Tensor Square Representations of Lie Algebras
Or: Symmetry Decides Simulability

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Joint work with R. Zeier, T. Schulte-Herbrüggen and D. Burgarth

Wigner Research Centre for Physics

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- The representation theory of compact groups is a well-understood subject into which one can embed another classic field, the representations of compact Lie algebras. One hears usually the following statements:
  - all the questions concerning the representation theory of semi-simple (and compact) Lie algebras are already solved.
  - There are no interesting and 'natural' properties of the representation theory of compact Lie algebras that makes it very distinct from that of compact groups.
  - All the group theory needed for physics have been already worked out. Group theory has no more practical importance for physics. Revival of the *Gruppenpest* argument of Slater. (Interview with E. Wigner by Lillian Hoddeson, Gordon Baym and Frederick Seitz at the New Yorker Hotel January 24, 1981)

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- Any continuous representation of a compact group is equivalent to a unitary representation.
- Any continuous irreducible representation (irrep) of a compact group is finite dimensional.
- Any continuous representation of a compact group is completely reducible.
- For any two irreps can define through the Clebsch-Gordan series (or the direct-product fusion rules)

$$\lambda_i \times \lambda_j \cong \oplus_k N_{ij}^k \lambda_k .$$

The representation ring of the compact group. (More precisely it is a ring with a basis/ordered ring/rig/semi-ring.)

- Given $H < G$ (H is a closed subgroup of the compact group G), and an irrep λ^G of G,

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Dynkin solved all representation theoretic problems concerning semi-simple Lie algebras...

- A baby version of the famous **P** $\neq$ **NP** conjecture is Valiant's conjecture **VP** $\neq$ **VNP**.
- Consider $\mathfrak{h} = \mathfrak{su}(d_1) \oplus \mathfrak{su}(d_2)$ and $\mathfrak{g} = \mathfrak{su}(d_1 d_2)$ with the canonical embedding $\mathfrak{h} < \mathfrak{g}$, and the restriction fusion rules

$$\lambda^{\mathfrak{g}}_{\mathfrak{h}} \cong \oplus_k N_k \lambda^{\mathfrak{h}}_k$$

- Solving **VP** $\neq$ **VNP** is equivalent to deciding whether there exists a polynomial algorithm (in d_1, d_2, and the dimension of $\lambda^{\mathfrak{g}}$) for obtaining the above N_k.
- This question gave rise to a whole field of modern mathematics called Geometric Complexity Theory, with many subquestions, e.g., is there a polynomial algorithm for deciding $N_k \neq 0$.

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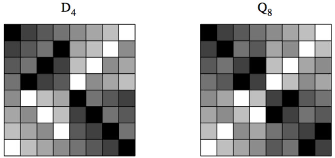
The representation rings of compact Lie algebras have no distinct features...

- Consider the dihedral and quaternion groups, D_4 and Q_8



D_4	1	1	2	2	2
	1	z	r	s	t
1	1	1	1	1	1
λ_r	1	1	1	−1	−1
λ_s	1	1	−1	1	−1
λ_t	1	1	−1	−1	1
δ	2	−2	0	0	0
ρ	8	0	0	0	0

Q	1	1	2	2	2
	1	−1	i	j	k
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- D_4 and Q_8 are not isomorphic, but isomorphic isomorphic representation rings.
- Handelman's theorem: the representation ring $(\lambda_i \times \lambda_j \cong \oplus_k N_{ij}^k \lambda_k)$ uniquely determines a semi-simple Lie algebra/simply connected compact Lie group. (J. R. McMullen, Math. Z. 185 539 (1984); D. Handelman, Int. J., 4 59 (1993); D. Kazdhan, M. Larsen, Y. Varshavski, Algebra & Number Theory 8 243 (2014).). The theorem was proved using the classification of semi-simple Lie algebras and their representations.

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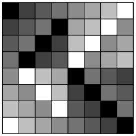
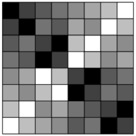


$\mathbf{Q_8}$



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1	1	1	1	1	1
λ_i	1	1	1	−1	−1
λ_j	1	1	−1	1	−1
λ_k	1	1	−1	−1	1
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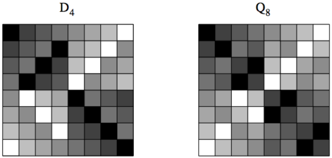

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- Consider two irreps λ, μ of a compact group G, then $\|\lambda \times \mu\|_2 = \|\lambda \times \overline{\mu}\|_2$ holds. Furthermore, $\|\lambda \times \mu\|_1 = \|\lambda \times \overline{\mu}\|_1$ holds if G is simply connected and compact. (R. Coquereaux, J.-B. Zuber J. Phys. A 44 295208 (2011); Sigma 9 039 (2013); J. Phys. A: Math. Theor. 47 455202 (2014))

On sums of tensor and fusion multiplicities

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Abstract

The total multiplicity in the decomposition into irreducibles of the tensor product $\lambda \otimes \mu$ of two irreducible representations of a simple Lie algebra is invariant under conjugation of one of them $\sum_\nu N_{\lambda\mu}{}^\nu = \sum_\nu N_{\overline{\lambda}\mu}{}^\nu$. This also applies to the fusion multiplicities of affine algebras in conformal WZW theories. In that context, the statement is equivalent to a property of the modular S matrix, viz $\Sigma(\kappa) := \sum_\lambda S_{\lambda\kappa} = 0$ if κ is a complex representation. Curiously, this vanishing of $\Sigma(\kappa)$ also holds when κ is a quaternionic representation. We provide proofs of all these statements. These proofs rely on a case-by-case analysis, maybe overlooking some hidden symmetry principle. We also give various illustrations of these properties in the contexts of boundary conformal field theories, integrable quantum field theories and topological field theories.

Of course the Theorem is non-trivial only in cases where $\mathfrak{g}$ has complex representations, *i.e.* $\mathfrak{g} = A_n$, $D_{n=2s+1}$ or E_6. Although this looks like a classroom exercise in group theory, we couldn't find either a reference in the literature or a simple and compact argument and we had to resort to a case by case analysis, see Sect 2 below. Note also that this property is not a trivial consequence of the general representation theory of groups; in particular, it does not hold in general in finite groups, see Sect 7 below for counterexamples based on finite subgroups of SU(3).

The representation rings of compact Lie algebras have no distinct features...

- Consider two irreps λ_i, λ_j of a compact group G, and the Clebsch-Gordan series $\lambda_i \times \lambda_j \cong \oplus_k N_{ij}^k \lambda_k$. The relation $N_{ij}^k f_i f_j f_k > 0$ holds if G is simply connected. (E. P. Wigner, On representations of certain finite groups, Amer. J. Math., 63 (1941), 57-63.)
- Consider the irreps $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \nu_m$ of a simply connected compact group G. If $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n \cong \nu_1 \times \nu_2 \times \ldots \times \nu_m$, then $n = m$ and there exists a permutation $\pi \in S_n$ such that $\lambda_k = \nu_{\pi(k)}$.
- All of these theorems have classification dependent proofs.

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Elementary unitary quantum control theory I

- Assume that we can implement interactions from a given set $\mathcal{I} = \{iH_1, iH_2, \ldots\}$ of Hamiltonians with tunable control parameters: $H(t) = \sum_j \alpha_j(t) H_j$. This generates a unitary of the form

$$U = \mathcal{T} \int_{t=0}^{1} \exp\left[\sum_{j=1}^{m} i\alpha_j(t) H_j\right]$$

- Two basic questions:
  - Which are the gates (unitaries) that we can generate?
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$$e^{[iH_k,iH_l]} = \lim_{n\to\infty}\left(e^{iH_k/\sqrt{n}}e^{iH_l/\sqrt{n}}e^{-iH_k/\sqrt{n}}e^{-iH_l/\sqrt{n}}\right)^n,$$
$$e^{-i(\alpha H_k+\beta H_l)} = \lim_{n\to\infty}\left(e^{-i(\alpha H_k/n)}e^{-i(\beta H_l/n)}\right)^n,$$
shows that one can obtain exponential of all commutators $[iH_k, iH_l]$, $[[iH_k, iH_l], iH_m], \ldots$(and their linear combinations), i.e., we end up with the full Lie algebra generated by $\mathcal{I}$:
$$i\tilde{H} \in \langle iH_1, iH_2, \ldots, iH_n\rangle_{Lie}$$
we can obtain $U = e^{i\tilde{H}}$.

Elementary unitary quantum control theory II

- Using the Lie -Trotter formulas, we have

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Unitary controllability, pure state controllability

- Full unitary controllability. Any unitary gate can be reached iff

$$\langle iH_1, iH_2, \ldots \rangle = \mathfrak{su}(d).$$

- Pure-state controllability:

$$\begin{aligned} \langle iH_1, iH_2, \ldots \rangle &= \mathfrak{su}(d) \quad \text{when } d \text{ is odd,} \\ \langle iH_1, iH_2, \ldots \rangle &\supset \mathfrak{usp}(d) \quad \text{when } d \text{ is even.} \end{aligned}$$

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Membership problems

- Is there an efficient way to determine whether $i\tilde{H} \in \langle iH_1, iH_2, \ldots, iH_n \rangle_{Lie}$ (or $\tilde{U} \in G$)?

- Discrete case: $\{U_1, U_2, \ldots U_n\}$ set of unitaries; G is the discrete (finite or infinite) group generated by this set. Is there an efficient way of determining whether $\tilde{U} \in G$?

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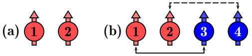
Simple symmetries

- For Unitary Gates:
  - If there exists a non-trivial symmetry S, such that $[S, U_i] = 0$ for all $\{U_1, U_2, \ldots, U_n\}$, but $[S, U] \neq 0$, then U cannot be generated.
- For Hamiltonians:
  - If there exists a non-trivial symmetry S, such that $[S, H_i] = 0$ for all $\{iH_1, iH_2, \ldots, iH_n\}$, but $[S, iH] \neq 0$, then iH cannot be generated.
- However, this is only a necessary, but not sufficient, condition.

A simple example

- The pair interaction $iH_{zz} := iZ_1Z_2$ cannot be simulated by the local interactions $\mathcal{P} = \{iX_1, iY_1, iX_2, iY_2\}$ of a two-qubit system in spite of coinciding (trivial) commutants $\mathcal{P}' = (\mathcal{P} \cup \{iH_{zz}\})' = \mathbb{C}\mathbb{1}_4$.



- However, we know that if we consider a 'doubled Hilbert space', then there are entanglement (or LU) invariants.

$$\langle\psi|\langle\psi|P^{(13)}|\psi\rangle|\psi\rangle =$$
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- Hence we should study higher order symmetries.

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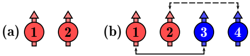

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Higher-order symmetries

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  - A non-trivial second-order symmetry $S^{(2)}$ on $\mathcal{H}^{\otimes 2}$ or a third-order symmetry $S^{(3)}$ on $\mathcal{H}^{\otimes 3}$ are operators that satisfy $[S^{(2)}, U_i \otimes U_i] = 0$ and $[S^{(3)}, U_i \otimes U_i \otimes U_i] = 0$ for all $\{U_1, U_2, \ldots, U_n\}$.
  - If for some n-th order symmetry $[S^{(n)}, U^{\otimes n}] \neq 0$, then U cannot be generated.
  - This cannot be a sufficient an necessary condition for any finite n - e.g. group designs provide counter examples.
- For Hamiltonians:
  - Second-order and third-order symmetries: $[S^{(2)}, iH_\ell \otimes \mathbb{1} + \mathbb{1} \otimes iH_\ell] = 0$ and $[S^{(3)}, iH_\ell \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes iH_\ell \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} iH_\ell] = 0$ for all $\{iH_1, iH_2, \ldots, iH_n\}$.
  - If $[S^{(2)}, iH \otimes \mathbb{1} + \mathbb{1} \otimes iH] \neq 0$ then $iH \notin \langle iH_1, iH_2, \ldots, iH_m \rangle_{\mathrm{Lie}}$.

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Our main theorem

Theorem

Given a subalgebra $\mathfrak{h}$ *of a compact semisimple Lie algebra* $\mathfrak{g}$ *and a faithful representation* ϕ *of* $\mathfrak{g}$*, then the following statements are equivalent:*

(1) $\mathfrak{h} = \mathfrak{g}$,
(2) $\dim(\mathfrak{com}[(\phi \otimes \bar{\phi})|_{\mathfrak{h}}]) = \dim(\mathfrak{com}[\phi \otimes \bar{\phi}])$,
(3) $\dim(\mathfrak{com}[(\phi \otimes \phi)|_{\mathfrak{h}}]) = \dim(\mathfrak{com}[\phi \otimes \phi])$,
(4) $\|(\phi \otimes \bar{\phi})|_{\mathfrak{h}}\|_2 = \|\phi \otimes \bar{\phi}\|_2$,
(5) $\|(\phi \otimes \bar{\phi})|_{\mathfrak{h}}\|_1 = \|\phi \otimes \bar{\phi}\|_1$,
(6) $\|(\phi \otimes \phi)|_{\mathfrak{h}}\|_2 = \|\phi \otimes \phi\|_2$.
(7) $\|(\phi \otimes \phi)|_{\mathfrak{h}}\|_1 = \|\phi \otimes \phi\|_1$.

Strengthening the theorem

Theorem

Let α be a simple and self-dual representation of a compact simple Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, then
(1) $\|(\alpha \otimes \alpha)|_{\mathfrak{h}}\|_1 \geq b(\alpha) + \|\alpha \otimes \alpha\|_1$,
(2) $\|(\alpha \otimes \alpha)|_{\mathfrak{h}}\|_2 \geq b(\alpha)^2 + \|\alpha \otimes \alpha\|_2$, *and*
(3) $\dim(\mathfrak{com}[(\alpha \otimes \alpha)|_{\mathfrak{h}}]) \geq b(\alpha)^2 + \dim(\mathfrak{com}[\alpha \otimes \alpha])$ *hold,*
where $b(\alpha)$ *denotes the number of non-vanishing components in the highest weight* $(\alpha_1, \ldots, \alpha_\ell)$ *corresponding to* α.

Our final main result for control theory

Consider two sets $\mathcal{P} := \{iH_1, \ldots, iH_p\}$ and $\mathcal{Q} := \{iH_{p+1}, \ldots, iH_q\}$ of (skew-hermitian) interactions, and let C_α denote elements of a linear basis spanning the ce
nter $\mathcal{C}$ of the commutant $(\mathcal{P} \cup \mathcal{Q})'$. For the central projections, define the matrix T by its entries $T_{\alpha\beta} := \mathrm{Tr}[C_\alpha^\dagger iH_\beta]$ for $1 \leq \alpha \leq \dim(\mathcal{C})$ and $1 \leq \beta \leq q$ as well as $\widetilde{T}$ by $\widetilde{T}_{\alpha\beta} := \mathrm{Tr}[C_\alpha^\dagger iH_\beta]$ for $1 \leq \beta \leq p$. Then $\mathcal{P}$ simulates $\mathcal{Q}$ in the sense $\langle \mathcal{P} \rangle = \langle \mathcal{P} \cup \mathcal{Q} \rangle$, if and only if both conditions (A) $\dim[\mathcal{P}^{(2)}] = \dim[(\mathcal{P} \cup \mathcal{Q})^{(2)}]$ and (B) $\mathrm{rank}(\widetilde{T}) = \mathrm{rank}(T)$ are fulfilled.

Central spin model

Consider a central spin interacting with $n-1$ surrounding spins via a star-shaped coupling graph (where the surrounding spins may be taken as uncontrolled spin bath) The interactions amount to a drift term (tunneling plus coupling) and just a local Z-control on the central spin, $\mathcal{P} := \{iX_1 + i\sum_{k=2}^{n} J_k(X_1X_k+Y_1Y_k+Z_1Z_k),\, iZ_1\}$. We ask whether the central spin can be fully controlled, i.e., if $\mathcal{Q} := \{iX_1\}$ can be simulated.

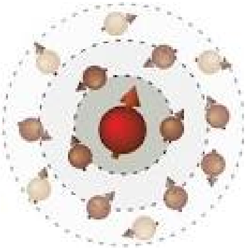

Table : Central spin model. number n of spins, Lie dimensions $\dim(\langle\mathcal{P}\rangle) = \dim(\langle\mathcal{P}\cup\mathcal{Q}\rangle)$, the isomorphy type, dimensions of second- and first-order symmetries (i.e. $\dim[\mathcal{P}^{(2)}] = \dim[(\mathcal{P}\cup\mathcal{Q})^{(2)}]$ and $\dim[\mathcal{P}'] = \dim[(\mathcal{P}\cup\mathcal{Q})']$), and ranks of the central projections (i.e. $\mathrm{rank}(\widetilde{T}) = \mathrm{rank}(T)$).

n	Lie-dim.	Isomorphy type	No. of symmetries 2nd	1st	Rank of proj.
case (a): $J_k = 1$					
2	15	$\mathfrak{su}(4)$	2	1	0
3	38	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)$	8	2	0
4	78	$\mathfrak{su}(4)\oplus\mathfrak{su}(8)$	50	5	0
5	137	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)\oplus\mathfrak{su}(10)$	392	14	0
6	221	$\mathfrak{su}(4)\oplus\mathfrak{su}(8)\oplus\mathfrak{su}(12)$	3528	42	0
case (b): $J_k = 2$ for even k and $J_k = 1$ otherwise					
2	15	$\mathfrak{su}(4)$	2	1	0
3	63	$\mathfrak{su}(8)$	2	1	0
4	158	$\mathfrak{su}(4)\oplus\mathfrak{su}(12)$	8	2	0
5	396	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)\oplus\mathfrak{su}(6)\oplus\mathfrak{su}(18)$	32	4	0
6	796	$\mathfrak{su}(4)\oplus\mathfrak{su}(8)\oplus\mathfrak{su}(12)\oplus\mathfrak{su}(24)$	200	10	0

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