Efficiency of template banks for highly-eccentric gravitational-wave sources

DÁNIEL BARTA

WIGNER RESEARCH CENTRE FOR PHYSICS, Budapest
Ph.D. Student, EÖTVÖS LORÁND UNIVERSITY, Budapest

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Motivation:

- Data analysis of targeted search techniques uses a **large number** of predicted waveform templates upon matched filtering. Waveforms for inspirals are parametrized by a set of intrinsic physical quantities (*masses, magnitudes and orientations of eccentricity*) result an **multi-dimensional parameter space**.
- Price of high dimensionality: parameter estimation and modeling are **prohibitively expensive** and **computationally unfeasible** with most methods!

Method:

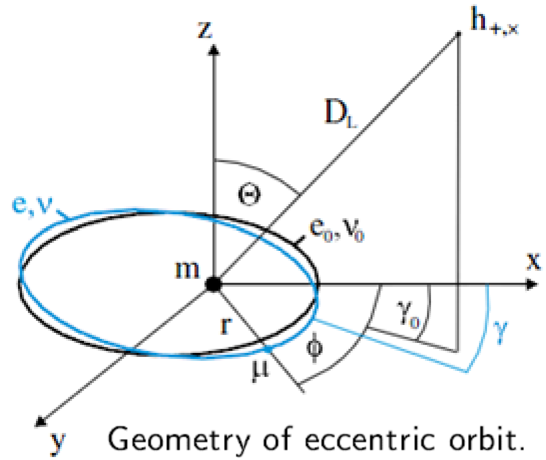
- High-accuracy reduced basis representations have to be constructed that determines a relatively small set of the most relevant waveforms.
- One of the most common techniques is to eliminate the redundancy of the original data – wherein the waveform is orthogonal base-mediated – by singular value decomposition methods (SVD).
- In the parameter space of few, relevant waveforms (basis), any other waveform can be accurately expressed by interpolation techniques.

Decomposition techniques (SVD and *greedy basis*) have been preformed for waveforms of merging binary black holes:

- Field et al. (2011); Phys. Rev. Lett. 106, 221102; *Reduced basis catalogs for gravitational wave templates*
- Field et al. (2012); Phys. Rev. D 86, 084046; *Towards beating the curse of dimensionality for gravitational waves using reduced basis*
- Sathyaprakash et al. (2009); Phys. Rev. D 80, 084043; *Comparison of post-Newtonian templates for compact binary inspiral signals in gravitational-wave detectors*
- M. Pürrer (2014); arXiv:1402.4146 [gr-qc]; *Frequency domain reduced order models for gravitational waves from aligned-spin black-hole binaries*

...however for elliptic sources such a method is yet to be developed.

1. Eccentricity is often overlooked, arguing that it evaporates anyway in isolated systems, on the course of its evolution. Its role of eccentricity – like of spin – is fundamental for the prediction of waveform, hence it must be treated alike.
2. In star clusters, and in the vicinity of galactic nuclei, a system cannot be regarded as being isolated: substantial eccentricity!



Geometry of eccentric orbit.

- Reduced mass: $\mu = M_1 M_2 / M$
- Separation: $r = a_0(1 - e_0^2/(1 + e_0 \cos\phi)$
- Semi-major axis: $a_0 = M^{1/3}(2\pi\nu_0)^{2/3}$

Equations of motion[1] to be integrated:

$$\dot{r} = -\frac{16}{15}\frac{M_1 M_2 M}{a^3(1-e^2)^5}\left[1 + e\cos\phi\right]^4 \left[12 + (1 + e\cos\phi)^2 + e^2\cos^2\phi\right]^4$$

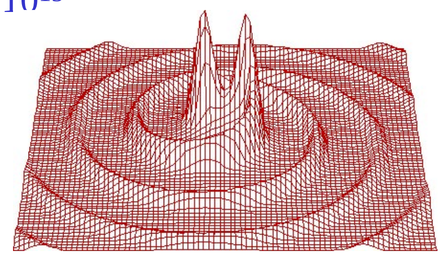
$$\dot{e} = -\frac{8}{15}\frac{M_1 M_2 M}{a^4 e(1-e^2)^4}\left[1 + e\cos\phi\right]^3 \left[\left(12 + (1 + e\cos\phi)^2 + e^2\sin^2\phi\right)\right. \\ \left.(1 + e\cos\phi) - 3\left(1 - e^2\right)\left(4 + 6e\cos\phi + e^2\left(3\cos^2\phi - 1\right)\right)\right.$$

$$\dot{\phi} = \frac{M^{1/2}}{a^{3/2}(1-e^2)^{3/2}}(1 - e\cos\phi)^2$$

[1] leading order Keplerian orbit and quadrupole radiation

Adaptive stepsize int. →

Parametric *NDSolve:*
- Cut off, if the separation reaches $6M_\odot$ (Schw. ISCO)
- MaxSteps = 10^{15}

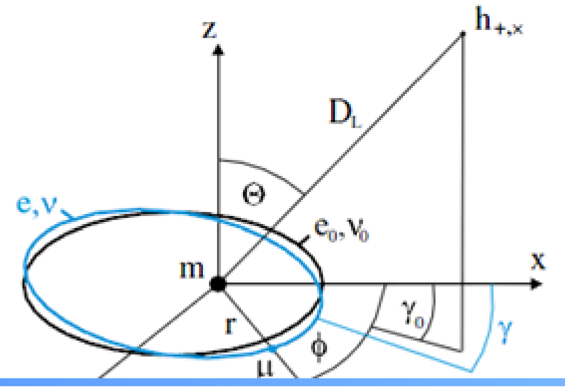


Waveform (2.5PN):

$$h_\times(\phi) = -\frac{\mu m\cos\Theta}{a(1-e^2)D_L} \times\left[(5e\sin\phi + 4\sin 2\phi + e\sin 3\phi)\cos 2\gamma - (5e\cos\phi + 4\cos 2\phi + e\cos 3\phi + 2e^2)\sin 2\gamma\right],$$

$$h_+(\phi) = -\frac{\mu m(1+\cos^2\Theta)}{a(1-e^2)D_L}\left[\left(\frac{5e}{2}\cos\phi + 2\cos 2\phi + \frac{e}{2}\cos 3\phi + e^2\right)\cos 2\gamma + \left(\frac{5e}{2}\sin\phi + 2\sin 2\phi + \frac{e}{2}\sin 3\phi\right)\sin 2\gamma + (e\cos\phi + e^2)\frac{\sin^2\Theta}{1+\cos^2\Theta}\right]$$

Numerical integration
Orbital and waveform evolution



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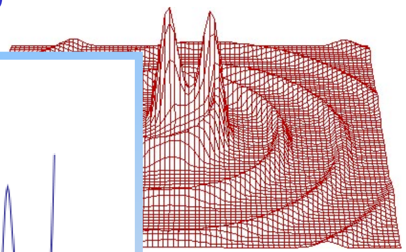

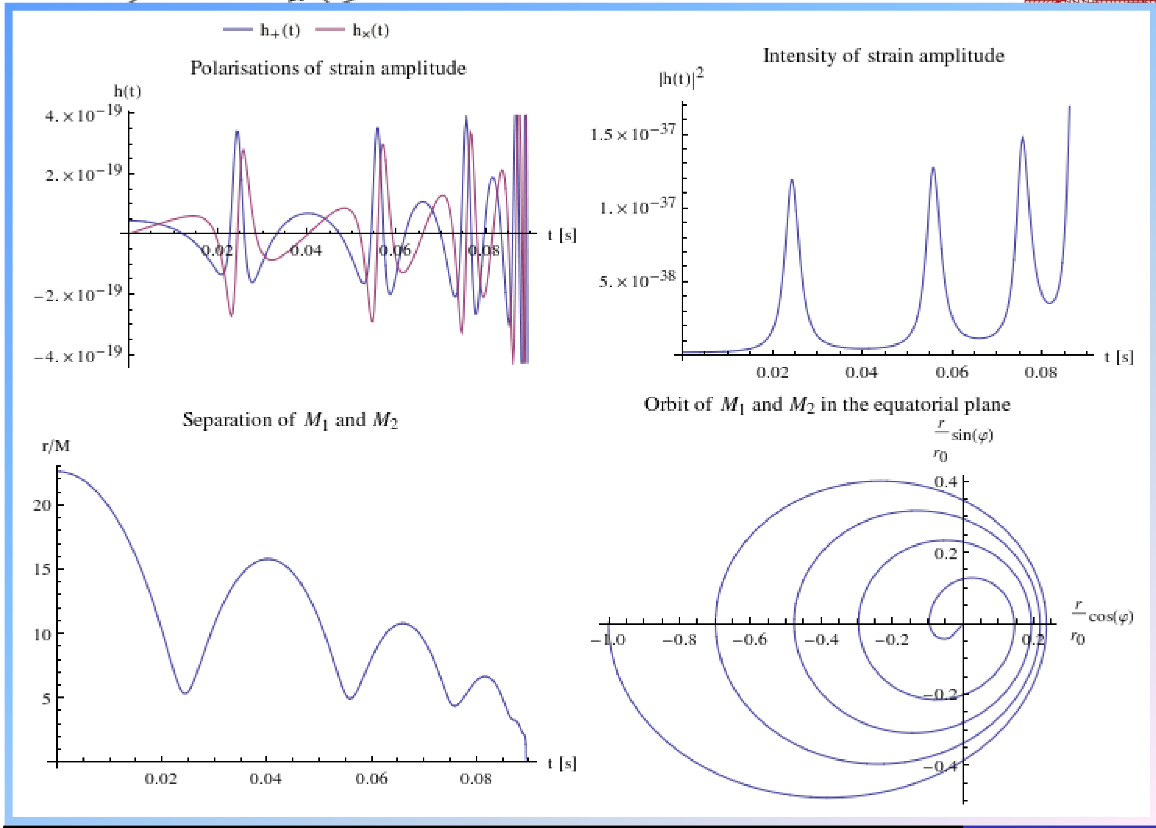

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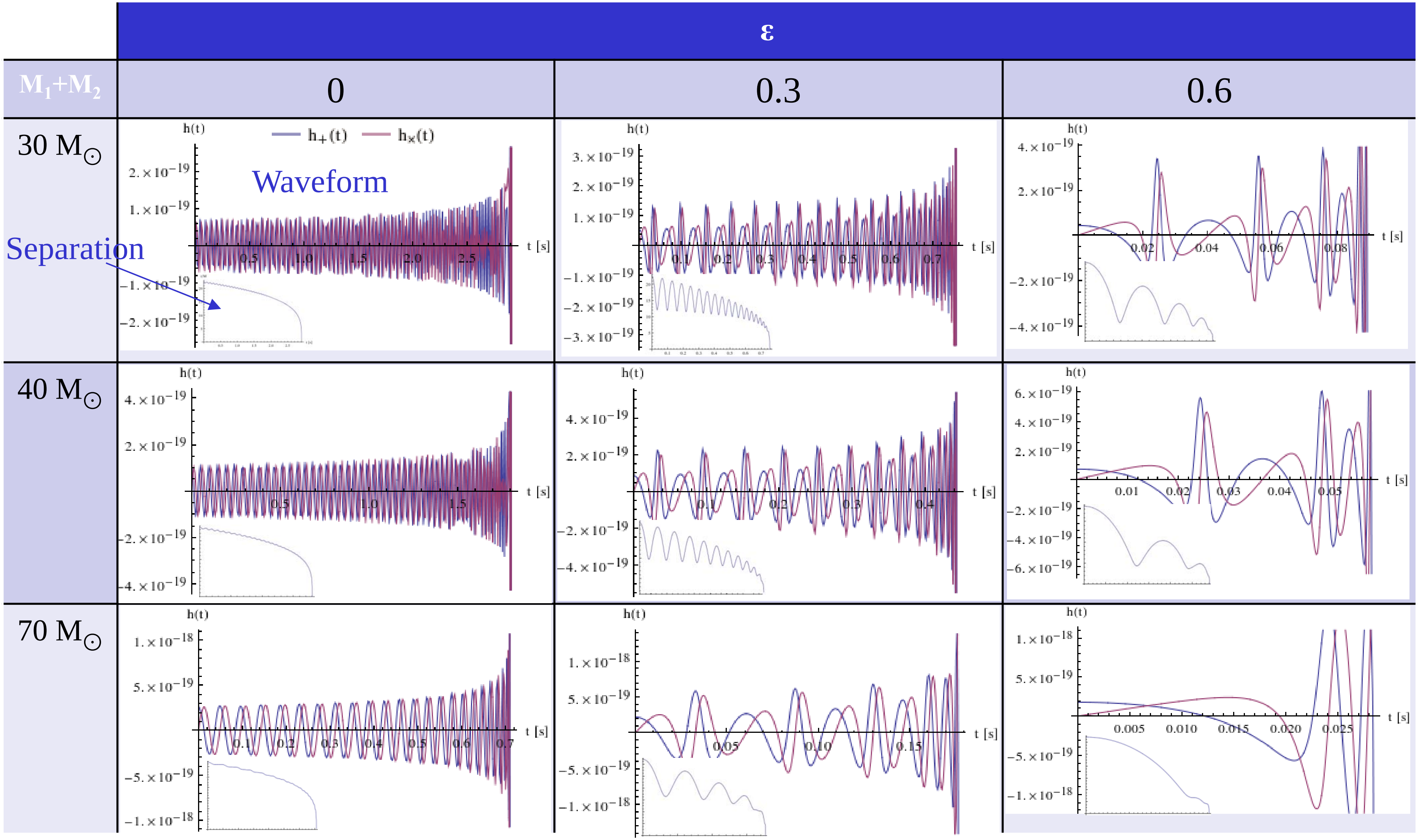
Numerical integration

Construction of data basis

Data to be stored:

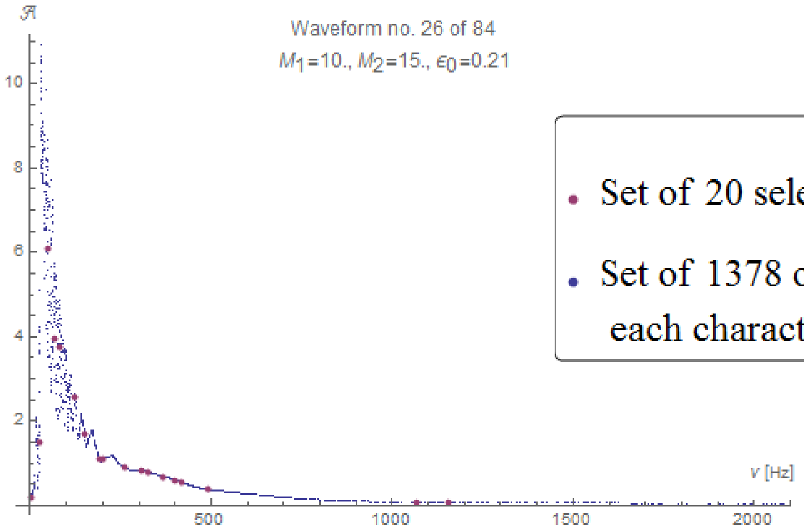
- Setting the sample rate: $M_{start} = 5.00$; $M_{end} = 35.00$; $M_{step} = 5.00$; $\varepsilon_{start} = 0.01$; $\varepsilon_{end} = 0.50$; $\varepsilon_{step} = 0.20$;
- Waveforms
- Time to reach Schwarzschild ISCO ($6M$): t_{end}



The model following the footsteps of the code *SEOBNRv1* for merging black holes, described in [Pürrer (2014)] with some modifications:

I. Generate a set of n input waveforms that cover the multi-dimensional (in our case: 3) parameter space domain of interest, as densely as desired:
$\{M_1, M_2\} \in (5M_\odot, 35M_\odot), \varepsilon \in (0, 1)$

1. Choose starting and ending frequencies (10 és 8000 Hz), total mass and sampling rate so that the fast Fourier Transforms (FFTs) of the time-domain (TD) waveforms cover a suitable frequency range.
2. Generate n waveforms on a regular grid over the multi-dimensional parameter space
3. Store the amplitude (original and normalized), the phase and the interpolating functions fitted to them for each and every sample waveforms.



Datasets

- Set of 20 selected frequency points characteristic for the specific waveform
- Set of 1378 other points in the same frequency range (among all 1680), each characteristic for one of the other waveforms

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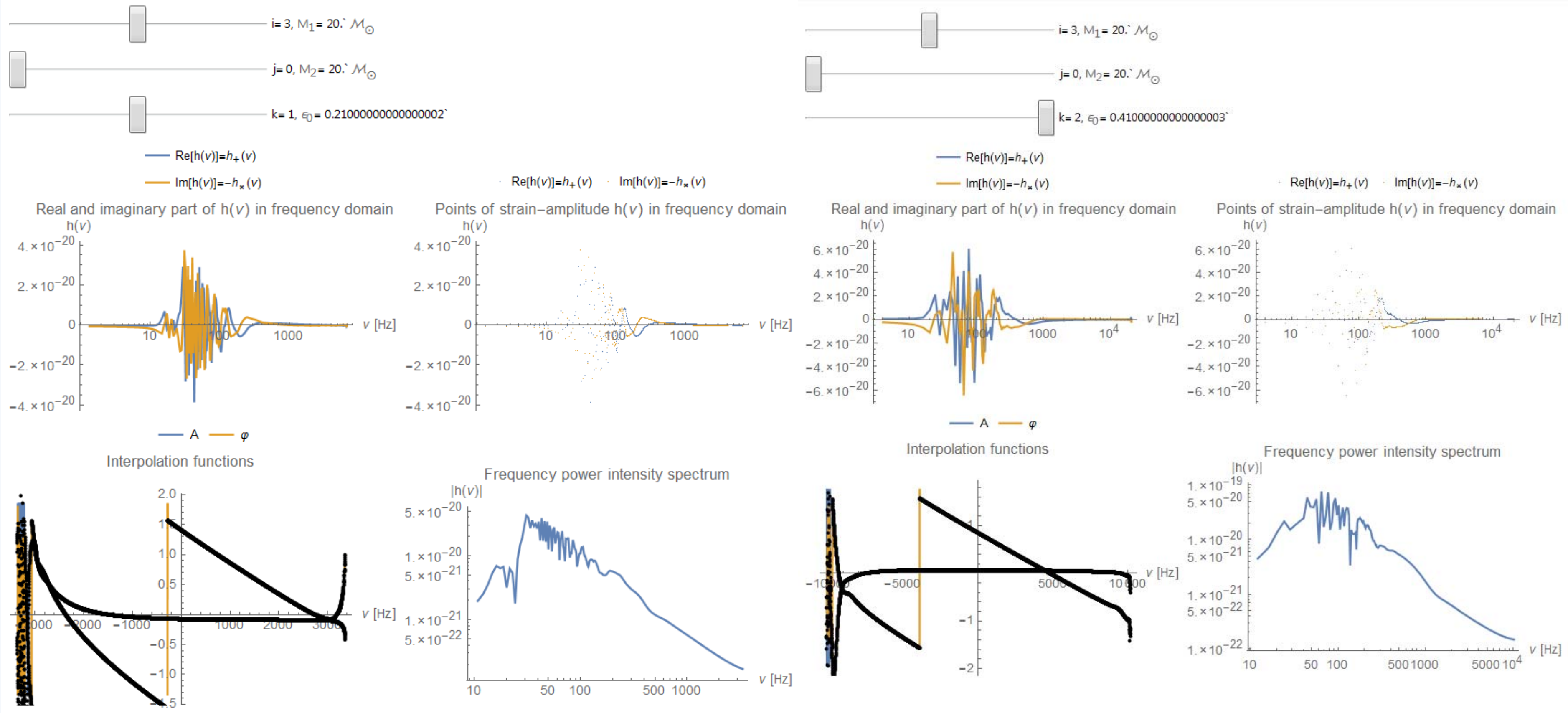

'Recipe'
Blueprint for building reduced order models

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3. Store the amplitude (original and normalized), the phase and the interpolating functions fitted to them for each and every sample waveforms.

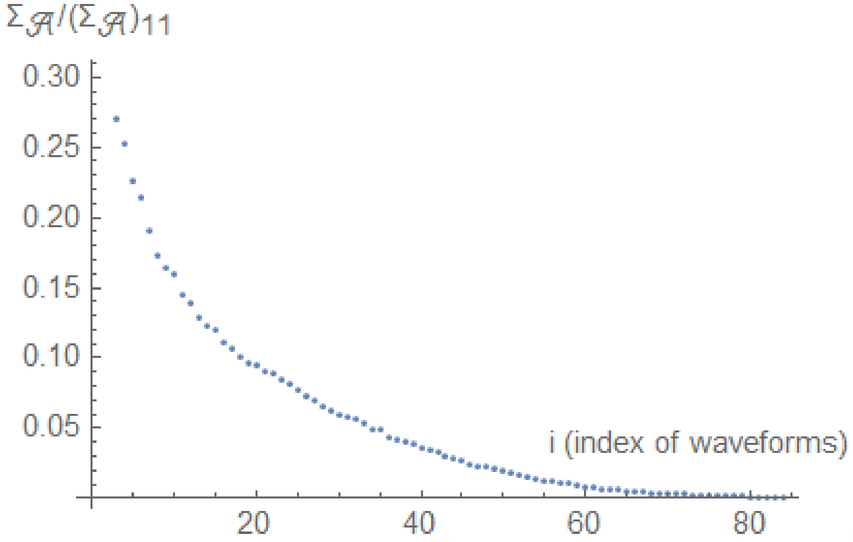
II. Define frequency grids separately for amplitudes and phases.

1. Find m ($100 - 200$) db. sparse amplitude and frequency points.
2. The preferred prescription to generate the frequency points keeps the error of cubic spline interpolation $I_f[\cdot]$ approximately constant in frequency.
3. Interpolate input amplitudes and phases onto the sparse frequency grids.

III. Compute *reduced bases* for the amplitudes and phases with the SVD:

1. Pack the input amplitudes and phases into the columns of matrices $T_A, T_\Phi \in \mathbb{R}^{m \times n}$.
2. Take the SVD of the input amplitude T_A and phase T_Φ matrices.
3. Obtain orthonormal amplitude B_A and phase bases B_Φ and a list of decreasing singular values $\sigma_{A,i}, \sigma_{\Phi,i}$ indicating the relative importance of the SVD-modes.

Compression: The basis of amplitude/phase is given in column vectors B_i where $B \equiv \begin{cases} V_n \in \mathbb{R}^{m \times n}, \text{ if } m > n \\ V \in \mathbb{R}^{m \times m}, \text{ if } m \leqslant n \end{cases}$ and we desire a full rank approximation $\Rightarrow$ *thin SVD*. If $m < n$, i.e. the number m of grid points is smaller than the number n of waveforms we compute a full SVD.



IV. Interpolation over the parameter space:

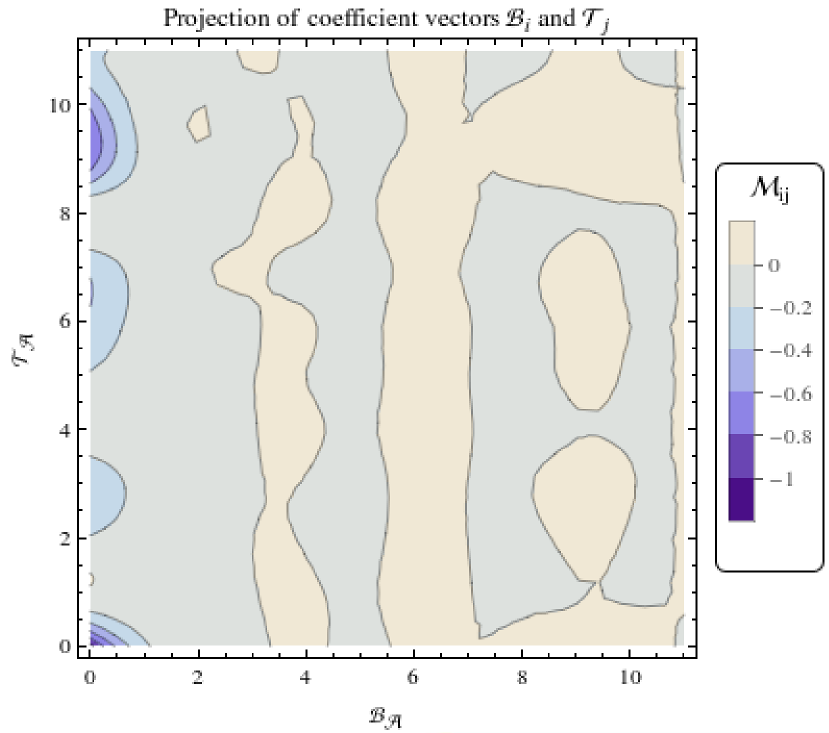
1. Calculate projection coefficients μ of all input waveforms $\tilde{h} \in \mathbb{R}^m$ in:

   $\mu(\tilde{h}) := B^T \tilde{h} \in \mathbb{R}^m$

   Collect projection coefficients μ in matrices M_A and M_Φ (for all input waveforms):

   $M_{ji} = \mu_j(\tau_i) = (B^T T)_{ji} \in \mathbb{R}^{k \times n}$.

   It has been proved that $M = B^T T = \Sigma U^T$ is full rank for basis $B = V$.

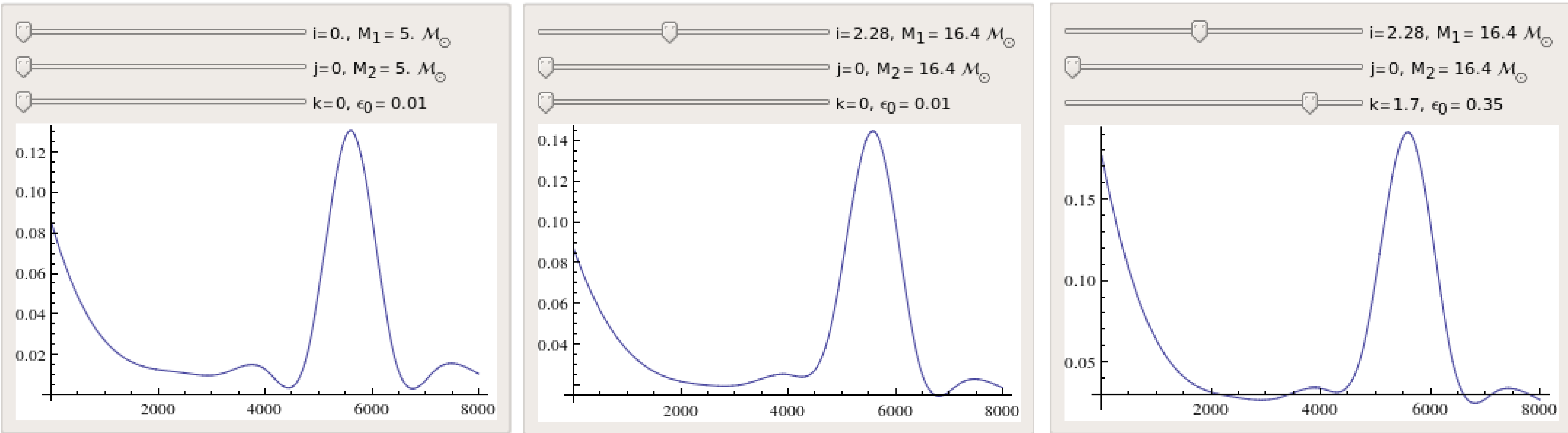


'Recipe'
Blueprint for building reduced order models

2. Unpack from M and interpolate the projection coefficients over the parameter space using tensor product spline interpolation $I_{\otimes}[M]$ where $I[M](M_1, M_2, \varepsilon) \in \mathbb{R}^k$.
3. Interpolate the amplitude normalization factors over the parameter space, i.e. restore the original order of magnitudes.

V. Assemble the frequency domain surrogate model:

$$\tilde{h}_m(q,\varepsilon;M;f) := A_0(q,\varepsilon,M)\, I_f\left[\mathcal{B}_{\mathcal{A}} \cdot I_{\otimes}[\mathcal{M}_{\mathcal{A}}](q,\varepsilon)\right]\, \exp\left\{i\, I_f\left[\mathcal{B}_{\Phi} \cdot I_{\otimes}[\mathcal{M}_{\Phi}](q,\varepsilon)\right]\right\}$$



Thank you for your kind attention!

Dániel Barta
E-mail: barta.daniel@wigner.mta.hu

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