### Hadronization via Coherent States

#### Superstatistics due to Quantum Properties

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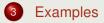
- to construct particular Coherent States
- designed to describe NB *n* distribution
- and non-extensive Tsallisean statistical weights











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#### Statistical Weight from n-distribution

2 Nonlinear coherent states

#### 3 Examples

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### Summary of ideal reservoir fluctuations 1

Ideal gas formula and thermodynamical limit:

$$w_E^{\text{LIM}}(\omega) = \lim_{\substack{n \to \infty \\ E \to \infty \\ E/n = T}} \left(1 - \frac{\omega}{E}\right)^n = e^{-\omega/T}.$$
 (1)

Poisson average on *n* at fix *E*:

$$w_E^{\text{POI}}(\omega) = \mathrm{e}^{-\langle n \rangle \, \omega/E}.$$
 (2)

Negative binomial (NB) average:

$$w_E^{\text{NBD}}(\omega) = \left(1 + \frac{\langle n \rangle}{k+1} \frac{\omega}{E}\right)^{-(k+1)}.$$
 (3)

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## Summary of ideal reservoir fluctuations 2

# Expanding up to variance in n we obtain for general n fluctuations

Exact for Poisson, Bernoulli, Negative Binomial

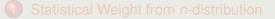
$$T = \frac{E}{\langle n \rangle}$$
, and  $q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}$  (4)

In general *T* and *q* are related to expectation values of derivatives of the EoS S(E) over reservoir fluctuations.

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Outline





#### 2 Nonlinear coherent states

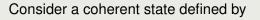
- State labels
- Operator eigenstate

#### 3 Examples

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Biró Hadronization: NBS

State labels



$$|z\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(t)} e^{in\Theta} |n\rangle$$
(5)

with  $z = \sqrt{t}e^{i\Theta}$ .

Definition

("nonlinear coherent state")

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It overlaps with the *n*-quantum state:

$$|\langle n|z\rangle|^2 = p_n(t) \ge 0.$$
(6)



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Statistical Weight from n-distribution Nonlinear coherent states Examples

Summary

State labels Operator eigenstate



#### Normalization

It is normalized,

$$\langle z|z\rangle = \sum_{n,m} \langle m|\sqrt{p_m p_n} e^{i(n-m)\Theta}|n\rangle = \sum_n p_n(t) = 1.$$
 (7)

The expectation value of a function of the number operator is

$$\langle z | \varphi(N) | z \rangle = \sum_{n} \varphi(n) p_{n}(t).$$
 (8)

This ensures that  $p_n(t)$  is a probability distribution in n!

To what is it an eigenstate?

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State labels Operator eigenstate



#### Completeness

We construct a complete set:

$$\int \frac{d^2 z}{\pi} |z\rangle \langle z| = \int_{0}^{\infty} dt \int_{0}^{2\pi} \frac{d\Theta}{2\pi} \sum_{n,m} \sqrt{p_n p_m} e^{i(n-m)\Theta} |m\rangle \langle n|$$
$$= \int_{0}^{\infty} dt \sum_{n} p_n(t) |n\rangle \langle n| = \sum_{n} |n\rangle \langle n| = 1.$$
(9)

It is satisfied only if  $\int_{0}^{\infty} dt p_n(t) = 1$ .

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#### This makes $p_n(t)$ to a probability distribution function of t!

State labels Operator eigenstate



#### CS as eigenstate to some operator

Eigenstate with eigenvalue z to

$$F |z\rangle = ag(\hat{n}) |z\rangle = z |z\rangle.$$
 (10)

Here *a* is an annihilating ( $a^{\dagger}$  is a creating) operator, and  $\hat{n} = a^{\dagger}a$  is the number operator. Its action on the CS:

$$F |z\rangle = \sum_{n=1}^{\infty} g(n) \sqrt{np_n} e^{in\Theta} |n-1\rangle, \qquad (11)$$

can be re-indexed to

$$F | z \rangle = \sum_{n=0}^{\infty} g(n+1) \sqrt{(n+1)p_{n+1}} e^{i(n+1)\Theta} | n \rangle, \qquad (12)$$

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Biró Hadronization: NBS

#### Nonlinear coherent states Examples

Statistical Weight from n-distribution

# **Recursion law**

Compare this with

$$z |z\rangle = \sqrt{t} e^{i\Theta} \sum_{n=0}^{\infty} \sqrt{p_n} e^{in\Theta} |n\rangle, \qquad (13)$$

Operator eigenstate

to conclude that

$$p_n(t) = \frac{t}{ng(n)^2} p_{n-1}(t).$$
 (14)

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Statistical Weight from n-distribution Nonlinear coherent states Examples Operator eigenstate Summary wigner *n*-probability Symmetric generalized binomial distributions, J.Math.Phys. 54 (2013) 123301 The recursion is solved by  $p_n(t) = p_0(t) \frac{t^n}{n!} \prod_{i=1}^n g(i)^{-2}.$ (15)Here  $p_0(t)$  can be obtained from the normalization condition.

Also the completeness constraint,  $\int_{0}^{\infty} dt p_n(t) = 1$ , has to be checked.

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Outline

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2 Nonlinear coherent states

#### 3 Examples

- Glauber states
- Phase states: negative binomial
- NB states

Glauber states Phase states: negative binomial NB states



### Traditional CS

The most known CS is defined by g(n) = 1.

#### This results in a **Poisson** in *n* and **Euler-Gamma** in *t*:

$$p_n(t) = \frac{t^n}{n!} e^{-t}, \qquad (16)$$

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and  $|z\rangle$  is an eigenstate to the F = a annihilator.

Phase states: negative binomial NB states



### 7 sources of NBD

- Phase space cell statistics (Biró)
- Squeeze parameter (Varró, Jackiw)
- Wave packet statistics (Pratt, Csörgő, Zimányi)
- Temperature superstatistics (Beck, Wilk)
- S KNO + pQCD (Dokshitzer, Dremin, Hegyi, Carruthers)
- Tsallis/Rényi entropy canonical state (Rényi, Tsallis, ...)
- Ø Glittering Glasma (Gelis, Lappi, McLerran)

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Summary

Glauber states Phase states: negative binomial NB states



# 1. Phase Space Cell Statistics

Probability to find n particles in k cells, if altogether we have thrown N particles into K cells:

#### Pólya distribution

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$$\mathsf{P}_n = \frac{\binom{k+n}{n}\binom{K-k+N-n}{N-n}}{\binom{K+N+1}{N}}.$$

Necessary limit:  $K \to \infty$ ,  $N \to \infty$  while f = N/K kept finite.

Prediction:  $k + 1 = \langle n \rangle / f \propto N_{part}$ , if f is universal.

Here k is the number of observed phase space cells: from which the detected n particles seem to come.

Summary

Glauber states Phase states: negative binomial NB states



## 2. Negative Binomial State (NBS)

Neg.Binom. state as a nonlinear coherent state:

$$|z,k\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k)} e^{in\Theta} |n\rangle$$
(17)

with

$$p_n(k) = \binom{k+n}{n} f^n (1+f)^{-n-k-1}.$$
 (18)

#### NBS annihilated

$$a | z, k \rangle = \sqrt{f(k+1)} e^{i\Theta} | z, k+1 \rangle.$$
 (19)

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Summary

Glauber states Phase states: negative binomial NB states

# 3. Wave packet statistics

small size limit: k = 0 NBS

HBT with onefold filled bosonic states gives a correlation factor of 2 at zero relative momentum. With M-fold occupation of the same state it reduces to

$$C_2(0) = 1 + \frac{1}{M} = 1 + \frac{1}{k+1}.$$
 (20)

The logarithmic cumulants for an NBS, defined by  $G(z) = \sum p_n z^n$  and  $\ln G(z) = \sum_{n=1}^{\infty} C_n(z^n - 1)$ , are

$$C_n = \frac{k+1}{n} \left(\frac{f}{1+f}\right)^n.$$
 (21)

This is a (k + 1)-fold overload of the simple Bose case, given if k = 0.



Statistical Weight from n-distribution Nonlinear coherent states Phase states: negative binomial Examples Summary wigner 4. Superstatistics Beck, Wilk Thermodynamical  $\beta$ -fluctuation and *n*-fluctuations are related by Poisson transform:  $\infty$ 

$$\int_{0}^{\infty} \gamma(\beta) e^{-\beta \omega} d\beta = \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^{n} P_{n}(E).$$
 (22)

In this way  $\Delta \beta^2 / \langle \beta \rangle^2 = 1/(k+1)$ .

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Summary

Wigner



Dokshitzer, Dremin, Hegyi, Carruthers

Phase states: negative binomial

KNO scaling + GLAP give nearly NBD with constant k, related to  $\Lambda_{QCD}$  and expressed by *n*-variance.

NBD is in fact slightly violated.

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Glauber states Phase states: negative binomial NB states



## 6. Canonical

Accepting Tsallis' or Rényi entropy as a formula, the usual canonical constraint on the average energy leads to

$$w(\omega) = \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$
(23)

Using further assumptions about reservoir fluctuations, further entropy formulas can be constructed, as expectation values of formal logarithms, behaving additively (ARC).

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Summary

Glauber states Phase states: negative binomial NB states

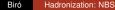
7. Glittering Glasma: k-fold ropes

Wigner

Gelis, Lappi, McLerran

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 $k = \kappa (N_c^2 - 1)Q_s^2 R^2/2$  is about the number of tubes, makes NBS with this parameter.



Summary

Glauber states Phase states: negative binomial NB states



#### NB coherent states

The negative binomial distribution (NBD),

$$p_n(t) = {\binom{n+k}{n}} (t/k)^n (1+t/k)^{-n-k-1},$$
 (24)

is well normalized in n and, as an Euler-Beta distribution, also in t. From the recursion one obtains

$$g(n)^2 = \frac{t}{n} \frac{p_{n-1}}{p_n} = \frac{k+t}{k+n},$$
 (25)

so this state satisfies

$$a\sqrt{\frac{k+|z|^2}{k+a^{\dagger}a}} |z\rangle = z |z\rangle.$$
 (26)

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Summary

Glauber states Phase states: negative binomial NB states



# su(1,1) structure in NBD

Rapidity-like notation:  $t/k = \sinh^2 \zeta$ ;

$$p_{n}(t) = {\binom{k+n}{n}} \sinh^{2n} \zeta \cosh^{-2n-2k-2} \zeta.$$
$$|z\rangle = \cosh^{-k-1}(\zeta) \sum_{n=0}^{\infty} \sqrt{\binom{k+n}{n}} \left( \tanh(\zeta) e^{i\Theta} \right)^{n} |n\rangle. \quad (27)$$

Using velocity  $v = tanh(\zeta)$  the overlap between two NBD Coh.States:

$$|\langle z_1 | z_2 \rangle|^2 = \left[ 1 + \gamma_1^2 \gamma_2^2 \left| v_1 e^{i\Theta_1} - v_2 e^{i\Theta_2} \right|^2 \right]^{-k-1}.$$
 (28)

2+1 dim relative velocity vector separates

Summary

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#### 2+1 dim vector representation

$$\vec{K} = \gamma_1 \gamma_2 \left( \vec{v}_1 - \vec{v}_2 \right) = \frac{1}{\sqrt{k}} \left( \gamma_2 z_1 - \gamma_1 z_2 \right).$$
 (29)

Overlap written this way

$$|\langle z_1|z_2\rangle|^2 = \left[1 + \frac{1}{k}|\gamma_2 z_1 - \gamma_1 z_2|^2\right]^{-k-1} \to e^{-|z_1 - z_2|^2}.$$
 (30)

with  $\gamma_i = \sqrt{1 + |z_i|^2/k}$ . Particle properties of the vector  $\vec{K}$ :

$$\vec{K} = \frac{1}{m_1 m_2} \left( E_2 \vec{P}_1 - E_1 \vec{P}_2 \right)$$
(31)

Its parallel component does not Lorentz transform:  $\vec{v} \vec{K}_{i} = \vec{v} \vec{K}$ .

Summary

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## Negative Binomial States

in our own notation

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$$z_{k}, k \rangle = \sum_{n=0}^{\infty} \sqrt{p_{n}(k)} e^{in\Theta} | n \rangle$$
(32)

with  $z_k = \sqrt{kf} e^{i\Theta}$  and

$$p_n(k) = \binom{k+n}{n} f^n (1+f)^{-n-k-1}$$
(33)

provides an average photon number  $\langle n \rangle = f(k + 1)$ .

- excited geometric state: n over k
- intermediate number state: f/(1 + f)
- eigenstate with complex eigenvalue



Summary

Glauber states Phase states: negative binomial NB states



# The Effect of Ladder Operator

Effect of annihilating a quantum:

$$a | z_k, k \rangle = \sum_{n=1}^{\infty} \sqrt{p_n} e^{in\Theta} \sqrt{n} | n-1 \rangle = \sum_{n=0}^{\infty} \sqrt{(n+1)p_{n+1}} e^{i\Theta} e^{in\Theta} | n \rangle.$$
(34)

#### Consider now

$$(n+1)\binom{k+n+1}{n+1}f^{n+1}(1+f)^{-(n+1)-k-1} = f(k+1)\binom{k+1+n}{k+1}f^{n}(1+f)^{-n-(k+1)-1}$$
(35)

Here we recognize  $z_{k+1} = \sqrt{f(k+1)}e^{i\Theta}$  as a factor and arrive at NBS annihilated

$$a | z_k, k \rangle = z_{k+1} | z_{k+1}, k+1 \rangle.$$
 (36)

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Summary

Glauber states Phase states: negative binomial NB states



## NBS as eigenstate of what?

The action of another operator

$$\sqrt{\hat{N}+k+1} | z,k \rangle = \sqrt{(k+1)(1+f)} | z,k+1 \rangle,$$
 (37)

based on the relation

$$(k+1+n)\binom{k+n}{k} = (k+1)\binom{k+1+n}{k+1}.$$
 (38)

This helps to recognize:

NBS eigenvalue equation

$$\left(\sqrt{f}\left(\hat{n}+k+1\right)-\sqrt{1+f}\,e^{-i\Theta}\,\sqrt{\hat{n}+k+1}\,a\right)\,|\,z,k\,\rangle\,=\,0.$$
(39)

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Summary

Glauber states Phase states: negative binomial NB states



### Our NBS algebra

In the previous equation the following operator occurs:

$$K_{-} = \sqrt{\hat{n} + k + 1} a, \qquad K_{+} = K_{-}^{\dagger} = a^{\dagger} \sqrt{\hat{n} + k + 1}.$$

The commutator,

$$[K_{-}, K_{+}] = (\hat{n} + 1)(\hat{n} + k + 1) - \hat{n}(\hat{n} + k) = 2\hat{N} + k + 1 = 2K_{0}$$
(40)  
defines  $K_{0} = \hat{n} + (k + 1)/2$ .  
With  $\alpha = \sqrt{(1 + f)/f} e^{-i\Theta}$  we get

OUR eigenvalue equation

$$(\alpha K_{-} - K_{0}) |z, k\rangle = \frac{k+1}{2} \cdot |z, k\rangle.$$
(41)

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Summary

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# SU(1,1) algebra for NBS

The commutators form an SU(1,1) algebra:

#### Commutators

$$[K_0, K_+] = K_+$$
  

$$[K_0, K_-] = -K_-$$
  

$$[K_-, K_+] = 2K_0$$
(42)

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The Casimir operator is given as:  $Q = K_0^2 - K_0 - K_+ K_-$ .

Summary

Glauber states Phase states: negative binomial NB states



# $|NBS\rangle$ created from $|0\rangle$ :

1. preliminaries

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Operator identity

 $e^{A+B} = e^{-\lambda/2} e^A e^B$  if  $[A, B] = \lambda$  (const.) (43)

We choose  $A = \alpha zg(\hat{n}) a^{\dagger}$  and  $B = -\beta z^* a \frac{1}{g(\hat{n})} \neq -A^{\dagger}$ .

Commutator:

$$\lambda = [A, B] = -|z|^2 \alpha \beta g(\hat{n}) a^{\dagger} a \frac{1}{g(\hat{n})} + |z|^2 \alpha \beta a \frac{1}{g(\hat{n})} g(\hat{n}) a^{\dagger} = |z|^2 \alpha \beta.$$
(44)

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Summary

Glauber states Phase states: negative binomia NB states

# $|NBS\rangle$ created from $|0\rangle$ :

Wigner

2. evolution operator

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$$U = e^{\Phi/2 + A + B} = e^{(\Phi - \lambda)/2} e^A e^B.$$
 (45)

Here  $e^{B} | 0 \rangle = | 0 \rangle$  due to  $B | 0 \rangle = 0$ .

$$U | 0 \rangle = e^{(\Phi - \alpha\beta |z|^2)/2} \sum_{n=0}^{\infty} \frac{\alpha^n z^n}{n!} \left( g(\hat{n}) a^{\dagger} \right)^n | 0 \rangle.$$
 (46)

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Summary

Glauber states Phase states: negative binomial NB states

# 3. n-photon state

# $|\textit{NBS}\rangle$ created from $|\,0\,\rangle$ :

We have

$$\left(g(\hat{n})a^{\dagger}\right)^{n}|0\rangle = g(n)\cdot\ldots\cdot f(1)\sqrt{n!}|n\rangle.$$
 (47)

#### Regarding the form

$$U|0\rangle = \sum_{n=0}^{\infty} \sqrt{u_n} e^{in\Theta} |n\rangle, \qquad (48)$$

with  $z = \sqrt{t} e^{i\Theta}$ , and using  $g(\hat{n}) = \xi \sqrt{\hat{n} + k}$  we obtain

$$u_n = e^{\Phi - \alpha\beta t} \left( \alpha^2 \xi^2 t \right)^n \binom{k+n}{n}.$$
 (49)

wigner

Summary

Glauber states Phase states: negative binomial NB states

# $|NBS\rangle$ created from $|0\rangle$ :

Wigner

## 4. normalization

For  $||U|0\rangle||^2 = 1$  one needs

$$\sum_{n=0}^{\infty} u_n = e^{\Phi - \alpha \beta t} \left( 1 - \alpha^2 \xi^2 t \right)^{-(k+1)} = 1.$$
 (50)

#### From this we express

$$\alpha^{2}\xi^{2}t = 1 - e^{\frac{1}{k+1}(\Phi - \alpha\beta t)} = 1 - w,$$
 (51)

#### and gain

#### negative binomial distribution

$$u_n = \binom{n+k}{n} w^{k+1} (1-w)^n.$$
 (52)

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Summary

Glauber states Phase states: negative binomial NB states

# $|NBS\rangle$ created from $|0\rangle$ :

5. superstatistics

For the superstatistics normalization, we utilize

$$\int_{0}^{\infty} dt \, u_n(t) = \binom{k+n}{n} \int_{0}^{\infty} dt \, w^{k+1} (1-w)^n = \binom{k+n}{n} k \int_{0}^{1} dw \, w^{k-1} (1-w)^n = 1.$$
(53)

This is achieved if

$$dt = -k \frac{dw}{w^2}, \qquad \longrightarrow \qquad t = k \left(\frac{1}{w} - 1\right), \qquad (54)$$

or expressing w(t) if

$$w = \frac{1}{1+t/k} = \frac{k}{t+k}.$$
 (55)

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Summary

Glauber states Phase states: negative binomial NB states

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# $|NBS\rangle$ created from $|0\rangle$ :

6. conclusion

In this way  $\alpha, \beta$  and  $\xi$  remain undetermined. A purposeful choice is  $\alpha = \beta = 1/\sqrt{t} = 1/|z|$  with  $\xi = 1/\sqrt{t+k}$ .

The log of the evolution operator becomes in this case

#### neither hermitic nor anti-hermitic

$$\ln U = -\frac{k+1}{2}\ln(1+t/k) + \frac{1}{2} + e^{i\Theta}\sqrt{\frac{\hat{n}+k}{t+k}}a^{\dagger} - e^{-i\Theta}a\sqrt{\frac{t+k}{\hat{n}+k}}.$$
(57)

#### U may be connected to a Hamiltonian...

Summary

Glauber states Phase states: negative binomial NB states



 $|NBS\rangle$  created from  $|0\rangle$ :

7. physics behind

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The anti-hermitic part gives a guess for the Hamiltonian

$$\frac{i}{\hbar}\int Hd\tau = \frac{1}{2}\left(\ln U - (\ln U)^{\dagger}\right) = e^{i\Theta}\left(f(\hat{n}) + \frac{1}{f(\hat{n})}\right)a^{\dagger} - e^{-i\Theta}a\left(f(\hat{n}) + \frac{1}{f(\hat{n})}\right).$$
(58)

The hermitic part means non-conserved poarticle number...





- A class of coherent states designed for a given n pdf is also a superstatistics in t = |z|<sup>2</sup>;
- The NB coherent state is an eigenstate for the regularized phase operator.
- An su(1,1) structure is inherent in the NB coherent state.
- Hamiltonians combined from K<sub>-</sub>, K<sub>0</sub> and K<sub>+</sub> operators are likely to produce NB distributed bosons.