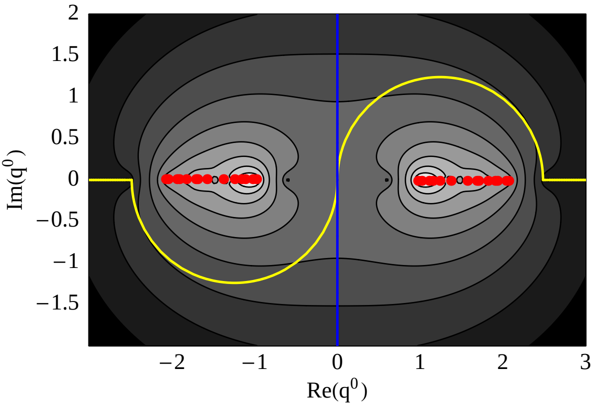
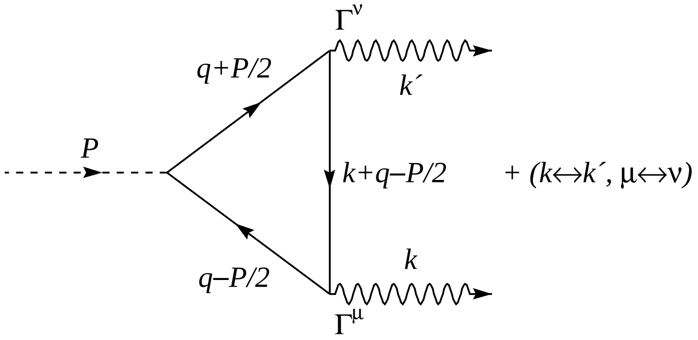


Figure : Contour plot of the integrand and path of integration in the complex q^0–plane.

$\Rightarrow f_\pi = 67\ \mathrm{MeV}$ and $2\frac{f_\pi - f_\pi}{f_\pi + f_\pi} \sim 10^{-5}$

$\pi^0 \to \gamma\gamma$

- anomaly nonrenormalization
- Adler–Bardeen theorem (for QED originally)
- cancelations of higher order corrections
- role of the soft limit!



$\pi^0 \to \gamma\gamma$

$$
\begin{aligned}
& T^{\mu\nu}(k,k') \\
= \quad & -N_c \frac{Q_u^2 - Q_d^2}{2} \int \frac{d^4 q}{(2\pi)^4} \mathrm{tr}\{\Gamma^\mu(q - \frac{P}{2}, k + q - \frac{P}{2}) S(k + q - \frac{P}{2}) \\
\times \quad & \Gamma^\nu(k + q - \frac{P}{2}, q + \frac{P}{2}) S(q + \frac{P}{2}) \left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right) S(q - \frac{P}{2})\} \\
+ \quad & (k \leftrightarrow k', \mu \leftrightarrow \nu)
\end{aligned}
$$

$$
T^{\mu\nu}(k,k') = \varepsilon^{\alpha\beta\mu\nu} k_\alpha k'_\beta T(k^2, k'^2)
$$

$\pi^0 \to \gamma\gamma$

Simple constituent quark model: $S^{-1}(q) = \not{q} + M_q$, $\Gamma^\mu(p', p) = \gamma^\mu$

$$T(k^2, k'^2) = \frac{M_q^2}{2\pi^2 f_\pi} C_0(k^2, k'^2, M_\pi^2, M_q^2, M_q^2, M_q^2)$$

Soft pion limit ($k \to 0$, $k' \to 0$)

$$T(0,0) = \frac{1}{4\pi^2 f_\pi}$$

$\pi^0 \to \gamma\gamma$ decay width

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi\alpha^2 M_\pi^3}{4} |T(0,0)|^2 = \frac{\alpha^2}{64\pi^3} \frac{M_\pi^3}{f_\pi^2} = (7.760 \pm 0.024)\ \mathrm{eV}$$

Experimental value $\Gamma_{\mathrm{exp}}(\pi^0 \to \gamma\gamma) = (7.63 \pm 0.16)\ \mathrm{eV}$

$\pi^0 \to \gamma\gamma$

More general quark propagator:

$$S^{-1}(q) = A(q^2)\not{q} + B(q^2)$$

Ball-Chiu vertex:

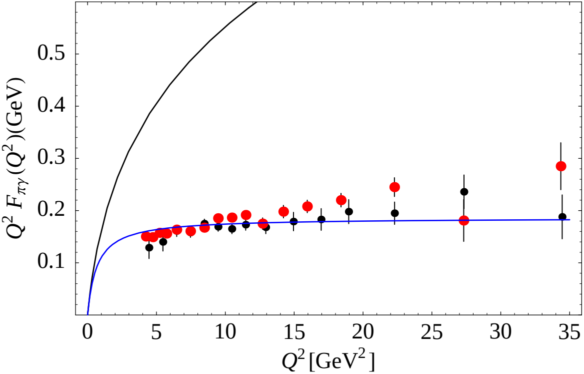
$$\Gamma^\mu(p', p) = \frac{1}{2}[A(p'^2) + A(p^2)]\gamma^\mu$$
$$+ \quad \frac{(p'+p)^\mu}{(p'^2 - p^2)}\{[A(p'^2) - A(p^2)]\frac{(\not{p}' + \not{p})}{2} - [B(p'^2) - B(p^2)]\}$$

Again, the same amplitude in the soft limit:

$$T(0,0) = \frac{1}{2}\int \frac{d^4 q_E}{(2\pi)^4} \frac{M_E(x)(M_E(x) - 2xM_E'(x))}{(x + M_E^2(x))^3}$$
$$= \frac{1}{2\pi^2 f_\pi}\int_0^{+\infty} \frac{y\,dy}{(1+y)^3} = \frac{1}{4\pi^2 f_\pi}$$

π^0 transition form factor

$$F(Q^2) = |T(-Q^2, 0)|$$



- BaBar
- Belle

— $A(q^2) = 1,\ B(q^2) = M_q = 300$ MeV quark propagator

Extension to finite density

- In the vacuum, *i.e.* with no medium, the symmetry of the theory allows propagators with only two tensor structures, 4-vector and scalar:
- general (dressed) quark propagator: $S^{-1}(p) = i\not{p}A(p^2) + B(p^2)$,

But in medium, the original $O(4)$ symmetry is broken to $O(3)$ symmetry. Then, in medium with density and chemical potential $\mu > 0$, the most general form of the dressed quark propagator has four independent tensor structures:

$$S^{-1}[\mu](p) = i\vec{\gamma}\cdot\mathbf{p}\,\mathcal{A}(p^2, u\cdot p) - \mu\,\gamma_4\mathcal{C}(p^2, u\cdot p) + \mathcal{B}(p^2, u\cdot p) - \mu\,\gamma_4\,\vec{\gamma}\cdot\mathbf{p}\,\mathcal{D}(p^2, u\cdot p)\,,$$

where u_μ denotes the relative velocity of the medium, which in the rest frame of the medium can be written $u_\mu = (\vec{0}, 1)$.

- Looks quite complicated!

Extension to finite density simplified

- However, Zong *et al.*, PRC 71 (2005) 015205, assuming analyticity of the dressed quark propagator at $\mu = 0$ and neglecting the μ-dependence of the dressed gluon propagator, argued that within the rainbow approximation, the dressed quark propagator at $\mu \neq 0$ is obtained from the $\mu = 0$ one by a simple shift $q_4 \to q_4 + i\mu \equiv \tilde{q}_4$:

$$S[\mu](q) = -\sigma_V(\tilde{q}^2)\tilde{\not{q}} - \sigma_S(\tilde{q}^2)$$

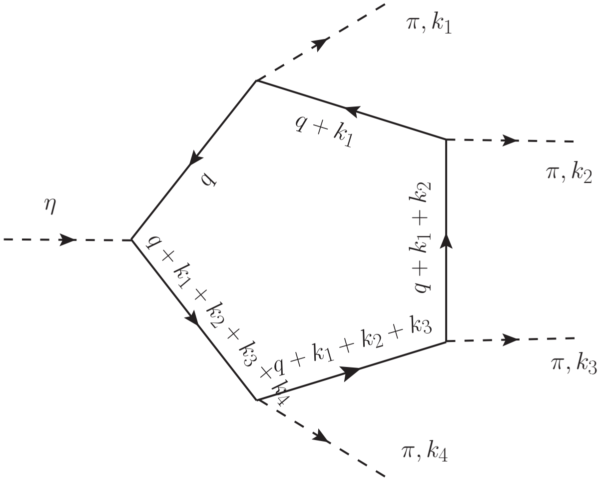
- Only two tensor structures again, in spite of medium! Looks like a very severe truncation!
- Nevertheless, Jiang *et al.*, PRD 78 (2008) 116005, used this simplification in their $\mu > 0$ extension of the present Ansatze 3R and 2CC.
- They obtained reasonable μ-dependence of f_π and m_π, in accordance with other, independent predictions on general grounds (Halasz *et al.*, PRD 58 (1998) 096007).
- $\Rightarrow$ The present Minkowski-vs-Euclidean analysis probably can be extended to $\mu > 0$ in this simplifed way (to be done).

Summary

- We have seen that the knowledge of the analytic behavior in the whole complex plane is needed.
- But even for physically motivated dressed quark mass function $M(p^2)$, with good analytic structure, it is very hard to predict and control the analytic structure of the corresponding nonperturbative quark propagator.
- $\Rightarrow$ the analytic structure of for quark propagators in the nonperturbative regime of QCD has been investigated for certain Ansaetze.
- A propagator form allowing the Wick rotation and enabling equivalent calculations in Minkowski and Euclidean spaces is presented.
- In spite of its simplicity, this model yields good qualitative and semi-quantitative description of some pseudoscalar meson processes.
- Extension to finite densities and temperatures is possible.

Additional slides

$\eta, \eta' \to 4\pi$



$\eta, \eta' \to 4\pi$

$$
\begin{aligned}
S_{fi} &= (2\pi)^4 \delta^{(4)}(P - k_1 - k_2 - k_3 - k_4)\, \mathrm{tr}(I_{\mathrm{color}}) \\
&\times \frac{\cos\phi_P}{2^5} \mathrm{tr}\left((\sqrt{\frac{2}{3}}\lambda^0 - \frac{1}{\sqrt{3}}\lambda^8)\lambda^3\lambda^3\lambda^3\lambda^3 \right) \left(-\frac{2M_q}{f_\pi} \right)^5 \\
&\times (-) \int \frac{d^4 q}{(2\pi)^4} \mathrm{tr}\Big(\gamma_5\, iS(q)\gamma_5\, iS(q+k_1)\gamma_5\, iS(q+k_1+k_2)\gamma_5 \\
&\times iS(q+k_1+k_2+k_3)\gamma_5\, iS(q+k_1+k_2+k_3+k_4) \Big) \\
&+ 23 \text{ permutations}
\end{aligned}
$$

$\eta, \eta' \to 4\pi$

Loop integration gives the Passarino–Veltman five-point function E_0

$$\int d^4q \,\mathrm{tr}\Big(\gamma_5\, iS(q)\gamma_5\, iS(q+k_1)\gamma_5\, iS(q+k_1+k_2)\gamma_5$$
$$\times iS(q+k_1+k_2+k_3)\gamma_5\, iS(q+k_1+k_2+k_3+k_4)\Big)$$
$$= 4i\pi^2 M_q\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} E_0(a)$$

where $a = (M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2, (k_1+k_2)^2, (k_2+k_3)^2, (k_3+k_4)^2, (k_4+k_5)^2, (k_5+k_1)^2, M_q^2, M_q^2, M_q^2, M_q^2, M_q^2)$

- E_0 calculated using Golem95 library
- phase space integration, Kumar and Vegas

$\eta, \eta' \to 4\pi$

	PDG2011	$M_q = 0.3$	$M_q = 0.4$
$\eta \to 4\pi^0$	$< 6.9 \cdot 10^{-7}$	$1.31 \cdot 10^{-27}$	$2.12 \cdot 10^{-32}$
$\eta' \to 4\pi^0$	$< 5 \cdot 10^{-4}$	$6.68 \cdot 10^{-7}$	$2.03 \cdot 10^{-8}$
$\eta' \to \pi^-\pi^+2\pi^0$	$< 2.6 \cdot 10^{-3}$	$1.50 \cdot 10^{-3}$	$5.13 \cdot 10^{-3}$
$\eta' \to 2(\pi^-\pi^+)$	$< 2.4 \cdot 10^{-4}$	$8.59 \cdot 10^{-4}$	$7.11 \cdot 10^{-5}$

Table : η and $\eta' \to 4\pi$ branching ratios. The $\eta' \to \pi^-\pi^+2\pi^0$ branching ratios are calculated with $M_\pi = (M_{\pi^0}^{\mathrm{exp}} + M_{\pi^\pm}^{\mathrm{exp}})/2$.

$\eta, \eta' \to 4\pi$

	BESIII	$M_q = 0.3$	$M_q = 0.4$
$\eta \to 4\pi^0$	N/A	$1.31 \cdot 10^{-27}$	$2.12 \cdot 10^{-32}$
$\eta' \to 4\pi^0$	N/A	$6.68 \cdot 10^{-7}$	$2.03 \cdot 10^{-8}$
$\eta' \to \pi^-\pi^+2\pi^0$	$(1.82 \pm 0.39) \cdot 10^{-4}$	$1.50 \cdot 10^{-3}$	$5.13 \cdot 10^{-3}$
$\eta' \to 2(\pi^-\pi^+)$	$(8.53 \pm 0.94) \cdot 10^{-5}$	$6.58 \cdot 10^{-4}$	$2.24 \cdot 10^{-3}$

Table : η and $\eta' \to 4\pi$ branching ratios.

- $\eta' \to 4\pi^0$, $\eta' \to \pi^-\pi^+2\pi^0$, and $\eta' \to \pi^-\pi^+2\pi^0$ branching ratios are calculated using $M_\pi = M_{\pi^0}^{\exp}$, $M_\pi = (M_{\pi^0}^{\exp} + M_{\pi^\pm}^{\exp})/2$, and $M_\pi = M_{\pi^\pm}^{\exp}$, respectively.
- BESIII column contains the results of BESIII collaboration