Second order dissipative fluid dynamics from kinetic theory

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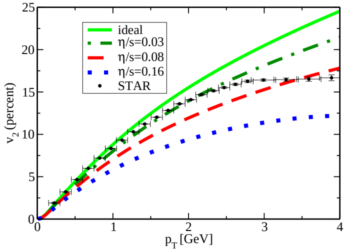
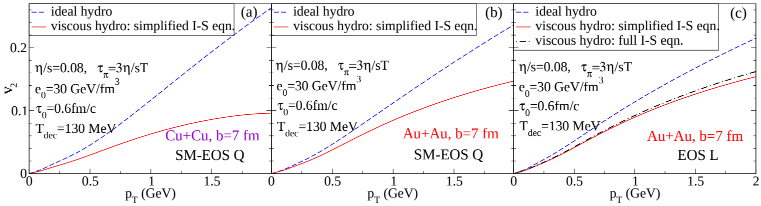


Romatschke and Romatschke (2007)



Song and Heinz (2008)

Motivation II

- We are interested in the study of non-equilibrium phenomena (transient or stationary) of long wavelength (fluid dynamic limit), where the dissipative fluxes, heat flux, diffusion flux and viscous pressure tensor, play an important role, e.g., Navier-Stokes equations, since the 1820's.
- The study of non-equilibrium phenomena from kinetic theory started since the 1850's, Maxwell, Boltzmann, Hilbert, Chapman, Enskog, Grad, etc. The resulting equations and coefficients are calculable from (relativistic) kinetic theory, e.g., in the dilute gas limit.
- We aim for a macroscopic theory consistent with the principles of relativity and thermodynamics, since the 1940's, Eckart, Taub, Landau, Lifsitz, Müller, Israel, Stewart, etc.
- Nowadays, we aim to understand the role of dissipation in high-energy heavy-ion collisions!

Notation

- We work in flat space-time, the metric is $g^{\mu\nu} \equiv g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- The normalized 4-flow of matter is denoted by $u^\mu(t, \mathbf{x})$, where $u^\mu u_\mu = 1, (c^2 = 1)$.
- The local rest frame (LRF) is defined as, $u^\mu = (1, 0, 0, 0)$.
- We define the projection tensor, perpendicular to the 4-flow of matter, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, where $\Delta^{\mu\nu} u_\mu = 0$, and $\Delta^{\mu\nu}\Delta_{\mu\nu} = 3$.
- The comoving time-derivative or time-derivative in LRF of A is denoted by, $\dot{A} = u^\mu \partial_\mu A$.
- The comoving spatial-derivative or gradient in LRF of A is denoted by, $\nabla^\mu A = \Delta^{\mu\nu}\partial_\nu A$.
- The symmetric, traceless and orthogonal part of a tensor is denoted by, $A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] A^{\alpha\beta}$.

Perfect Fluids I.

Conservation laws for a simple (single component) perfect fluid (no dissipation)

$$\partial_\mu N_0^\mu = 0 \quad \textbf{charge conservation} \quad \Rightarrow \textbf{1 eq.}$$

$$\partial_\mu T_0^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation} \quad \Rightarrow \textbf{4 eqs.}$$

Perfect fluid decomposition with respect to u^μ

$$N_0^\mu = n_0 u^\mu$$

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu}$$

$$n_0 = N_0^\mu u_\mu \quad \textbf{(net)charge density in LRF}$$

$$e_0 = T_0^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density in LRF}$$

$$p_0 = -\frac{1}{3}\Delta_{\mu\nu} T_0^{\mu\nu} \quad \textbf{isotropic pressure in LRF}$$

- We only have 5 equations for 6 unknowns: $n_0(1)$, $e_0(1)$, $p_0(1)$ and $u^\mu(3)$. These equations are *postulated*! but they are *not closed*!

Perfect Fluids II.

- The assumption of local thermal equilibrium! Provides closure:

Equation of State (EoS)

$$p_0 = p_0(e_0, n_0) \qquad \text{EoS} \Rightarrow \mathbf{1\ eq.}$$

and/or $p(T, \mu)$ or $s = s(e, n)$.

- Auxiliary, $S_0^\mu = s_0 u^\mu$, where $s_0 = S_0^\mu u_\mu$, and for continuous solutions

$$\partial_\mu S_0^\mu = 0$$

entropy is maximum in local thermal equilibrium, with no entropy production!

- The fundamental thermodynamic relations from $T\partial_\mu(su^\mu) = \partial_\mu(eu^\mu) + p(\partial_\mu u^\mu) - \mu\partial_\mu(nu^\mu)$

$$\begin{aligned} Ts &= e + p - \mu n \\ T\dot{s} &= \dot{e} - \mu\dot{n} \end{aligned}$$

Dissipative Fluids I.

Conservation laws for a simple (single component) dissipative fluid

$$\partial_\mu N^\mu = 0 \quad \textbf{charge conservation} \quad \Rightarrow \textbf{1 eq.}$$
$$\partial_\mu T^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation} \quad \Rightarrow \textbf{4 eqs.}$$

Dissipative fluid decomposition with respect to u^μ

$$N^\mu = n u^\mu + V^\mu$$
$$T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}$$
$$n = N^\mu u_\mu \quad \textbf{charge density in LRF}$$
$$e = T^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density in LRF}$$
$$p + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \quad \textbf{isotropic + bulk viscous pressure in LRF}$$
$$V^\mu = \Delta^{\mu\alpha} N_\beta \quad \textbf{charge flow in LRF}$$
$$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta \quad \textbf{energy-momentum flow in LRF}$$
$$q^\mu = W^\mu - \frac{e+p}{n} V^\mu \quad \textbf{heat flow in LRF}$$
$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} \quad \textbf{stress tensor in LRF}$$

- We only have 5 equations for 18 unknowns, $n(1)$, $e(1)$, $p(1)$, $u^\mu(3)$ and $\Pi(1)$, $V^\mu(3)$, $W^\mu(3)$, $\pi^{\mu\nu}(5)$.

Dissipative Fluids II.

Simplifications: Matching conditions and EoS

$$n = n_0$$
$$e = e_0$$
$$p(e,n) = p_0(e_0, n_0) \qquad \text{EoS} \Rightarrow \textbf{1 eq.}$$

- These are the most convenient since they extend $T = T_0$ and $\mu = \mu_0$ for non-equilibrium, but the entropy changes! For other matching possibilities see for example, Pavon, Biró, Ván.

Fixing the LRF

$$u^\mu_E = N^\mu / n \quad \Leftrightarrow V^\mu = 0 \quad \Rightarrow q^\mu = W^\mu \qquad \textbf{Eckart} \Rightarrow \textbf{3 eqs.}$$
$$u^\mu_L = T^{\mu\nu} u_{L\nu} / e \quad \Leftrightarrow W^\mu = 0 \quad \Rightarrow q^\mu = -\frac{e+p}{n} V^\mu \qquad \textbf{Landau \& Lifsitz} \Rightarrow \textbf{3 eqs.}$$

- One of these choices eliminates, but at the same time relates u^μ to, V^μ or W^μ.
- We are still left with 14 unknowns! $n(1), e(1), u^\mu(3)$ and $\Pi(1), q^\mu(3), \pi^{\mu\nu}(5)$.

Dissipative Fluids III.

The relativistic Navier-Stokes theory

- The definition of entropy is also modified $S^\mu \equiv S_0^\mu + \delta S^\mu = (s_0 + \delta s) u^\mu + \Phi^\mu$, where $s \equiv S^\mu u_\mu = (s_0 + \delta s)$ and $\Phi^\mu = \Delta^{\mu\nu} S_\nu$

2nd law of thermodynamics (Eckart's frame)

$$\partial_\mu S^\mu \equiv \partial_\mu \left[\Phi^\mu - \left(\frac{q^\mu}{T} \right) \right] - \frac{q^\mu}{T} \left(\frac{1}{T} \partial_\mu T - \dot{u}_\mu \right) - \frac{\Pi}{T} \partial_\mu u^\mu + \frac{\pi^{\mu\nu}}{T} \partial_\mu u_\nu \geq 0$$

- Assuming that $s(e, n) = s_0(e_0, n_0)$, $\Phi^\mu = q^\mu / T$ and gradients are small

Relativistic Navier-Stokes values

$$\begin{aligned} \Pi_{NS} &= -\zeta \nabla_\mu u^\mu \\ \pi^{\mu\nu}_{NS} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle} \\ q^\mu_{NS} &= -\kappa T \frac{T n}{e + p} \nabla^\mu \left(\frac{\mu}{T} \right) \end{aligned}$$

- $\zeta \geq 0$, $\eta \geq 0$ bulk and shear viscosity, $\kappa \geq 0$ thermal conductivity, coefficients.
- Now the eqs. are closed, but the rel. Navier-Stokes theory lead to accausal signal propagation and stability issues

Dissipative Fluids IV.

2nd order theories of Müller (1967), Israel (1976) and Stewart (1971, 1977)

- The previously mentioned issues are "cured" if:

Entropy current generalization (Eckart's frame)

$$\begin{aligned} S^\mu &\equiv s_0 u^\mu + \frac{q^\mu}{T} - \left(\beta_0 \Pi^2 - \beta_1 q^\mu q_\mu + \beta_2 \pi^{\mu\nu}\pi_{\mu\nu}\right)\frac{u^\mu}{2T} \\ &- \frac{\alpha_0 \Pi q^\mu}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{T} + O_3 \end{aligned}$$

where the newly introduced coefficients $\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1$ are related to the relaxation time/length of bulk viscosity, heat conductivity and shear viscosity

$$\begin{aligned} \tau_\Pi &= \zeta\beta_0 \\ \tau_q &= \kappa T \beta_1 \\ \tau_\pi &= 2\eta\beta_2 \\ l_{\Pi q} &= \zeta\alpha_0 \\ l_{q\Pi} &= \kappa T \alpha_0 \\ l_{q\pi} &= \kappa T \alpha_1 \\ l_{\pi q} &= 2\eta\alpha_1 \end{aligned}$$

- The $\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1$ coefficients are frame dependent and remain undetermined in phenomenological theories!

Dissipative Fluids V.

2nd order equations from entropy production

Relaxation equations in Eckart's frame (Israel 1976)

$$\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi &= \Pi_{NS} + l_{\Pi q} \nabla_\mu q^\mu \\
\tau_q \Delta^\mu_\alpha \dot{q}^\alpha + q^\mu &= q^\mu_{NS} + l_{q\Pi} \nabla^\mu \Pi - l_{q\pi} \Delta^\mu_\alpha \partial_\nu \pi^{\alpha\nu} \\
\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{NS} + l_{\pi q} \nabla^{\langle\mu} q^{\nu\rangle}
\end{aligned}$$

- The equations determine the time evolution of Π, q^μ, and $\pi^{\mu\nu}$
- The Navier-Stokes theory appears if the relaxation times and length scales $\tau_i \to 0$, $l_i \to 0$ with ζ, η and κ_q fixed
- Later O_2 corrections ($\sim \nabla_\mu \alpha_i, \sim \nabla_\mu \beta_i, \ldots$, etc.) were added by Israel and Stewart (1977-1979), Hiscock and Lindblom (1983), Relativistic Extended Thermodynamics of Liu, Müller and Ruggeri (1983) etc.
- Higher order O_3 corrections in the entropy current, by A. El et. al. (2008), A. Muronga (2009)

We need kinetic theory to motivate and determine the above introduced phenomenological equations self-consistently!

The relativistic Boltzmann equation

In a dilute gas, the space-time evolution of the single-particle distribution function $f_{\mathbf{k}} = f(t, \mathbf{x}, k^0, \mathbf{k})$ due to particle motion and binary collisions is given by the

The relativistic Boltzmann equation

$$k^{\mu}\partial_{\mu} f_{\mathbf{k}} = \frac{1}{2}\int dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right)$$

where $k^{\mu} = (k^0, \mathbf{k})$ is the four-momenta of particles with energy $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ and m is the mass of particles. The inv. phase-space element is, $dK = g d^3\mathbf{k} / \left[(2\pi)^3 k^0\right]$. The transition rate W is invariant to final state momenta $W_{\mathbf{kk'}\to\mathbf{pp'}} = W_{\mathbf{kk'}\to\mathbf{p'p}}$ and to the time reversal symmetry of the collision $W_{\mathbf{kk'}\to\mathbf{pp'}} = W_{\mathbf{pp'}\to\mathbf{kk'}}$.

Conservation laws from the Boltzmann equation

$$\partial_{\mu} N^{\mu} \equiv \int dK\, k^{\mu}\partial_{\mu} f_{\mathbf{k}} = \int dK\, C[f_{\mathbf{k}}] = 0 \qquad \textbf{charge cons.}$$

$$\partial_{\mu} T^{\mu\nu} \equiv \int dK\, k^{\nu} k^{\mu}\partial_{\mu} f_{\mathbf{k}} = \int dK\, k^{\nu} C[f_{\mathbf{k}}] = 0 \qquad \textbf{energy-momentum cons.}$$

Conservation laws are obtained, but we still need the solution of the Boltzmann equation $f_{\mathbf{k}}$!

The hierarchy of moments and balance equations

Moments of the single-particle distribution function and collision integral

$$N^{\mu}(t,\mathbf{x}) \equiv \int dK\, k^{\mu} f_{\mathbf{k}} = \langle k^{\mu} \rangle$$ **charge current**

$$T^{\mu\nu}(t,\mathbf{x}) \equiv \int dK\, k^{\mu} k^{\nu} f_{\mathbf{k}} = \langle k^{\mu} k^{\nu} \rangle$$ **energy-momentum tensor**

$$F^{\mu_1\cdots\mu_n}(t,\mathbf{x}) \equiv \int dK\, k^{\mu_1} \ldots k^{\mu_n} f_{\mathbf{k}} = \langle k^{\mu_1} \ldots k^{\mu_n} \rangle$$ **n-rank moment**

$$P^{\mu_1\cdots\mu_n}(t,\mathbf{x}) \equiv \int dK\, k^{\mu_1} \ldots k^{\mu_n} C\left[f_{\mathbf{k}}\right] = \langle C^{\mu_1\cdots\mu_n} \rangle$$ **n-rank production term**

where $\langle p^{\mu_1} p^{\mu_2} \ldots p^{\mu_{n-1}} p^{\mu_n} \rangle \Rightarrow \int dK\, p^{\mu_1} \ldots p^{\mu_n} f_{\mathbf{k}}$.
The mass-shell condition $k^{\mu} k_{\mu} = m^2$ gives

$$F^{\mu_1\cdots\mu_n\lambda}_{\lambda} = m^2 F^{\mu_1\cdots\mu_n}$$

$$P^{\mu_1\cdots\mu_n\lambda}_{\lambda} = m^2 P^{\mu_1\cdots\mu_n}$$

$$\partial_{\lambda} F^{\mu_1\cdots\mu_n\lambda} = P^{\mu_1\cdots\mu_n}$$ **balance of fluxes**

These balance equations are never closed but the solution of the Boltzmann equation is also a solution of the hierarchy!

Kinetic decomposition and matching

Decompose the momenta, $k^\mu = E_\mathbf{k} u^\mu + k^{\langle\mu\rangle}$, where $E_\mathbf{k} = k^\mu u_\mu$ and $k^{\langle\mu\rangle} = \Delta^\mu_\nu k^\nu$

Decompositions

$$N^\mu = \langle E_\mathbf{k} \rangle u^\mu + \langle k^{\langle\mu\rangle} \rangle$$

$$T^{\mu\nu} = \langle E_\mathbf{k}^2 \rangle u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle + \langle E_\mathbf{k} k^{\langle\mu\rangle} \rangle u^\nu + \langle E_\mathbf{k} k^{\langle\nu\rangle} \rangle u^\mu + \langle k^{\langle\mu} k^{\nu\rangle} \rangle$$

Definitions

$$n \equiv \langle E_\mathbf{k} \rangle$$

$$e \equiv \langle E_\mathbf{k}^2 \rangle$$

$$V^\mu \equiv \langle k^{\langle\mu\rangle} \rangle$$

$$p + \Pi \equiv -\frac{1}{3}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle$$

$$W^\mu \equiv \langle E_\mathbf{k} k^{\langle\mu\rangle} \rangle$$

$$\pi^{\mu\nu} \equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle$$

Local thermal equilibrium

In local thermal equilibrium $f_{\mathbf{k}} \to f_{0\mathbf{k}}$, $f_{0\mathbf{k}}(x^\mu, k^\mu) = f(\alpha, \beta, E_{\mathbf{k}})$

$$f_{0\mathbf{k}} \equiv \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathbf{k}}\right) + a\right]^{-1}$$

where $\alpha_0 = \mu_0/T_0$ is the chemical potential, $\beta_0 = 1/T_0$ is the inverse temperature, $E_{\mathbf{k}} = k^\mu u_\mu$, and $a = 0$ for Boltzmann, $a = 1$ for Fermi, $a = -1$ for Bose statistics. $f_{0\mathbf{k}}$ is not a solution of the Boltzmann eq., since $C\left[f_{0\mathbf{k}}\right] = 0$ but $k^\mu \partial_\mu f_{0\mathbf{k}} \neq 0$

$$\partial_\mu \langle k^\mu \rangle_0 \equiv \left(\partial_\mu \alpha_0\right) \langle k^\mu \rangle_0 + \left(u_\nu \partial_\mu \beta_0\right) \langle k^\mu k^\nu \rangle_0 - \left(\beta_0 \partial_\mu u_\nu\right) \langle k^\mu k^\nu \rangle_0 = 0$$

where $\langle p^{\mu_1} \dots p^{\mu_n} \rangle_0 \Rightarrow \int dK \, p^{\mu_1} \dots p^{\mu_n} f_{0\mathbf{k}}$.

Laws of equilibrium thermodynamics: Boltzmann gas (a=0)

$$\dot{n}_0 + n_0 \nabla_\mu u^\mu \equiv n_0 \dot{\alpha}_0 - e_0 \dot{\beta}_0 + \beta_0 p_0 \nabla_\mu u^\mu$$

$$n_0 = \beta_0 p_0 \qquad \textbf{(ideal gas law)}$$

$$\dot{n}_0 = n_0 \dot{\alpha}_0 - e_0 \dot{\beta}_0$$

Introducing the entropy $s_0 = (e_0 + p_0)\beta_0 - n_0 \alpha_0$ leads to $\dot{s}_0 = \beta_0 \dot{e}_0 - \alpha_0 \dot{n}_0$.

Out of equilibrium

For systems out of equilibrium $f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$

Equilibrium fields

$$
\begin{aligned}
n_0 &\equiv \langle E_{\mathbf{k}} \rangle_0 \\
e_0 &\equiv \langle E_{\mathbf{k}}^2 \rangle_0 \\
V^\mu &\equiv \langle k^{\langle\mu\rangle} \rangle_0 = 0 \\
p_0 &\equiv -\frac{1}{3} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_0 \\
\Pi &= 0\,, \\
W^\mu &\equiv \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle_0 = 0 \\
\pi^{\mu\nu} &\equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle_0 = 0
\end{aligned}
$$

Out of equilibrium fields

$$
\begin{aligned}
n &\equiv \langle E_{\mathbf{k}} \rangle \\
e &\equiv \langle E_{\mathbf{k}}^2 \rangle \\
V^\mu &\equiv \langle k^{\langle\mu\rangle} \rangle = \langle k^{\langle\mu\rangle} \rangle_\delta \\
p + \Pi &\equiv -\frac{1}{3} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle \\
\Pi &= -\frac{1}{3} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_\delta\,, \\
W^\mu &\equiv \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle = \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle_\delta \\
\pi^{\mu\nu} &\equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle = \langle k^{\langle\mu} k^{\nu\rangle} \rangle_\delta
\end{aligned}
$$

where $\langle \ldots \rangle_\delta \equiv \langle \ldots \rangle - \langle \ldots \rangle_0 = \int dK \ldots f_{\mathbf{k}} - \int dK \ldots f_{0\mathbf{k}} = \int dK \ldots \delta f_{\mathbf{k}}$

Matching: Non-equilibrium to equilibrium

$$
\begin{aligned}
\langle E_{\mathbf{k}} \rangle_\delta = 0 \quad &\Rightarrow \quad n = n_0 \\
\langle E_{\mathbf{k}}^2 \rangle_\delta = 0 \quad &\Rightarrow \quad e = e_0 \\
-\frac{1}{3} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_\delta = \Pi \quad &\Rightarrow \quad p(e, n) = p_0(e_0, n_0)
\end{aligned}
$$

Approximate solutions: Grad's method of moments

The previous definitions from kinetic theory are only useful if we can specify $f_{\mathbf{k}}$ or $\delta f_{\mathbf{k}}$

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \simeq f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

where $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}} \ll 1$ close to equilibrium.

Relativistic Grad's approach: Stewart (1972), Israel and Stewart (1977-1979)

$$\phi_{\mathbf{k}} \equiv \sum_{l=0}^{\infty} \epsilon_{(l)}^{\mu_1 \dots \mu_l} k_{\mu_1} \dots k_{\mu_l} = \epsilon_{(0)} + \epsilon_{(1)}^{\mu} k_{\mu} + \epsilon_{(2)}^{\mu\nu} k_{\mu} k_{\nu} + \dots$$

Assuming that in $\phi_{\mathbf{k}}$ only $l = 0, 1, 2$-rank tensors appear $\lambda(1)$, $\lambda^{\mu}(4)$ and $\lambda^{\mu\nu}(9)$ where $\lambda_{\mu}^{\mu} = 0$, we provide a closure for the hierarchy (truncation) and be able to determine the 14 moments of dissipative fluid dynamics!

Applying the 14-method

Non-equilibrium 14-moment ansatz

$$f_{\mathbf{k}} \equiv f_{0\mathbf{k}}\Big[1+\epsilon_{(0)}+E_{\mathbf{k}}\epsilon^{\mu}_{(1)}u_\mu+\epsilon^{\langle\mu\rangle}_{(1)}k_{\langle\mu\rangle}+E^2_{\mathbf{k}}\epsilon^{\mu\nu}_{(2)}u_\mu u_\nu+\frac{1}{3}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)\Delta_{\mu\nu}\epsilon^{\mu\nu}_{(2)}$$
$$+\ E_{\mathbf{k}}\epsilon^{\langle\mu\rangle\nu}_{(2)}k_{\langle\mu\rangle}u_\nu+E_{\mathbf{k}}\epsilon^{\langle\mu\rangle\nu}_{(2)}k_{\langle\nu\rangle}u_\mu+\epsilon^{\langle\mu\nu\rangle}_{(2)}k_{\langle\mu}k_{\nu\rangle}\Big]$$

which is basically an expansion in $E^n_{\mathbf{k}}$ and $k_{\langle\mu_1}\dots k_{\mu_n\rangle}$

Decomposition and matching

$$\epsilon_{(0)}-\alpha_0 = A_0\Pi$$
$$\epsilon^{\mu}_{(1)}u_\mu-\beta_0 = A_1\Pi$$
$$\epsilon^{\mu\nu}_{(2)}u_\mu u_\nu = A_2\Pi$$
$$\epsilon^{\langle\mu\rangle}_{(1)} = B_0 q^\mu + B_1 V^\mu$$
$$\epsilon^{\langle\mu\rangle\beta}_{(2)}u_\beta = C_0 q^\mu + C_1 V^\mu$$
$$\epsilon^{\mu\nu}_{(2)} = A_3(3u^\mu u^\nu-\Delta^{\mu\nu})\Pi+2C_0 u^{(\mu}q^{\nu)}+2C_1 u^{(\mu}V^{\nu)}+D_0\pi^{\mu\nu}$$
$$\epsilon^{\langle\mu\nu\rangle}_{(2)} = D_0\pi^{\mu\nu}$$

The coefficients $\epsilon_{(0)}$, $\epsilon^{\mu}_{(1)}$, $\epsilon^{\mu\nu}_{(2)}$ expressed in terms of dissipative fields, Π, $q^\mu(V^\mu)$, $\pi^{\mu\nu}$

The relaxation equations

Using the 14-moment ansatz, the 3rd moment, $\langle k^\mu k^\nu k^\lambda \rangle = F(e, n, p, u^\mu, \Pi, q^\mu, \pi^{\mu\nu})$, therefore, using the Boltzmann transport eq., $\partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle = \langle C^{\mu\nu} \rangle$, we get:

Israel-Stewart (1979)

$$\begin{aligned}
u_\mu u_\nu \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= u_\mu u_\nu \langle C^{\mu\nu} \rangle && \text{Bulk eq.} \\
\Delta^\alpha_\mu u_\nu \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= \Delta^\alpha_\mu u_\nu \langle C^{\mu\nu} \rangle && \text{Heat-flow eq.} \\
\Delta^{\alpha\beta}_{\mu\nu} \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= \Delta^{\alpha\beta}_{\mu\nu} \langle C^{\mu\nu} \rangle && \text{Shear viscosity eq.}
\end{aligned}$$

The linearised collision integral

$$\begin{aligned}
u_\mu u_\nu \langle C^{\mu\nu} \rangle &= C_\Pi \Pi \\
\Delta^\alpha_\mu u_\nu \langle C^{\mu\nu} \rangle &= C_q q^\alpha \\
\Delta^{\alpha\beta}_{\mu\nu} \langle C^{\mu\nu} \rangle &= C_\pi \pi^{\alpha\beta}
\end{aligned}$$

where $C_\Pi, C_q, C_\pi \sim \langle \sigma \rangle$

The "exact" relaxation equations

The original Israel-Stewart derivation neglected several terms. Later correction have been added: Muronga (2007), Betz et. al. (2009).

$$\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi &= \Pi_{NS} + \tau_{\Pi q}\, q\cdot\dot{u} - \ell_{\Pi q}\, \partial\cdot q - \zeta\, \hat{\delta}_0\, \Pi\, \theta \\
&\quad + \lambda_{\Pi q}\, q\cdot\nabla\alpha + \lambda_{\Pi\pi}\, \pi^{\mu\nu}\sigma_{\mu\nu} \\
\tau_q\, \Delta^{\mu\nu}\dot{q}_\nu + q^\mu &= q^\mu_{NS} - \tau_{q\Pi}\, \Pi\, \dot{u}^\mu - \tau_{q\pi}\, \pi^{\mu\nu}\, \dot{u}_\nu \\
&\quad + \ell_{q\Pi}\, \nabla^\mu \Pi - \ell_{q\pi}\, \Delta^{\mu\nu}\, \partial^\lambda \pi_{\nu\lambda} + \tau_q\, \omega^{\mu\nu}\, q_\nu - \frac{\kappa}{\beta}\, \hat{\delta}_1\, q^\mu\, \theta \\
&\quad - \lambda_{qq}\, \sigma^{\mu\nu}\, q_\nu + \lambda_{q\Pi}\, \Pi\, \nabla^\mu\alpha + \lambda_{q\pi}\, \pi^{\mu\nu}\, \nabla_\nu\alpha \\
\tau_\pi\, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{NS} + 2\,\tau_{\pi q}\, q^{<\mu}\dot{u}^{\nu>} \\
&\quad + 2\,\ell_{\pi q}\, \nabla^{<\mu} q^{\nu>} + 2\,\tau_\pi\, \pi_\lambda{}^{<\mu}\omega^{\nu>\lambda} - 2\,\eta\, \hat{\delta}_2\, \pi^{\mu\nu}\, \theta \\
&\quad - 2\,\tau_\pi\, \pi_\lambda{}^{<\mu}\sigma^{\nu>\lambda} - 2\,\lambda_{\pi q}\, q^{<\mu}\nabla^{\nu>}\alpha + 2\,\lambda_{\pi\Pi}\, \Pi\, \sigma^{\mu\nu}
\end{aligned}$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341
W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341
A. Muronga, PRC 76 (2007) 014909; A. Muronga, ibid.
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Further corrections

Take into account 2nd order corrections in the collision integral, i.e., $C[f_0 + \delta f] \sim \delta f + (\delta f)^2$

Full collision integral: Rischke et.al. (2010)

$$
\begin{aligned}
u_\mu u_\nu \langle C^{\mu\nu} \rangle &\sim C_\Pi \Pi + A_0 \Pi^2 + A_1 V^\mu q_\mu + A_2 q^\mu q_\mu + A_3 \pi^{\mu\nu} \pi_{\mu\nu} \\
\Delta^\alpha_\mu u_\nu \langle C^{\mu\nu} \rangle &\sim C_q q^\alpha + B_0 \Pi q^\alpha + B_1 \Pi V^\alpha + B_2 q_\mu \pi^{\mu\alpha} + B_3 V_\mu \pi^{\mu\alpha} \\
\Delta^{\alpha\beta}_{\mu\nu} \langle C^{\mu\nu} \rangle &\sim C_\pi \pi^{\alpha\beta} + C_0 \Pi \pi^{\alpha\beta} + C_1 q^{\langle\alpha} V^{\beta\rangle} + C_2 q^{\langle\alpha} q^{\beta\rangle} + C_3 \pi^\alpha_\lambda \pi^{\beta\lambda}
\end{aligned}
$$

some terms also by Moore (2009)

Problem with the traditional moment method

$$
\begin{aligned}
u_{\mu_1} \ldots u_{\mu_n} \partial_\lambda \langle k^{\mu_1} \ldots k^{\mu_n} k^\lambda \rangle &= u_{\mu_1} \ldots u_{\mu_n} \langle C^{\mu_1 \cdots \mu_n} \rangle \\
\Delta^\alpha_{\mu_1} u_{\mu_2} \ldots u_{\mu_n} \partial_\lambda \langle k^{\mu_1} \ldots k^{\mu_n} k^\lambda \rangle &= \Delta^\alpha_{\mu_1} u_{\mu_2} \ldots u_{\mu_n} \langle C^{\mu_1 \cdots \mu_n} \rangle \\
\Delta^{\alpha\beta}_{\mu_1\mu_2} u_{\mu_3} \ldots u_{\mu_n} \partial_\lambda \langle k^{\mu_1} \ldots k^{\mu_n} k^\lambda \rangle &= \Delta^{\alpha\beta}_{\mu_1\mu_2} u_{\mu_3} \ldots u_{\mu_n} \langle C^{\mu_1 \cdots \mu_n} \rangle
\end{aligned}
$$

Parts of higher order moments may contribute for $\Pi, q^\mu, \pi^{\mu\nu}$, so why stop ?

New method: Denicol et.al. (2010)

Rel. orthogonal polynomials: Stewart (1976), GeGroot (1980), Denicol et. al. (2010)

$$\phi_{\mathbf{k}} \equiv \sum_{l=0}^{\infty} \lambda_{(l)}^{\langle\mu_1\cdots\mu_l\rangle} k_{\langle\mu_1}\cdots k_{\mu_l\rangle} = \lambda_{(0)} + \lambda_{(1)}^{\langle\mu\rangle} k_{\langle\mu\rangle} + \lambda_{(2)}^{\langle\mu\nu\rangle} k_{\langle\mu} k_{\nu\rangle} + \ldots$$

$$\lambda_{(l)}^{\langle\mu_1\cdots\mu_l\rangle} = \sum_{n=0}^{\infty} c_{(n)}^{\langle\mu_1\cdots\mu_n\rangle} \sum_{r=0}^{n} a_{(nr)}^{(l)} E_{\mathbf{k}}^{r}$$

Orthogonal and energy dependent

The 14-moment closure: $c_{(0,1,2)} \sim \Pi$, $c_{(0)}^{\langle\mu\rangle} \sim V^\mu$, $c_{(1)}^{\langle\mu\rangle} \sim V^\mu + q^\mu$ and $c_{(0)}^{\langle\mu\nu\rangle} \sim \pi^{\mu\nu}$

Generalized irreducible moments and exact equations of motion: Denicol et. al. (2010)

$$\rho_{(r)}^{\mu_1\cdots\mu_l} = \int dK\, E_{\mathbf{k}}^{r} k^{\langle\mu_1}\cdots k^{\mu_l\rangle} \delta f_{\mathbf{k}}$$

$$\dot{\rho}_{(r)} = \frac{d}{d\tau} \int dK\, (E_{\mathbf{k}})^r\, \delta f_{\mathbf{k}},$$

$$\dot{\rho}_{(r)}^{\langle\mu\rangle} = \Delta^{\mu}_{\nu} \frac{d}{d\tau} \int dK\, (E_{\mathbf{k}})^r\, k^{\langle\nu\rangle} \delta f_{\mathbf{k}},$$

$$\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \frac{d}{d\tau} \int dK\, (E_{\mathbf{k}})^r\, k^{\langle\alpha} k^{\beta\rangle} \delta f_{\mathbf{k}}$$

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C[f]$$

Conclusions

- The 14-moments closure leads to the equations of motion for the dissipative fields consistent with the phenomenological 2nd order theories of fluid dynamics
- We have taken into account all 2nd order terms some of which have been neglected in the early days (Betz et. al. 2009)
- 2nd order terms form the collision integral (Rischke et. al. 2010), now the 2nd order theory of Israel and Stewart can be done even better
- Different methods and assumptions lead to formally the same equations of motions (formal consistency) but the coefficients are different (Denicol et. al. 2010) ($\int dK \rightarrow \int dK E^r_{\mathbf{k}}$)
- Now, we have an even large family of dissipative fluid dynamical models!
- Important! All 2nd order theories relax to the Navier-Stokes form, hence all novelties are not expected to be observed on large time-scales, only the important ones ζ, κ and nowadays favourite η
- Remember, we only calculate dilute gases from the Boltzmann equation