Influence of the Polyakov loop on the chiral phase transition in the two flavour chiral quark model

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- Motivation
- $\mathbb{Z}_3$ symmetry, the Polyakov loop effective potential
- Coupling of the Polyakov loop to the fields of the chiral quark model
- Formalism, approximations and the parametrization of the model
- The $\mu_B - T$ phase diagram in the chiral limit and
  for the physical pion mass obtained in arXiv:1006.0212 [hep-ph]
- Conclusions and outlook

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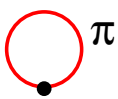
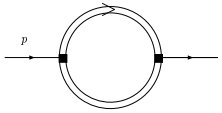
Technical motivation

1.) Going beyond some of the approximations used to solve the χQM

– in a previous work a local approximation to the self-consistent pion propagator obtained with a large-N_f technique was used in the chiral limit

Jakovác et al., PLB582 (2004) 179

the pion propagator was parametrized as $G_\pi(p) = \dfrac{i}{p^2 - M^2}$ with M^2 determined from the gap-equation

$$m^2 + \frac{\lambda}{6}v^2 + \frac{\lambda}{6}$$  $+$  $\Big|_{p^2=M^2} = M^2$

– in B.-J. Schaefer et al., PRD 76 (2007) 074023 a quasi-particle approximation was used, that is the vacuum fluctuations were omitted

– recently fluctuations were taken into account using FRG techniques in

V. Skokov, et al., PRC 82, 015206 (2010) T. K. Herbst, et al., arXiv:1008.0081 [hep-ph]

2.) study the renormalization, that is find a way to determine the counterterms

3.) include the effect of the Polyakov loop in our large-N_f treatment along the lines of B.-J. Schaefer et al., PRD 76 (2007) 074023

Hansen et al., PRD 75 (2007) 065004

Physical motivation

In QCD at finite T deconfinement phase transition and χ symmetry restoration are the main phenomena to be understood with lattice studies and analytically using DSE's or FRG methods.

The interplay between chiral and deconfinement transitions which occur in the cleanest way in opposite sectors of the QCD is a subject of continuous interest.

- $m_q = 0$: chiral condensate $\langle \bar{\psi}\psi \rangle$ acts as OP for the χ phase transition
  low T: $\langle \bar{\psi}\psi \rangle \neq 0$ high T: $\langle \bar{\psi}\psi \rangle = 0$

- $m \to \infty$: as OP the thermal expectation value of the traced Polyakov loop

$\Phi(\vec{x}) = \dfrac{\mathrm{Tr}_c L(\vec{x})}{N_c}$ and conjugate $\bar{\Phi}(\vec{x}) = \dfrac{\mathrm{Tr}_c \bar{L}(\vec{x})}{N_c}$ $\quad L(x) = \mathcal{P} \exp\left[i \int_0^\beta d\tau A_4(\vec{x}, \tau) \right]$

signals center symmetry $\mathbb{Z}_3$ breaking at the deconfinement transition

low T: confined phase, $\langle \Phi(\vec{x}) \rangle, \langle \bar{\Phi}(\vec{x}) \rangle = 0$

high T: deconfined phase, $\langle \Phi(\vec{x}) \rangle, \langle \bar{\Phi}(\vec{x}) \rangle \neq 0$

Physical interpretation of the Polyakov loop is given in terms of the free energy of static quarks McLerran & Svetitsky, PRD 24, (1981) 450

Thermal expectation value of the (charge conjugate) Polyakov loop:

$$\langle \Phi(\boldsymbol{x}) \rangle = \frac{Z_q}{Z} = e^{-\beta(F_q - F_0)} \qquad \left\langle \bar{\Phi}(\boldsymbol{x}) \right\rangle = \frac{Z_{\bar{q}}}{Z} = e^{-\beta(F_{\bar{q}} - F_0)}$$

$\langle \Phi(\boldsymbol{x}) \rangle$ ($\left\langle \bar{\Phi}(\boldsymbol{x}) \right\rangle$) measures the free energy of the external quark (antiquark) relative to the case without it.

For a static quark-antiquark pair:

$$|\langle \Phi(0) \rangle|^2 = \lim_{|\boldsymbol{r}| \to \infty} \left\langle \Phi(0) \bar{\Phi}(\boldsymbol{r}) \right\rangle = \exp(-\Delta F_{q\bar{q}}(r \to \infty)/T)$$

$\Delta F_{q\bar{q}}$ is the difference of the free energies of the quark-gluon plasma with and without the static quark-antiquark pair

Including dynamical fermions, the fermionic determinant breaks the $\mathbb{Z}_3$ symmetry explicitly.

However, the transition from the low to the high T phase is characterized by a substantial increase of the expectation value of the Polyakov loop.

F. Karsch, E. Laermann PRD 50 (1994) 6954

Connection between $\langle\bar{\Psi}\Psi\rangle$ and $\langle\Phi\rangle$ provided by the spectral density $\rho(\lambda)$ of the Dirac operator.

Casher's argument: in the vacuum confinement implies χ symmetry breaking.

At $T = \mu_q = 0$ the spectral density of the Dirac operator in the deep IR is proportional to the quark condensate (Banks-Casher relation) $\rho(0) = -\langle\bar{\Psi}\Psi\rangle/\pi$

At $\mu_q = 0$ and $T \neq 0$ the infrared part of $\rho(\lambda)$ undergoes a pronounced change as one crosses from the confining to the deconfined phase.

$\langle\Phi\rangle$ can be expressed as a spectral sum of eigenvalues and eigenvectors of the Dirac operator with different BC. C. Gattringer, Phys. Rev. Lett. 97, 032003 (2006)
The main contribution of arg $\langle\Phi\rangle$ is from the IR end of $\rho(\lambda)$.

It is generally true for $N_c = 2$ and $N_c = 3$ that above T_c the fermion determinant favors the sector where the Polyakov loop lies along the positive real axis.
G. Edwards et al., PRD 61, 074504 (2000) T. G. Kovács, PoS LATTICE2008, 198 (2008)

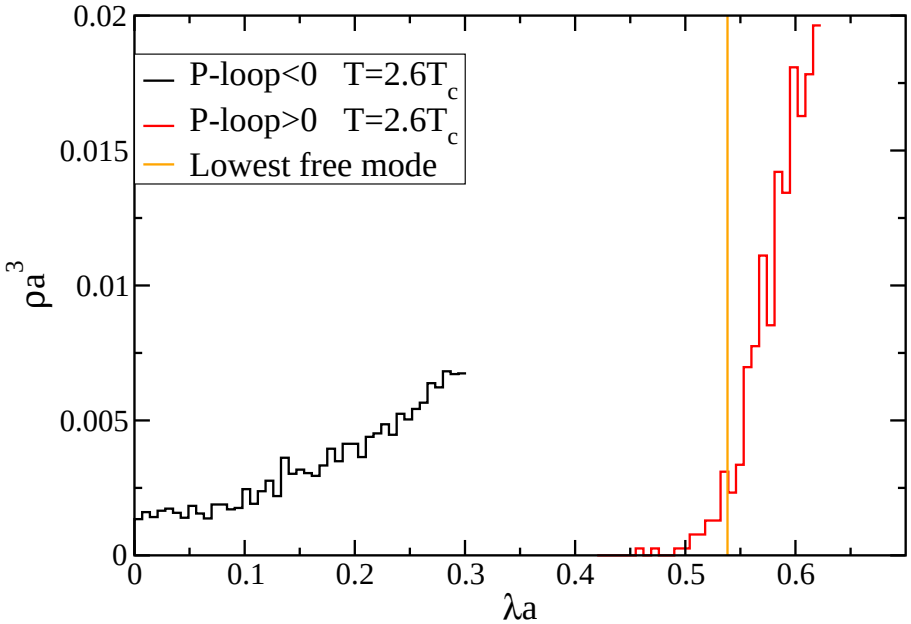
For configurations in which arg $\langle\Phi\rangle = 0$ the chiral symmetry is restored:
a sizable gap develops in the spectral density of the Dirac operator $\Longrightarrow \langle\bar{\Psi}\Psi\rangle = 0$

For configurations in which arg $\langle\Phi\rangle \neq 0$ the chiral symmetry is not restored.
S. Chandrasekharan, N. Christ, Nucl. Phys. Proc. Suppl. 47 (1996) 527
M. A. Stephanov, Phys. Lett. B375 (1996) 249

$SU(2)_c$ lattice simulation at $\mu_q = 0$ is in line with Casher's argument

In different $\mathbb{Z}_2$ sectors the eigenvalue problem for the Dirac equation is different because of different boundary condition, that is PBC or APBC.

T. G. Kovács, PoS LATTICE2008, 198 (2008)



under twisted gauge transformations $g(\tau+\beta, \boldsymbol{x})=hg(\tau, \boldsymbol{x})$ with $h \in \mathbb{Z}_3$

$^g\Phi(\boldsymbol{x})=z\,\Phi(\boldsymbol{x}), \quad ^g\Psi(\tau+\beta, \boldsymbol{x})= - z\,^g\Psi(\tau, \boldsymbol{x})$

for $SU(2)_c$ pure gauge $\mathbb{Z}_2=\{-1, 1\}$ changing from APBC to PBC for fermions $\Longleftrightarrow$ flipping the sign of the Polyakov loop

Same in $SU(3)_c$: arg $\langle\Phi\rangle = 0$ gives the larges possible twist to the fermions in the time direction, largest first eigenvalue $\Longrightarrow$ chiral symmetry restored in this case.

At finite density Casher's argument could fail. L. Ya. Glozman, PRD 80, 037701 (2009)

$\Longrightarrow$ possibility for the existence of a dense phase in which at a given temperature the chiral symmetry is restored while quarks remain confined.

New phase of the QCD at finite T and μ_q, the quarkyonic phase, suggested based on a large N_c analysis. L. McLerran and R. D. Pisarski, NPA796, 83 (2007)

Polyakov Gauge

Polyakov loop operator and its conjugate

$$L(\vec{x}) = \mathcal{P}\exp\left[i\int_0^\beta d\tau A_4(\tau,\vec{x})\right], \qquad L^\dagger(\vec{x}) = \mathcal{P}\exp\left[-i\int_0^\beta d\tau A_4^*(\tau,\vec{x})\right]$$

Polyakov gauge: the temporal component of the gauge field is time independent and can be gauge rotated to a diagonal form in the color space

$$A_{4,d}(\vec{x}) = \phi_3(\vec{x})\lambda_3 + \phi_8(\vec{x})\lambda_8$$

λ_3, λ_8 are the two diagonal Gell-Mann matrices.

In this gauge the Polyakov loop operator is

$$L(\vec{x}) = \text{diag}(e^{i\beta\phi_+(\vec{x})}, e^{i\beta\phi_-(\vec{x})}, e^{-i\beta(\phi_+(\vec{x})+\phi_-(\vec{x}))})$$

where $\phi_\pm(\vec{x}) = \pm\phi_3(\vec{x}) + \phi_8(\vec{x})/\sqrt{3}$

In a mean field treatment A_4, $\Phi = \frac{1}{N_c}\langle \text{Tr}_c L\rangle$, $\bar{\Phi} = \frac{1}{N_c}\langle \text{Tr}_c L^\dagger\rangle$ are $\vec{x}$-independent.

Mean field potential for the Polyakov loop

Idea: construction of a simple effective theory with desired symmetry properties for the study of deconfinement phase transition. R.D.Pisarski, PRD 62, 111501

I.) Simple polynomial potential invariant under $\mathbb{Z}_3$ and charge conjugation:

$$\frac{\mathcal{U}_{\text{poly}}\left(\Phi,\bar{\Phi}\right)}{T^4} = -\frac{b_2\left(T\right)}{2}\bar{\Phi}\Phi - \frac{b_3}{6}\left(\Phi^3+\bar{\Phi}^3\right) + \frac{b_4}{4}\left(\bar{\Phi}\Phi\right)^2$$

with $$b_2\left(T\right) = a_0 + a_1\frac{T_0}{T} + a_2\frac{T_0^2}{T^2} + a_3\frac{T_0^3}{T^3}$$

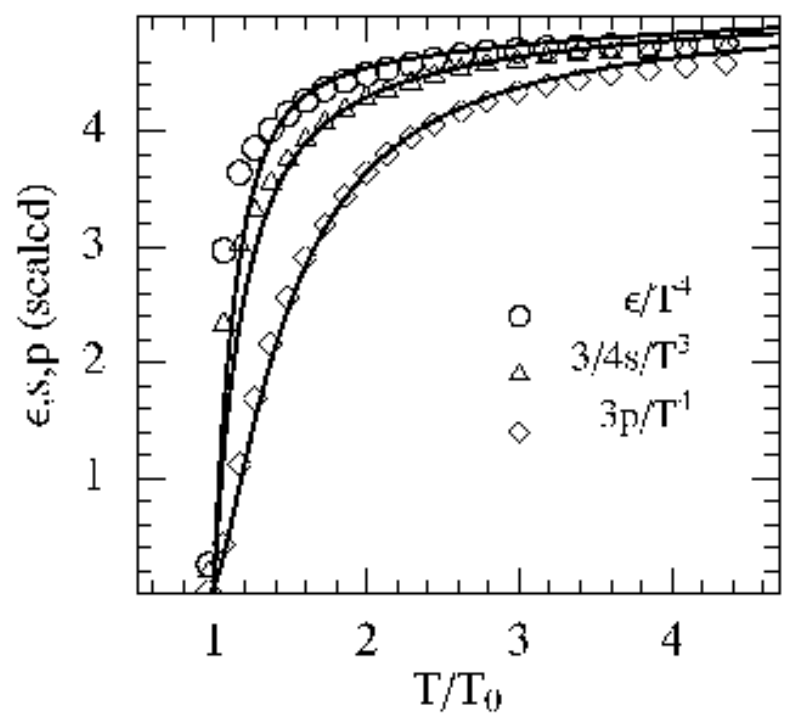
$\mathcal{U}\left(\Phi,\bar{\Phi}\right)$ models the free energy of a pure gauge theory.

coefficients determined such to reproduce $SU(3)$ pure-gauge lattice data from G. Boyd et al., NPB469, 419 (1996) for thermodynamics quantities $p = -\mathcal{U}$, $s = -\partial\mathcal{U}/\partial T$ and $\epsilon = -p + Ts$

$T_0 = 270$ MeV is the temperature of the 1st order phase transition

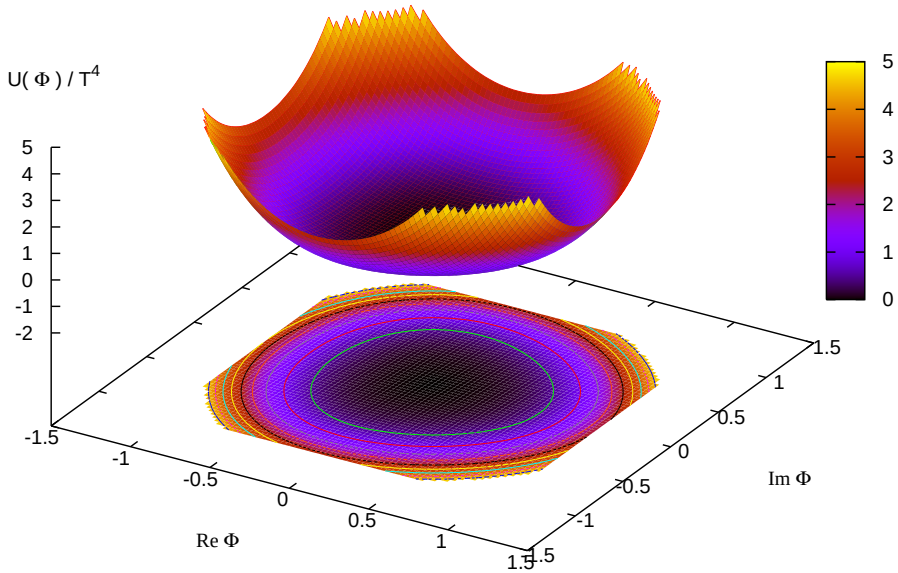
a_0	a_1	a_2	a_3	b_3	b_4
6.75	-1.95	2.625	-7.44	0.75	7.5

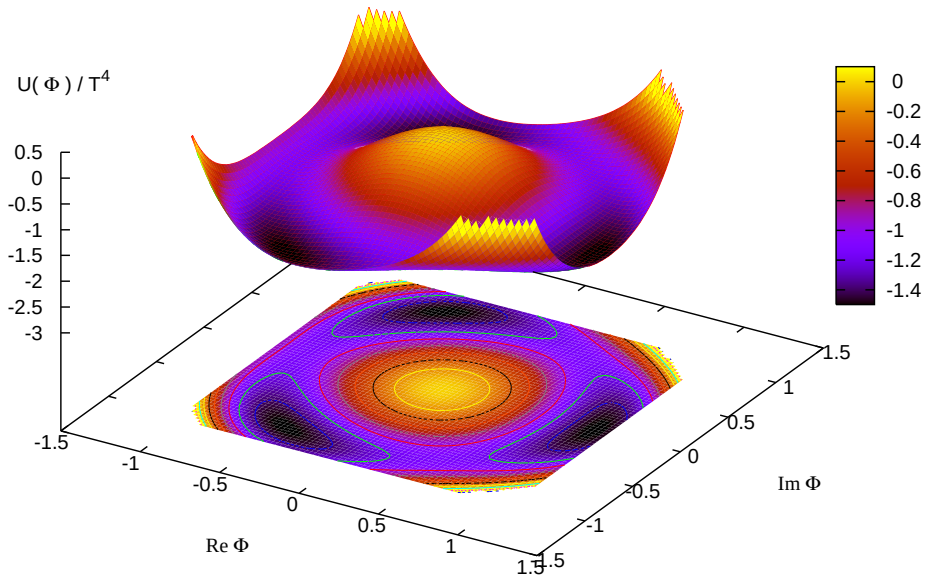
C. Ratti et al., PRD 73, 014019 (2006)



T=260 MeV$< T_0$ "Color confinement"
$\langle\Phi\rangle$ =0 $\longrightarrow$ no breaking of $\mathbb{Z}_3$
one minimum

T=1 GeV$> T_0$ "Color deconfinement"
$\langle\Phi\rangle \neq 0$ $\longrightarrow$ spontaneous breaking of $\mathbb{Z}_3$
minima at $0, 2\pi/3, -2\pi/3$
one of them spontaneously selected





from H. Hansen et al., PRD 75, 065004 (2007)

II.) Logarithmic potential coming from the $SU(3)$ Haar measure of group integration

K. Fukushima, Phys. Lett. B591, 277 (2004)

a) $$\frac{\mathcal{U}_{\text{log}}(\Phi,\bar{\Phi})}{T^4} = -\frac{1}{2}a(T)\Phi\bar{\Phi} + b(T)\ln\left[1 - 6\Phi\bar{\Phi} + 4\left(\Phi^3 + \bar{\Phi}^3\right) - 3\left(\Phi\bar{\Phi}\right)^2\right]$$

with $$a(T) = a_0 + a_1\frac{T_0}{T} + a_2\frac{T_0^2}{T^2}, \qquad b(T) = b_3\frac{T_0^3}{T^3}$$

the parameters are fitted to the pure gauge lattice data

a_0	a_1	a_2	b_3
3.51	-2.47	15.2	-1.75

C. Ratti, et al., Eur. Phys. J. C 49, 213 (2007)

b) $$\mathcal{U}_{\text{Fuku}}(\Phi,\bar{\Phi}) = -b\left[54e^{-a/T}\Phi\bar{\Phi} + \ln\left[1 - 6\Phi\bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi\bar{\Phi})^2\right]\right]$$

K. Fukushima, Prog. Theor. Phys. 153, 204 (2004)

a controls the temp. of the deconf. phase transition in the pure gauge theory

b controls the weight of gluonic effects in the transition

with $a = 664$ MeV, $b = (196.2\text{MeV})^3$ 1st order transition at $T \simeq 270$ MeV

K. Fukushima, PRD 77, 114028 (2008)

B.-J. Schaefer, et al., PRD 76, 074023 (2007)

The presence of dynamical quarks modifies the expansion coefficients.

The N_f and μ_q-dependence of T_0 estimated in B.-J. Schaefer et al., PRD 76, 074023

N_f	0	1	2	2+1	3
T_0[MeV]	270	240	208	187	178

estimation using renormalization group arguments:

$$T_0(\mu_q, N_f) = T_\tau \exp(-1/(\alpha_0 b(\mu_q)))$$

For further details see also T. K. Herbst, et al., arXiv:1008.0081 [hep-ph]

Chiral effective models with the Polyakov loop

Low energy effective chiral models have the same global symmetries as the QCD but know nothing about confinement. Some knowledge about confinement can be incorporated through the effective potential for the Polyakov loop $\boldsymbol{U(\Phi,\bar{\Phi};T)}$.

Usually, in models with fermions the Polyakov loop is coupled only to fermions:

Model	*Grand (thermodynamic) potential*
PNJL	$\Omega(\langle\bar{q}q\rangle,\Phi,\bar{\Phi};T,\mu)=\Omega_{\text{cond}}(\langle\bar{q}q\rangle;T)+\Omega_{\text{vac}}(\langle\bar{q}q\rangle)+\Omega_{q\bar{q}}(\langle\bar{q}q\rangle,\Phi,\bar{\Phi};T,\mu)+\boldsymbol{U(\Phi,\bar{\Phi};T)}$
	condensation & zero-point energy
PCQM	$\Omega(v,\Phi,\bar{\Phi};T,\mu)=\Omega_{\text{meson}}(v;T)+\Omega_{q\bar{q}}(v,\Phi,\bar{\Phi};T,\mu)+\boldsymbol{U(\Phi,\bar{\Phi};T)}$

Coupling obtained by considering propagation of fermions on spatially constant temporal background field ($A_\mu = \delta^4_\mu A_4$): $\partial\!\!\!/ \longrightarrow D\!\!\!\!/ = \gamma_\mu\partial^\mu - \gamma_0 A_4(\Phi)$

- This mimics the interaction of gluon fields with quarks.
- The effect is like having an imaginary chemical potential for quarks:

$$D^{-1}(p) = p\!\!\!/ + gv + \gamma_0(\mu - iA_4)$$

In models without fermions (LσM) meson-Polyakov loop interaction terms are given in A. Mocsy, F. Sannino & K. Tuominen, JHEP 0403 (2004) 044

E. Fraga & A. Mocsy, Braz. J. Phys. 37, 281 (2007)

Chiral constituent quark model

large N treatment of the $SU(2)_L \times SU(2)_R$ linear σ model $\qquad 2 \to N_f = \sqrt{N}$

$$L[\sigma, \pi^a, \psi] = L_M[\sigma, \pi^a] + L_F[\sigma, \pi^a, \psi] + \delta L_{ct}[\sigma, \pi^a, \psi],$$

broken symmetry phase: $\sigma(x) \to \sqrt{N} v + \sigma(x), \qquad m_q = gv, \qquad v(T=0) = f_\pi/2$

$$\begin{aligned}
L_M &= -\left[\frac{\lambda}{24} v^4 + \frac{1}{2} m^2 v^2\right] N - \left[\frac{\lambda}{6} v^3 + m^2 v + h\right] \sigma \sqrt{N} + \\
&+ \frac{1}{2}\left[(\partial^\mu \sigma)^2 + (\partial^\mu \pi^a)^2\right] - \frac{1}{2}\left[m^2 + \frac{\lambda}{2} v^2\right] \sigma^2 - \frac{1}{2}\left[m^2 + \frac{\lambda}{6} v^2\right] \pi^a \pi^a \\
&- \frac{\lambda}{6\sqrt{N}} v \left[\sigma^3 + \sigma \pi^a \pi^a\right] - \frac{\lambda}{24 N}\left[\sigma^2 + \pi^a \pi^a\right]^2 \\
L_F &= \bar{\psi}(x) \left[(i\partial^\mu + \delta^{\mu 0} A_0)\gamma_\mu - m_q - \frac{g}{\sqrt{N}}\left(\sigma(x) + i\sqrt{2N_f}\gamma_5 T^a \pi^a(x)\right)\right] \psi(x)
\end{aligned}$$

explicit symmetry breaking term h: $\mathcal{L} \to \mathcal{L} + \sqrt{N} h v$

fermions propagate on a constant A_0 background

$m_q = gv$ required to stay finite as $\boldsymbol{N_f = \sqrt{N} \to \infty}$

fermion contribution is $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ which precedes the $\mathcal{O}(N)$ NLO meson effects

Formalism

The grand partition function Z and the grand (thermodynamic) potential $\Omega(T,\mu_B)$

$$Z = \mathrm{Tr}\Big\{\exp\Big[-\beta\big(H_0(A_4) + H_{\mathrm{int}} - \mu_B Q_B\big)\Big]\Big\} = e^{-\beta\Omega}$$

$$H_0(A_4) = H_0 + \int d^3x \left[iu_i^\dagger(x)A_{4,ij}u_j(x) + id_i^\dagger(x)A_{4,ij}d_j(x)\right]$$

H_0 the free Hamiltonian at vanishing A_4

μ_B: baryon chemical potential $\quad Q_B = \frac{1}{3}\sum_{i=1}^{3}\underbrace{\big(N_{u,i}+N_{d,i}-N_{\bar{u},i}-N_{\bar{d},i}\big)}_{N_{q,i}}$: conserved charge

Since $A_4 = \mathrm{diag}(\phi_+,\phi_-,-(\phi_+ + \phi_-))$ and $N_{u,i} = \int d^3x\big(u_i^\dagger u_i + d_i^\dagger d_i\big)$, one can combine $H_0(A_4)$ and $\mu_B Q_B$ by introducing color-dependent chemical potential

$$\mu_{1,2} = \mu_q - i\phi_\pm, \quad \mu_3 = \mu_q + i(\phi_+ + \phi_-), \qquad \text{with} \quad \mu_q = \frac{\mu_B}{3}$$

C. P. Korthals Altes, et al., Phys. Rev. D 61, 056007 (2000)

Introducing $\mathcal{H} = H_0 - \sum_{i=1}^{3}\mu_i N_{q,i}$:

$$Z = e^{-\beta\Omega_0}\frac{\int\big[\mathcal{D}\Psi\big]\Big\{e^{-\beta\mathcal{H}}\mathcal{P}\exp\Big[-\int_0^\beta d\tau H_{\mathrm{int}}(\tau)\Big]\Big\}}{\int\big[\mathcal{D}\Psi\big]e^{-\beta\mathcal{H}}}, \qquad e^{-\beta\Omega_0} = \int\big[\mathcal{D}\Psi\big]e^{-\beta\mathcal{H}}$$

Grand potential in the Φ-derivable approximation at $\mathcal{O}(1/\sqrt{N})$:

J. M. Luttinger & J.C. Ward, Phys. Rev. 118 (1960) 1417

$$\beta\Omega[G_\pi, G_\sigma, G, v, \Phi, \bar{\Phi}] = U(\Phi, \bar{\Phi}) + \frac{N}{2} m^2 v^2 + N\frac{\lambda}{24} v^4 - N h v$$
$$-(N-1)\frac{i}{2}\int_k \left[\ln G_\pi^{-1}(k) + D_\pi^{-1}(k) G_\pi(k)\right] - \frac{i}{2}\int_k \left[\ln G_\sigma^{-1}(k) + D_\sigma^{-1}(k) G_\sigma(k)\right]$$
$$+\sqrt{N} i\, \mathrm{Tr}_{D,c} \int_k \left[\ln G^{-1}(k) + D^{-1}(k) G(k)\right] + \Gamma_{\text{2PI}}[G_\pi, G_\sigma, G, v, \Phi, \bar{\Phi}] + \text{c.t.}\,,$$

tree-level propagators D_π, D_σ, D

$$\Pi(k) = -i\int_p G_\pi(p) G_\pi(k+p)$$

$$\Gamma_{\text{2PI}} = N\frac{\lambda}{24}\left[\int_k G_\pi(k)\right]^2 + \frac{\lambda}{12}\int_k G_\pi(k)\int_p G_\sigma(p)$$
$$-\frac{\lambda}{6} v^2 \int_k G_\sigma(k) + \frac{\lambda}{6} v^2 \int_k \frac{G_\sigma(k)}{1 - \lambda\Pi(k)/6} - \frac{\lambda}{12} i \int_k \Pi(k) - \frac{i}{2}\int_k \ln\left[1 - \frac{\lambda}{6}\Pi(k)\right]$$
$$-\sqrt{N}\frac{g^2}{2} i\,\mathrm{Tr}_{D,c}\int_k\int_p \gamma_5 G(k)\gamma_5 G(k+p) G_\pi(p) + \frac{g^2}{2\sqrt{N}} i\,\mathrm{Tr}_{D,c}\int_k\int_p G(k) G(k+p) G_\sigma(p)$$

second line $\longrightarrow$

Equations obtained from stationary conditions:

$$\frac{\delta\Omega}{\delta G} = \frac{\delta\Omega}{\delta G_\pi} = \frac{\delta\Omega}{\delta G_\sigma} = \frac{\delta\Omega}{\delta v} = \frac{\delta\Omega}{\delta\Phi} = \frac{\delta\Omega}{\delta\bar{\Phi}} = 0$$

The fermion contribution is kept only at LO in the large-N_f expansion.

– LO fermion contribution is $\mathcal{O}(N^0)$ in:

$$iG^{-1}(K) = iD^{-1}(K) - ig^2 \int_P \gamma_5 G(P)\gamma_5 G_\pi(P-K)$$

– LO fermion contribution is $\mathcal{O}(1/\sqrt{N})$ in:

$$\begin{aligned}
iG_\pi^{-1}(K) &= iD_\pi^{-1}(K) - \frac{\lambda}{6}\int_P G_\pi(P) + \frac{ig^2}{\sqrt{N}}\mathrm{Tr}_{\mathrm{D,c}}\int_P \gamma_5 G(P)\gamma_5 G(K+P) \\
0 &= Nv\left[m^2 + \frac{\lambda}{6}v^2 + \frac{\lambda}{6}\int_K G_\pi(K) - \frac{g}{v\sqrt{N}}\mathrm{Tr}_{\mathrm{D,c}}\int_K G(K) - \frac{h}{v}\right] \\
iG_\sigma^{-1}(p) &= iD_\sigma^{-1}(p) + \frac{\lambda v^2}{3} - \frac{\lambda}{6}\int_k G_\pi(k) - \frac{\lambda v^2}{3}\frac{1}{1-\lambda\Pi(p)/6} \\
&\quad - \frac{ig^2}{\sqrt{N}}\mathrm{Tr}_{D,c}\int_k G(k)G(k+p)
\end{aligned}$$

– LO fermion contribution is $\mathcal{O}(\sqrt{N})$ in: $\quad \frac{\delta\Omega}{\delta\Phi} = \frac{\delta\Omega}{\delta\bar{\Phi}} = 0$

Approximations

1.) fermions are not treated self-consistently, the equation $\dfrac{\delta\Omega}{\delta G} = 0$ is not used

in the remaining five equations ones uses $G(K) \longrightarrow D(K) = \dfrac{i}{\not{p} - gv}$

2.) Two types of approximations are considered for the pion propagator $G_\pi(p)$:

- Local approximation $G_{\pi,l}(p) = \dfrac{i}{p^2 - M^2}$

– pole-mass M^2 determined from $G^{-1}_{\pi,l}(p_0^2 = M^2, \mathbf{p} = 0) = 0$ via the gap-equation

$$M^2 = m^2 + \frac{\lambda}{6}\left(v^2 + T_F(M)\right) - \frac{4g^2 N_c}{\sqrt{N}}\tilde{T}_F(m_q) + \frac{2g^2 N_c}{\sqrt{N}} M^2 \tilde{I}_F(M, \mathbf{0}; m_q).$$

– M^2 determined from $M^2 = -iG^{-1}_{\pi,l}(p = 0)$ through the gap-equation

$$M^2 = m^2 + \frac{\lambda}{6}\left(v^2 + T_F(M)\right) - 4\frac{g^2 N_c}{\sqrt{N}}\tilde{T}_F(m_q).$$

The two definitions of M^2 coincide in the chiral limit $h = 0$, where $M^2 = 0$.
Due to their self-consistent nature of the gap-equation a series containing all orders of $1/\sqrt{N}$ is in fact resummed.

- non-local approximation derived using the $1/\sqrt{N}$ expansion in $G_\pi(p)$

– The pion propagator is expanded in its non-local part only.

After exploiting the EoS:

$$G_\pi(p) = \frac{i}{p^2 - \frac{h}{v} - \frac{2g^2 N_c}{\sqrt{N}} p^2 \tilde{I}_F(p; m_q)} = \frac{i}{p^2 - \frac{h}{v}} + \frac{2g^2 N_c}{\sqrt{N}} \frac{i p^2 \tilde{I}_F(p; m_q)}{\left(p^2 - \frac{h}{v}\right)^2} + \mathcal{O}\left(\frac{1}{N}\right).$$

Using this form of the pion propagator in the EoS:

$$m^2 + \frac{\lambda}{6}\left(v^2 + T_F(M)\right) + \frac{2g^2 N_c}{\sqrt{N}} J_F(M, m_q) - \frac{4g^2 N_c}{\sqrt{N}} \tilde{T}_F(m_q) = \frac{h}{v},$$

$$M^2 = \frac{h}{v} \qquad\qquad J(M, m_q) = -i \int_p G^2_{\pi,l}(p) p^2 \tilde{I}_F(p; m_q)$$

When solving for v this approximation resums infinitely many orders in $1/\sqrt{N}$.

– The pion propagator is expanded in both its local and non-local parts.

$$iG_\pi^{-1}(k) = k^2 - M^2 - \frac{2g^2N_c}{\sqrt{N}}\,k^2\tilde{I}(k;m_q), \qquad M^2 = m^2 + \frac{\lambda}{6}v^2 + \frac{\lambda}{6}\int_p G_\pi(p) - \frac{4g^2N_c}{\sqrt{N}}\tilde{T}(m_q)$$

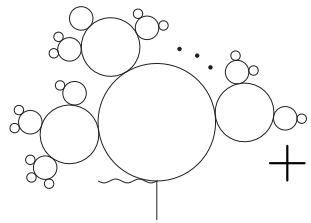
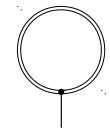
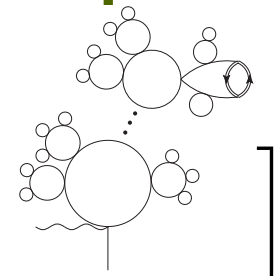
writing $\quad M^2 = M^2_{\text{LO}} + \frac{1}{\sqrt{N}}M^2_{\text{NLO}} \quad$ one has:

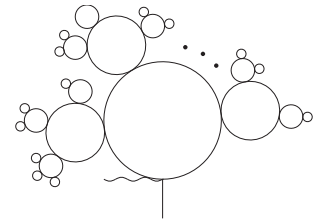
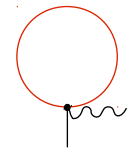
$$\begin{aligned} G_\pi(p) &= \frac{i}{p^2 - M^2_{\text{LO}} - \frac{1}{\sqrt{N}}M^2_{\text{NLO}} - \frac{2g^2N_c}{\sqrt{N}}\,p^2\tilde{I}(p;m_q)} \\ &= \frac{i}{p^2 - M^2_{\text{LO}}}\left[1 + \frac{M^2_{\text{NLO}}}{\sqrt{N}}\frac{1}{p^2 - M^2_{\text{LO}}} + \frac{2g^2N_c}{\sqrt{N}}\frac{p^2\tilde{I}(p;m_q)}{p^2 - M^2_{\text{LO}}}\right] + \mathcal{O}\left(\frac{1}{N}\right) \end{aligned}$$

$$\Longrightarrow \qquad G_\pi(p) = G_{\text{LO}}(p) - i\frac{M^2_{\text{NLO}}}{\sqrt{N}}G^2_{\text{LO}}(p) - \frac{2g^2N_c}{\sqrt{N}}iG^2_{\text{LO}}(p)\,p^2\tilde{I}(p;m_q)$$

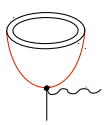
This approximation represent a “strict” expansion in $1/\sqrt{N}$, includes terms of no higher order than $\mathcal{O}(1/\sqrt{N})$.

Diagrammatics in case of a "strict" $1/\sqrt{N}$ expansion

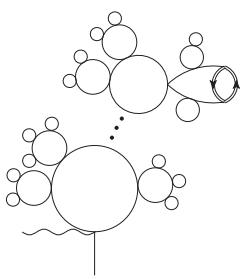
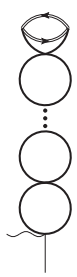
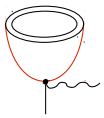
EoS: $v\sqrt{N}\left(m^2+\frac{\lambda}{6}v^2\right)+i\Bigg[\sum_{\text{loops}}$  $+$  $+\sum_{\text{loops}}$  $\Bigg]=h\sqrt{N}$

superdaisy resummation: $i\sum_{\text{loops}}$  $=i$  $=\frac{\lambda}{6}v\sqrt{N}T(M_{\text{LO}})$

$iG^{-1}_{\text{LO}}(p)=p^2-M^2_{\text{LO}}$ and $M^2_{\text{LO}}=m^2+\frac{\lambda}{6}\left[v^2+T(M_{\text{LO}})\right]$

i  $=-\frac{\lambda g^2}{6}vN_c\int_k G^2_{\text{LO}}(k)\int_p \text{Tr}_D[\gamma_5 D_\psi(p)\gamma_5 D_\psi(k+p)]=\frac{4\lambda g^2}{6}vN_c\left[-I(0;M_{\text{LO}})\tilde{T}(m_q)+\frac{1}{2}J(M_{\text{LO}},m_q)\right]$

$J(M_{\text{LO}},m_q)=\left[1+M^2_{\text{LO}}\frac{d}{dM^2_{\text{LO}}}\right]S(m_q,M_{\text{LO}})$

$i\sum_{\text{loops}}$  $=i\sum_{\text{skeleton loops}}$  $=\frac{1}{1-\frac{\lambda}{6}I(0;M^2_{\text{LO}})}\,i$ 

$$\Longrightarrow M^2_{\text{LO}}+\frac{1}{\sqrt{N}}M^2_{\text{NLO}}=\frac{h}{v},\qquad M^2_{\text{NLO}}=\frac{-4g^2N_c}{1-\frac{\lambda}{6}I(0;M_{\text{LO}})}\left[\tilde{T}(m_q)-\frac{\lambda}{12}J(M_{\text{LO}},m_q)\right]$$

Parametrization at $T = \mu_B = 0$

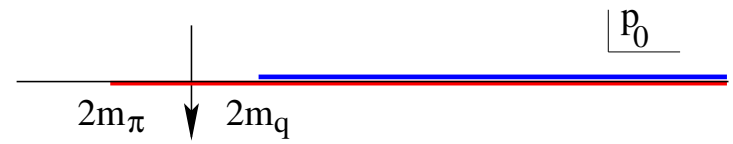
Six parameters to be fixed: $m^2, \lambda, g, v_0, h, M_{0B}$

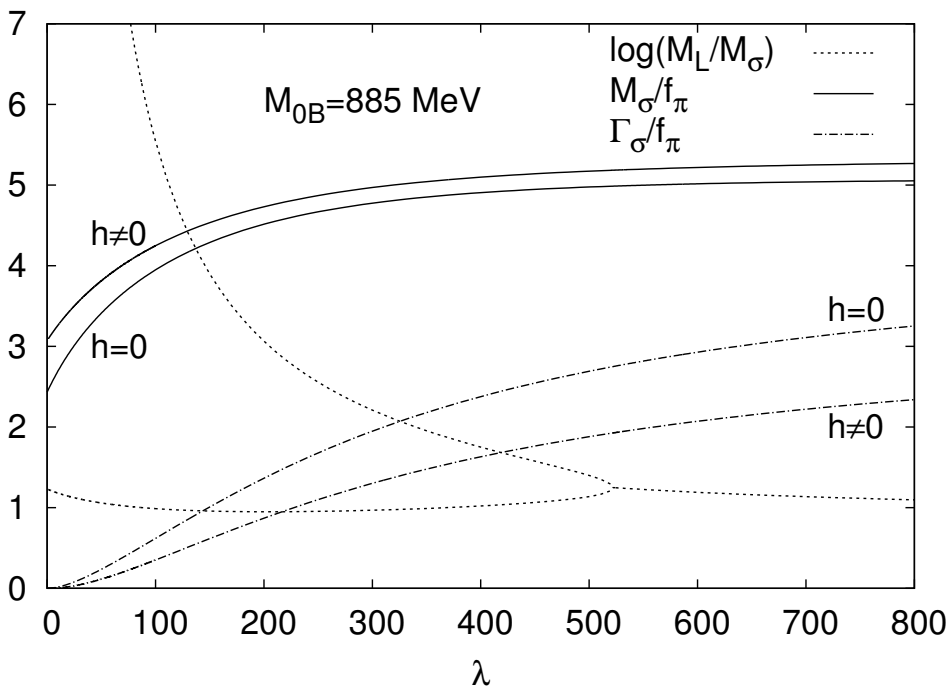
PCAC: $v_0 = \frac{f_\pi}{2}$ tree-level constituent quark mass: $g = \frac{m_q}{v_0} = \frac{2m_N}{3f_\pi} = 6.72$

The pole $p_0 = M_\sigma - i\Gamma_\sigma/2$ at rest $\mathbf{p} = 0$ is determined from the propagator

$$iG_\sigma^{-1}(p) = p^2 - \frac{h}{v} - \frac{\lambda v^2}{3}\frac{1}{1-\lambda I_F(p;m_\pi)/6} + \frac{2g^2N_c}{\sqrt{N}}(4m_q^2 - p^2)\tilde{I}_F(p;m_q)$$

analytically continued between the thresholds





– consistency of the σ-pole with the phenomenological values of m_σ, Γ_σ requires $\lambda \simeq 400$

– within the large N approximation σ is not broad enough

– $M_{0B} = 885$ MeV obtained from the requirement of having a large enough Landau ghost scale

m^2 and h determined from the EoS and the gap-equation (local approximations) or from $h = m_\pi^2 v_0$ (non-local approximations)

Doing the Matsubara and color sums

Calculation of the trace-log term $\omega_n = (2n+1)\pi T - i\mu_i, \quad E_k^2 = \mathbf{k}^2 + m_q^2$

$$i\,\mathrm{Tr}_{D,c}\int_k \ln D^{-1}(k) = -2T\sum_{i=1}^{N_c}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ln\left[\beta^2\left(\omega_n^2 + E_{\mathbf{k}}^2\right)\right]$$

$$= -2T\sum_{i=1}^{N_c}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\beta E_k + \ln\left(1+e^{-\beta(E_k-\mu_i)}\right) + \ln\left(1+e^{-\beta(E_k+\mu_i)}\right)\right]$$

After an integration by parts and introducing $\tilde{f}_i(E\mp\mu_i) = \dfrac{1}{\exp[\beta(E\mp\mu_i)]+1}$:

$$i\,\mathrm{Tr}_{D,c}\int_k \ln D^{-1}(k) = -2\sum_{i=1}^{N_c}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[E_k + \frac{k^2}{3E_k}\left(\tilde{f}(E_k-\mu_i) + \tilde{f}(E_k+\mu_i)\right)\right]$$

One defines:

$$\tilde{f}_\Phi^+(E) = \frac{1}{N_c}\sum_{i=1}^{N_c}\tilde{f}(E-\mu_i) = \frac{1}{N_c}\mathrm{Tr}_c\frac{1}{Le^{\beta(E-\mu_q)}+1}$$

$$\tilde{f}_\Phi^-(E) = \frac{1}{N_c}\sum_{i=1}^{N_c}\tilde{f}(E+\mu_i) = \frac{1}{N_c}\mathrm{Tr}_c\frac{1}{L^\dagger e^{\beta(E+\mu_q)}+1}$$

In Polyakov gauge, since $L = \text{diag}(a, b, c)$ the calculation is simple.

$$L, L^\dagger \in SU(3) \longrightarrow \det L^\dagger = \det L = 1 \quad \text{and} \quad L^\dagger L = 1$$

$$\Longrightarrow aa^* = bb^* = cc^* = 1,\, abc = a^* b^* c^* = 1$$

With $X = e^{\beta(E-\mu_q)}$:

$$\tilde{f}^+_\Phi(E) = \frac{1}{3}\text{Tr}_c \frac{1}{LX+1} = \frac{1}{3}\frac{X^2(ab+ac+bc)+2X(a+b+c)+3}{abcX^3+X^2(ab+ac+bc)+X(a+b+c)+1}$$

$$= \frac{1}{3}\,\frac{X^{-1}(a^*+b^*+c^*)+2X^{-2}(a+b+c)+3X^{-3}}{1+X^{-1}(a^*+b^*+c^*)+X^{-2}(a+b+c)+X^{-3}}$$

$$= \frac{X^{-1}\bar{\Phi}+2X^{-2}\Phi+X^{-3}}{1+3\left(\bar{\Phi}+X^{-1}\Phi\right)X^{-1}+X^{-3}}$$

Result: the inclusion of the Polyakov loop modifies the FD distribution function

$$f(E_p - \mu_q) \longrightarrow f^{+}_{\Phi}(E_p) = \frac{\left(\Phi + 2\bar{\Phi}e^{-\beta\left(E_p-\mu_q\right)}\right) e^{-\beta\left(E_p-\mu_q\right)} + e^{-3\beta\left(E_p-\mu_q\right)}}{1 + 3\left(\Phi + \bar{\Phi}e^{-\beta\left(E_p-\mu_q\right)}\right) e^{-\beta\left(E_p-\mu_q\right)} + e^{-3\beta\left(E_p-\mu_q\right)}}$$

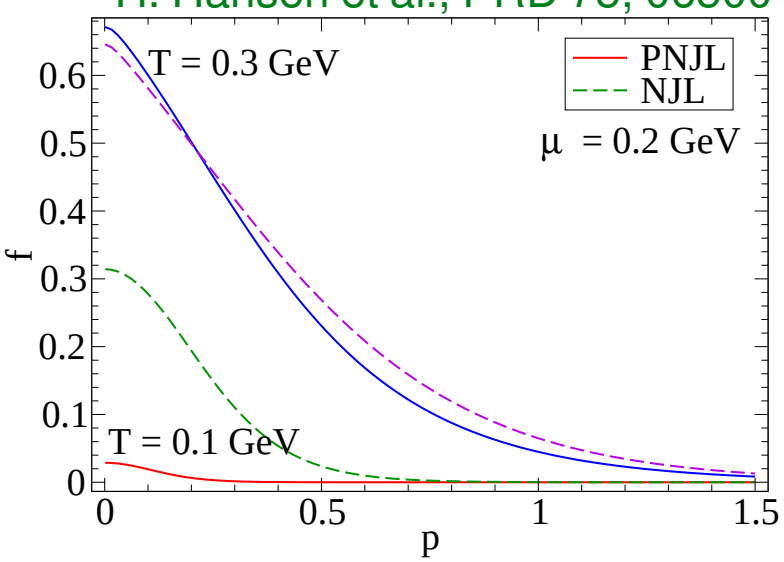
$$f(E_p + \mu_q) \longrightarrow f^{-}_{\Phi}(E_p) = \frac{\left(\bar{\Phi} + 2\Phi e^{-\beta\left(E_p+\mu_q\right)}\right) e^{-\beta\left(E_p+\mu_q\right)} + e^{-3\beta\left(E_p+\mu_q\right)}}{1 + 3\left(\Phi + \bar{\Phi}e^{-\beta\left(E_p+\mu_q\right)}\right) e^{-\beta\left(E_p+\mu_q\right)} + e^{-3\beta\left(E_p+\mu_q\right)}}$$

$f^{\pm}_{\Phi}(E_p) + f^{\pm}_{\Phi}(-E_p) + 1 = 0$ is still satisfied

$\Phi, \bar{\Phi} \to 0 \Longrightarrow f^{\pm}_{\Phi}(E_p) \to f(3(E_p \pm \mu_q))$ $\qquad$ $\Phi, \bar{\Phi} \to 1 \Longrightarrow f^{\pm}_{\Phi}(E_p) \to f(E_p \pm \mu_q)$

three-particle state appears: mimics confinement of quarks within baryons

H. Hansen et al., PRD 75, 065004



the effect of the Polyakov loop is more relevant for $T < T_c$

at $T = 0$ there is no difference between models with and without Polyakov loop: $\Theta(3(\mu_q - E_p)) \equiv \Theta(\mu_q - E_p)$

The color traces appearing in the setting-sun integral can be also expressed in terms of Φ and $\bar{\Phi}$:

$$\frac{1}{N_c}\sum_{i=1}^{N_c}\tilde{f}(E_1-\mu_i)\tilde{f}(E_2+\mu_i)=\frac{1}{N_c}\sum_{i=1}^{N_c}\mathrm{Tr}_c\left[\frac{1}{LX_+ + 1}\frac{1}{L^\dagger Y_- + 1}\right]$$

$$=\frac{1+X_+Y_-(1+X_+Y_-)+\bar{\Phi}\left[2Y_-(1+X_+Y_-)+X_+^2\right]+\Phi\left[2X_+(1+X_+)+Y_-^2\right]+3\Phi\bar{\Phi}X_+Y_-}{(X_+^3+3\bar{\Phi}X_+^2+3\Phi X_++1)(Y_-^3+3\Phi Y_-^2+3\bar{\Phi}Y_-+1)}$$

$$\frac{1}{N_c}\sum_{i=1}^{N_c}\tilde{f}(E_1-\mu_i)\tilde{f}(E_2-\mu_i)=\frac{1}{N_c}\sum_{i=1}^{N_c}\mathrm{Tr}_c\left[\frac{1}{LX_+ + 1}\frac{1}{LY_+ + 1}\right]$$

$$=\frac{3\bar{\Phi}^2X_+^2Y_+^2+6\Phi^2X_+Y_++(3\Phi\bar{\Phi}-1)X_+Y_+(X_++Y_+)+2\Phi(X_++Y_+-X_+^2Y_+^2)+\bar{\Phi}(X_+-Y_+)^2+1}{(X_+^3+3\bar{\Phi}X_+^2+3\Phi X_++1)(Y_+^3+3\bar{\Phi}Y_+^2+3\Phi Y_++1)}$$

$$E_1^2=\mathbf{k}^2+m_q^2,\ E_2^2=\mathbf{q}^2+m_q^2,\quad X_\pm=e^{\beta(E_1\mp\mu_q)},\ Y_\pm=e^{\beta(E_2\mp\mu_q)}$$

Setting $\Phi=\bar{\Phi}=0$, the second expression can be negative $\Longrightarrow$ it does not allow for an interpretation in terms of distribution functions.

Temperature dependence of v, Φ and $\bar{\Phi}$

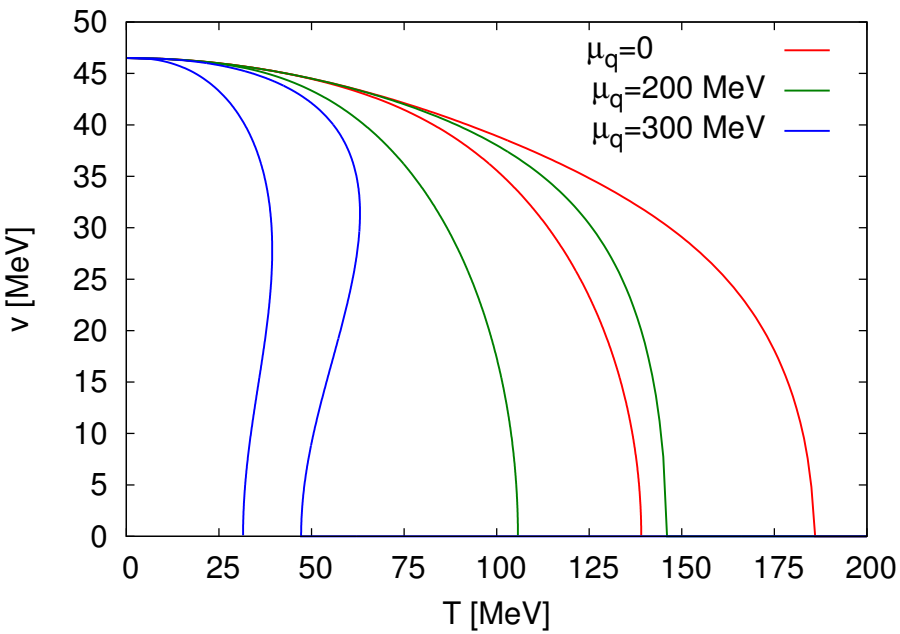
$$0 = \frac{dU(\Phi,\bar{\Phi})}{d\Phi} - 2N_c\sqrt{N}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{k^2}{3E_k}\left(\frac{d\tilde{f}^+_{\Phi}(E_k)}{d\Phi} + \frac{d\tilde{f}^-_{\bar{\Phi}}(E_k)}{d\Phi}\right)$$

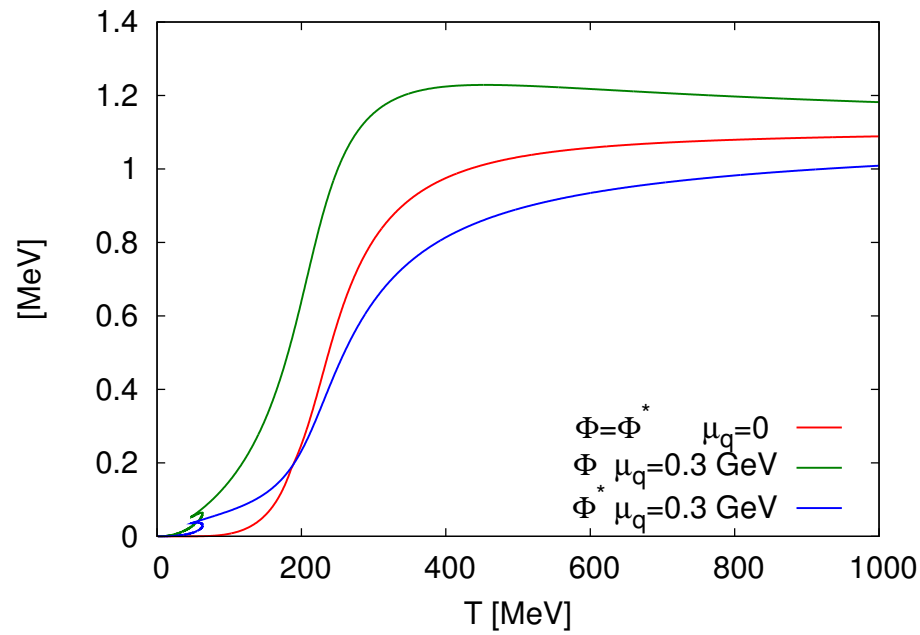
$$+\ g^2 N_c\sqrt{N}\left[2\left(\tilde{T}^0_F(m_q) - T_F(M)\right)\frac{d\tilde{T}^{\beta}(m_q)}{d\Phi} + \frac{d\tilde{T}^{\beta,2}_2(m_q)}{d\Phi} - M^2\,\frac{dS^{\beta,1+2}(M,m_q)}{d\Phi}\right]$$

$$0 = \frac{dU(\Phi,\bar{\Phi})}{d\bar{\Phi}} - 2N_c\sqrt{N}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{k^2}{3E_k}\left(\frac{d\tilde{f}^+_{\Phi}(E_k)}{d\bar{\Phi}} + \frac{d\tilde{f}^-_{\bar{\Phi}}(E_k)}{d\bar{\Phi}}\right)$$

$$+\ g^2 N_c\sqrt{N}\left[2\left(\tilde{T}^0_F(m_q) - T_F(M)\right)\frac{d\tilde{T}^{\beta}(m_q)}{d\bar{\Phi}} + \frac{d\tilde{T}^{\beta,2}_2(m_q)}{d\bar{\Phi}} - M^2\,\frac{dS^{\beta,1+2}(M,m_q)}{d\bar{\Phi}}\right]$$

$$0 = Nv\left[m^2 + \frac{\lambda}{6}\left(v^2 + \bigcirc^{\pi}\right) - 4g^2\frac{N_c}{\sqrt{N}}\tilde{T}_F(m_q)\right]$$





Results in CQM and PCQM in the chiral limit

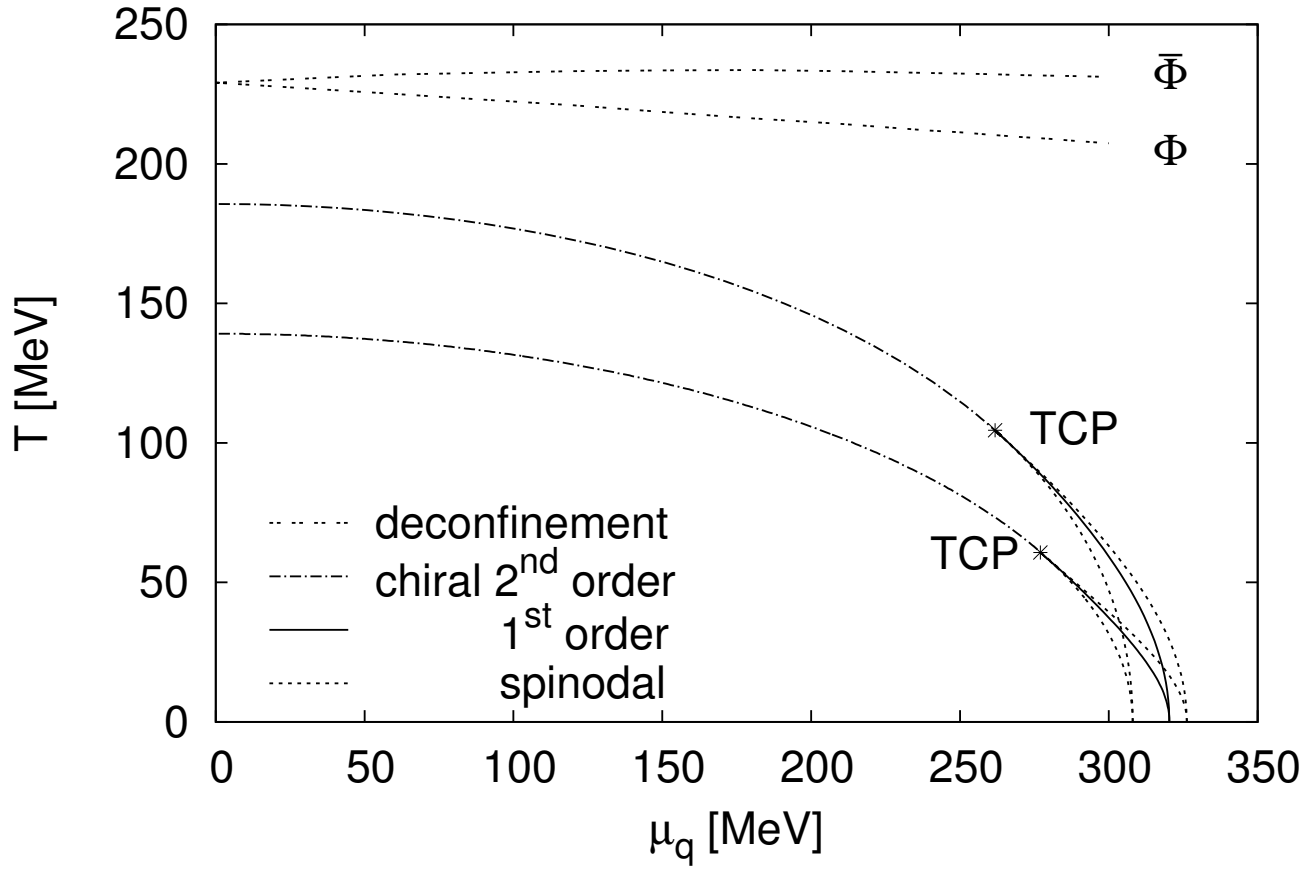
$U(\Phi,\bar{\Phi})$	T_0	$T_\chi(\mu_q=0)$	$T_d(\mu_q=0)$	T_{TCP}	μ_q^{TCP}
–	–	139.0	–	60.7	277.0
poly	270	185.6	229.0	104.5	261.8
poly	208	168.2	176.5	96.2	263.4
log	270	191.4	209.0	109.4	261.2
log	208	167.6	162.4	102.6	261.2
log	$T_0(\mu_q)$	167.9	162.8	84.3	266.9
Fuku	–	176.5	193.0	99.8	262.2

unit: MeV

for $U_{\text{log}}(\Phi,\bar{\Phi})$, $T_0 = 208$ MeV one has $T_\chi(\mu_q=0) > T_d(\mu_q=0)$

in all other cases $T_\chi(\mu_q=0) < T_d(\mu_q=0)$

Phase diagrams in CQM and PCQM in the chiral limit



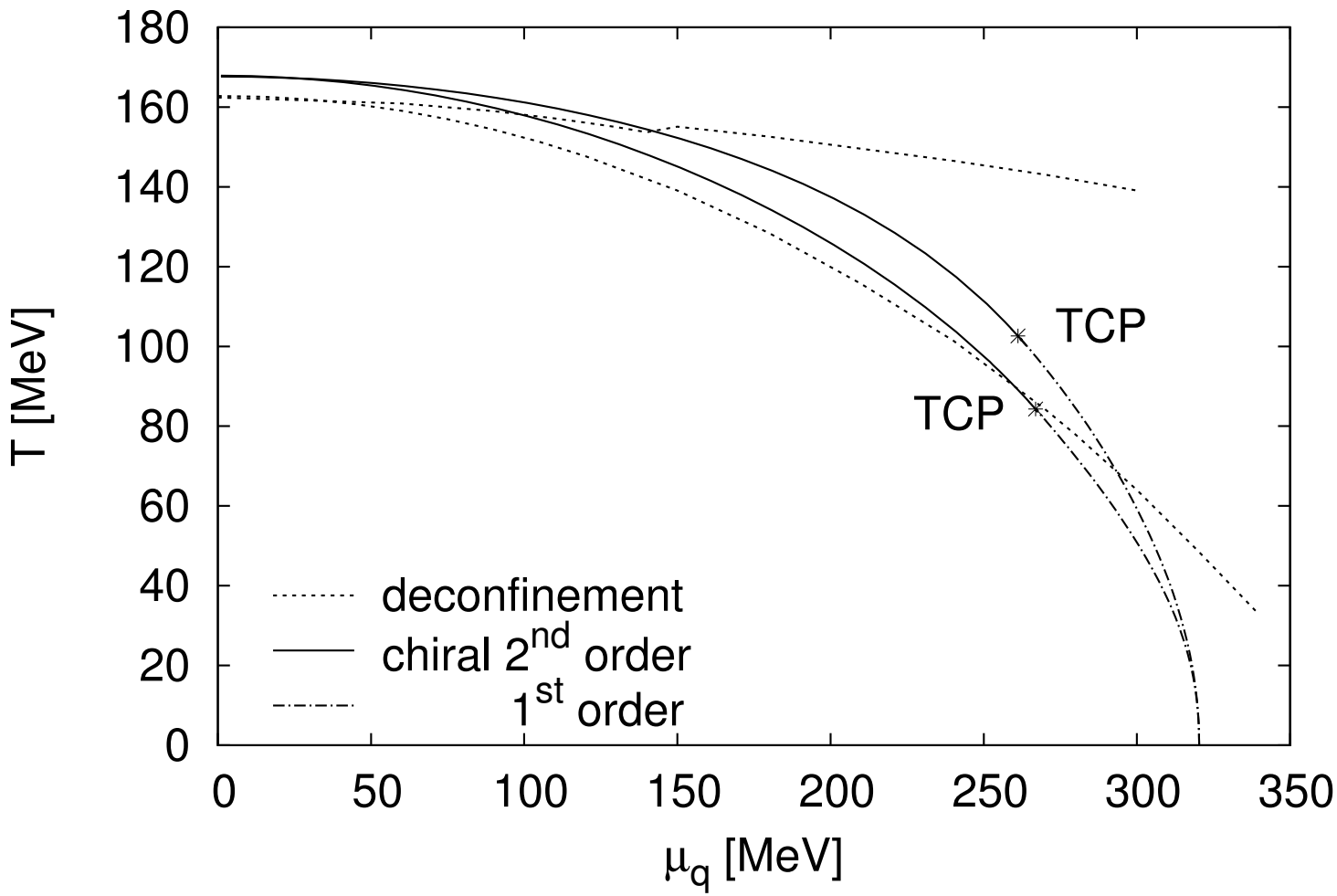
$U_{\text{poly}}(\Phi, \bar{\Phi})$
$T_0 = 270$ MeV

CQM $T_c(\mu_B = 0) = 139$ MeV $(\mu_B, T)_{\text{TCP}} = (831.0, 60.9)$ MeV
PCQM $T_c(\mu_B = 0) = 185.6$ MeV $(\mu_B, T)_{\text{TCP}} = (798.8, 100.0)$ MeV

$N_f = 2$ lattice: $T_c(\mu_B = 0) = 173(8)$ MeV Karsch et al., NPB 605, 579 (2001)

PNJL gives higher temperature for $T_c(\mu_B = 0)$

Chiral and deconfinement phase transitions obtained with $U_{\log}(\Phi, \bar{\Phi})$



$T_0 = 208$ MeV

$T_0(\mu_q)$

possibility for the existence of a region with quarkyonic phase which shrinks when $T_0(\mu_q)$ is used similarly as in H. Abuki, et al., Phys. Rev. D 78, 034034 (2008)

Results in CQM and PCQM for physical m_π

$T_\chi(\mu_q = 0)$ and $T_d(\mu_q = 0)$ obtained from the inflection points of $v(T)$ and $\Phi(T)$

Shown are those cases which gives $T_\chi(\mu_q = 0) \geq T_d(\mu_q = 0)$

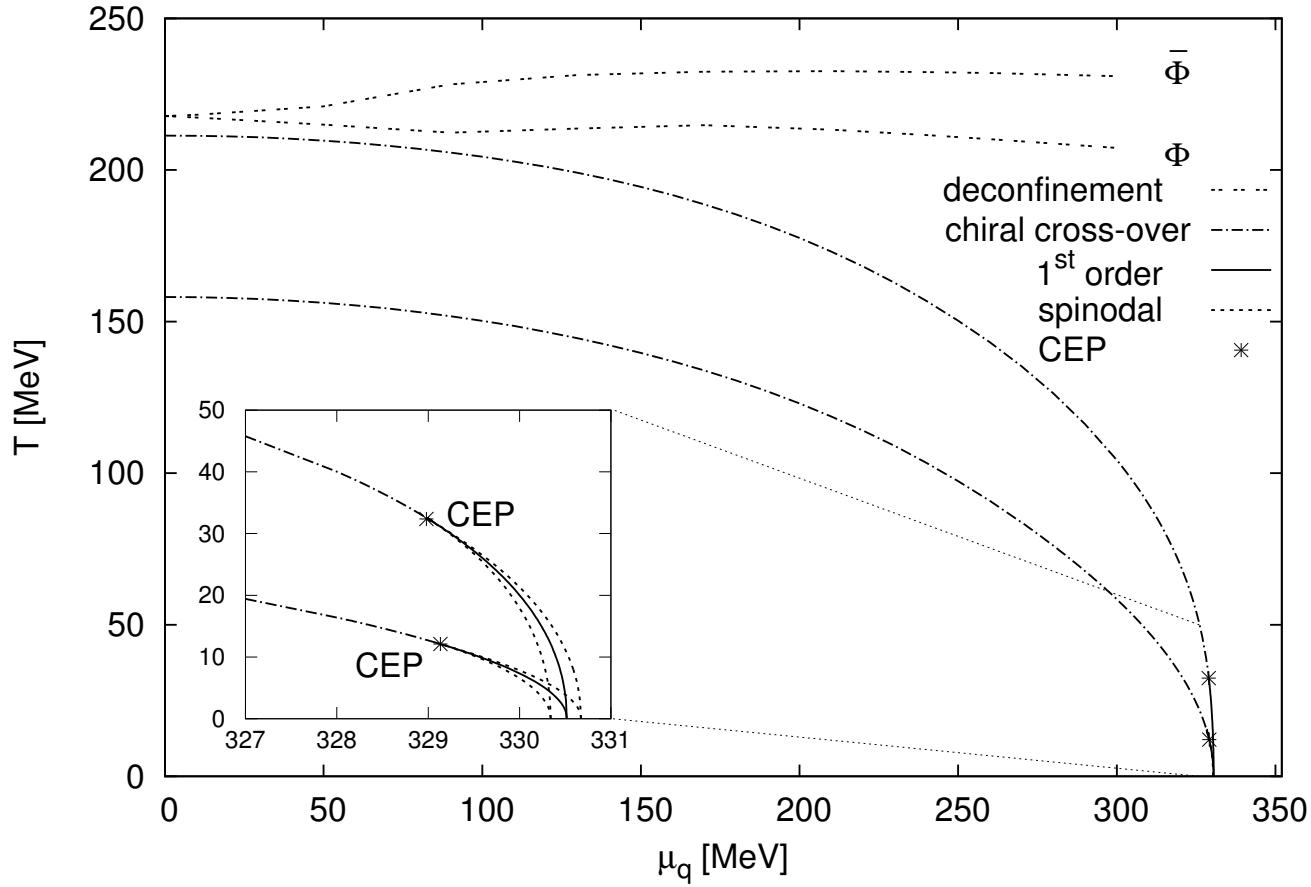
$U(\Phi, \bar{\Phi})$	T_0	$G_\pi(p)$	$T_\chi(\mu_q = 0)$	$T_d(\mu_q = 0)$	Γ_χ	T_{CEP}	μ_q^{CEP}
–	–	local, pole	152.8	–	37.6	14.4	327.1
–	–	local, $p = 0$	158.2	–	41.5	12.1	329.1
–	–	large-N	158.6	–	40.7	13.5	328.6
poly	208	local, pole	180.6	175.0	19.8	35.3	326.7
poly	208	local, $p = 0$	184.6	176.7	22.7	30.1	328.9
poly	208	large-N	184.6	176.8	22.3	30.6	328.8
log	208	local, pole	168.0	167.0	*30.3	37.9	326.9
log	208	local, $p = 0$	168.5	167.0	*42.8	32.7	329.0
log	$T_0(\mu_q)$	local, $p = 0$	168.9	167.4	*42.5	25.7	328.7
log	208	large-N	168.5	167.1	*43.0	33.0	328.9
Fuku	–	local, pole	191.0	188.7	19.8	36.2	326.8
Fuku	–	local, $p = 0$	195.3	191.2	21.2	31.2	328.9
Fuku	–	large-N	195.2	191.3	21.2	31.8	328.8
poly	208	large-N, full	188.1	183.1	21.4	32.2	329.0

unit: MeV

full: contribution of the quark-pion setting-sun is taken into account in the field equations for Φ and $\bar{\Phi}$

Phase diagrams in CQM and PCQM for physical m_π

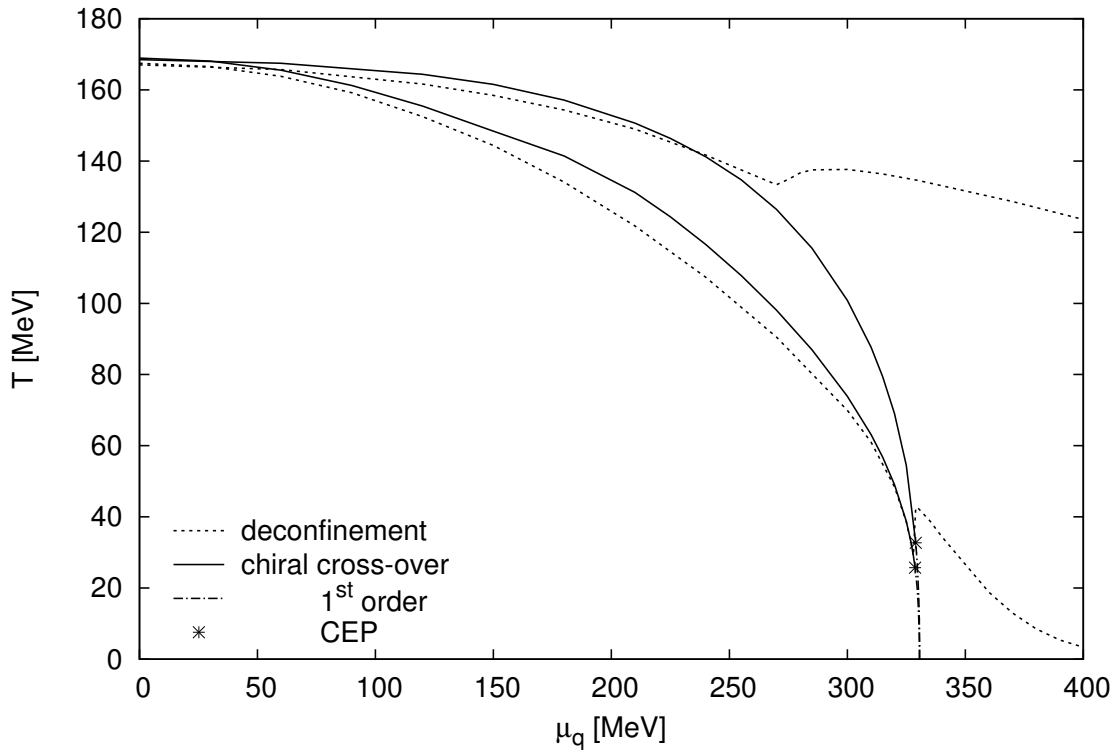
Local approximation for G_π with M^2 determined from $M^2 = -iG^{-1}_{\pi,l}(p=0)$



$U_{\text{poly}}(\Phi, \bar{\Phi})$
$T_0 = 270$ MeV

CQM	$T_c(\mu_B = 0) = 158.2$ MeV	$(\mu_B, T)_{\text{CEP}} = (987.3, 12.1)$ MeV
PCQM	$T_c(\mu_B = 0) = 211.4$ MeV	$(\mu_B, T)_{\text{CEP}} = (987.0, 32.6)$ MeV

Chiral and deconfinement phase transitions obtained with $U_{\text{log}}(\Phi, \bar{\Phi})$



$T_0 = 208$ MeV

$T_0(\mu_q)$

The use of $T_0(\mu_q)$ gives the lowest T_{CEP}.

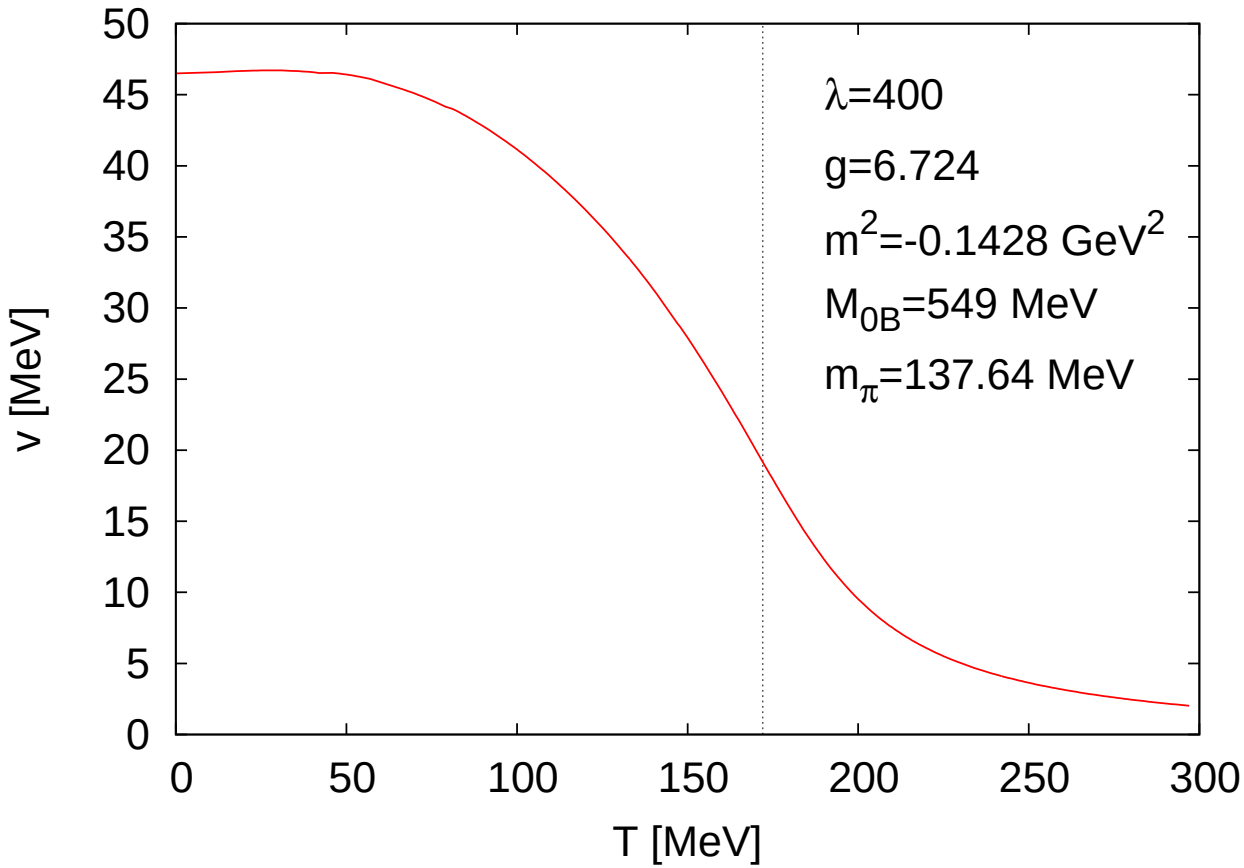
The confined region where the chiral symmetry is restored does not vanishes completely as in T. K. Herbst, et al., arXiv:1008.0081 [hep-ph].

Phase transition in case of a "strict" expansion in $1/\sqrt{N}$

One solves the finite EoS $\quad M_{\text{LO}}^2 + \frac{1}{\sqrt{N}} M_{\text{NLO}}^2 = \frac{h}{v} \quad$ at finite T and μ_q

$$M_{\text{LO}}^2 = m^2 + \frac{\lambda}{6}\left[v^2 + T_F(M_{\text{LO}})\right]$$

$$M_{\text{NLO}}^2 = \frac{4g^2 N_c}{1 - \frac{\lambda}{6} I_F(0; M_{\text{LO}})}\left[-\tilde{T}_F(m_q) + \frac{\lambda}{12}\left(1 + M_{\text{LO}}^2 \frac{d}{dM_{\text{LO}}^2}\right) S_F(m_q, M_{\text{LO}})\right]$$



Features:

– very weak crossover

$\Gamma_\chi \sim 60$ MeV

– no 1st order phase transition possible at reasonable values of μ_q (remains crossover for very large values of μ_q)

About the softening of the phase transition

The softening of the phase transition with the inclusion of quantum fluctuations can be simply demonstrated.

Results obtained using – $U_{\text{poly}}(\Phi, \bar{\Phi})$ with $T_0 = 208$ MeV
– the tree-level parametrization of the model

	$\tilde{T}_F^0$	T_F^0	$\tilde{T}^\beta$	T^β	$T_\chi(\mu_q = 0)$	Γ_χ	T_{CEP}	μ_q^{CEP}
QP	$-$	$-$	$+$	$-$	184.6	4.6	162.8	165.1
QP	$-$	$-$	$+$	$+$	180.2	8.6	145.3	204.3
QFT	$+$	$-$	$+$	$-$	173.0	26.9	91.3	241.1
QFT	$+$	$+$	$+$	$+$	170.1	30.3	85.5	243.5

first row: reproduced result of B.-J. Schaefer, et al., PRD 76, 074023 (2007)

$+$ ($-$) inclusion (not inclusion) of the given contribution
QP: quasiparticle approximation in which vacuum fluctuations are disregarded
QFT: quantum field theoretical treatment of the vacuum fluctuations, where the renormalization scales $M_{0F} = \sqrt{e} m_q$ and $M_{0B} = \sqrt{e} m_\pi$ are such that $T_F^0 = \tilde{T}_F^0 = 0$ only at $T = \mu_q = 0$

Inclusion of fluctuations pushes the CEP to higher μ_q similarly to the result of
E. Nakano, et al., PLB682, 401 (2010)
T. K. Herbst, et al., arXiv:1008.0081 [hep-ph]

Conclusions & Outlook

• Using large-N_f technique we solved with some approximations the $N_f = 2$ chiral quark model in the chiral limit and for the physical pion mass.

• The effect of the Polyakov loop on the $\mu_B - T$ phase diagram was studied.

– with the inclusion of the Polyakov loop the value of $T_c(\mu_B = 0)$ increases and becomes comparable with the value obtained on the lattice.

– the TCP/CEP moves to higher value of T and lower value of μ_B, but the change in μ_B is not significant.

• Expanding the pion propagator in $1/N_f$ while keeping the fermion propagator at its tree level is not a viable resummation, because due to a softening of the phase transition at $\mu_B = 0$ it does not lead to the expected phase diagram $\Longrightarrow$ self-consistency is needed

• The effect of a self-consistent fermion propagator has to be studied.

• One should attempt to solve the model self-consistently at $\mathcal{O}(1/N_f)$ order in the large-N_f expansion using:

– self-consistent pion and fermion propagators applying to the broken symmetry phase the renormalization method of U. Reinosa, NPA772, 138 (2006)

• Solve the more realistic $SU(3)_L \times SU(3)_R$ model using 2PI formalism.