Transport with unstable particles

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A. Jakovac and D. Nogradi, arXiv:0810.4181
A. Jakovac, arXiv:0901.2802
A. Jakovac, PhysRevD.81.045020 [arXiv:0911.3248]

Outlines

1. Introduction
   - Transport in plasma
   - Lower bound for η/s
2. eta/s in field theory
   - Transport coefficients
   - Entropy density
3. Model calculations
   - Small width case
   - Broad spectral function
   - System with zero mass excitations
4. Conclusions

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Quark matter near Tc

RHIC $\Rightarrow$ matter near T_c almost perfect fluid, $\eta/s \lesssim 0.1$

Hard to describe theoretically

- non-interacting particles form a free gas: free mean path is infinite $\Rightarrow$ transport coefficients are infinite (eg. η)
- in weakly interacting gas $\eta \sim 1/\sigma$ (cross section) $\sim 1/g^4 \times \log s$ $\Rightarrow$ η still large
- for small η we need large coupling constant $\Rightarrow$ non-perturbative system
- small η $\Rightarrow$ small mean free path $\Rightarrow$ fast decay $\Rightarrow$ for elementary excitations: width $\sim$ mass
  $\Rightarrow$ unstable quasiparticles (unparticles?)

MC studies

In principle exact method to measure η/s...

- measure $\langle T_{12}(x)T_{12}(0)\rangle$ correlator on lattice $\Rightarrow$ Euclidean discrete time
- we need the spectral function, which is related to the correlator as
  $$\int d^3\mathbf{x}\,\langle T_{12}(\tau,\mathbf{x})T_{12}(0)\rangle = \int_0^\infty \frac{d\omega}{\pi}\,C(\omega,\mathbf{k}=0)\,\frac{\cosh(\omega(\beta/2-\tau))}{\sinh\beta\tau/2}.$$
- invert this relation with the prior knowledge $C(\omega>0)>0$ Maximal Entropy Method, or ad hoc solutions
- too little sensitivity to small ω regime $\Rightarrow$ large systematical uncertainties; additional assumptions are needed
- best estimates $\eta/s = 0.102\,(56)$ at $T = 1.24\,T_c$

  (H. B. Meyer, Phys. Rev. D 76, 101701 (2007))

$\Rightarrow$ needs analytic control!

Conformal approach

Strongly interacting systems with large symmetry group (conformal symmetry) can be analytically treated: instead QCD we use $\mathcal{N} = 4$ SYM theory

- speactral functions are conituous
- duality, strongly interacting CFT can be mapped to weakly interacting gravity systems.
- in $N_c \gg 1$ and $\lambda = g^2 N_c \gg 1$ limit classical gravity can be used

But this method also has disadvantages

- QCD is not conformally symmetric
- at finite N_c and λ string corrections can be relevant.

We need generic statements about systems with large-width excitations

eta/s in different systems

- transport coefficients: diffusion constants of conserved quantities $\Rightarrow$ linear response theory should be used
  $C(x) = \langle [J_i(x), J_i(0)] \rangle \quad \Rightarrow \quad D = \lim_{\omega \to 0} \frac{C(\mathbf{k}=0,\omega)}{\omega}$
  for the shear viscosity $J_i \to T_{12}$
- in the perturbative regime Boltzmann equations or microcanonical approach can be used
- order of magnitude from Navier-Stokes equation: $\varrho \dot{v} \sim \eta \triangle v$
  $\Rightarrow \quad \eta \sim \varrho v^2 \tau \sim \epsilon\tau$

  (τ lifetime, ℓ mean free path, v velocity, ϵ energy density)

  and $s \sim n$ (particle density) $\quad \Rightarrow \quad \frac{\eta}{s} \sim E\tau$

Outlines

- argumentation: $\eta \sim \epsilon\tau,\ s \sim n \quad \Rightarrow \quad \frac{\eta}{s} \sim E\tau$

  In quasipatricle systems $E > \Delta E \quad \Rightarrow \quad \frac{\eta}{s} \gtrsim \Delta E\tau \gtrsim \hbar$

P. Danielewicz, M. Gyulassy, PRD 31, 53 (1985); P. Kovtun, D.T. Son, A.O. Starinets PRL 94, 111601 (2005).

- calculation: for $\mathcal{N} = 4$ SYM theory at $N_c \gg 1$, $\lambda = g^2 N_c \gg 1$ from graviton absorbtion in the dual 5D AdS gravity:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

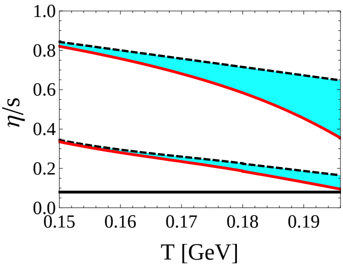
  (P. Kovtun, D.T. Son, A.O. Starinets JHEP 0310, (2003) 064.)

- for weaker coupling we expect larger ratio: indeed, first λ, N_c corrections are positive (R.C. Myers, M.F. Paulos, A. Sinha, arXiv:0806.2156)
- universal for a wide class of theories (A. Buchel, R.C. Myers, M.F. Paulos, A. Sinha, Phys.Lett.B669:364-370,2008.; M. Haack, A. Yarom, arXiv:0811.1794)
- so far we did not find counterexamples experimentally

Theoretical caveats

- $\mathcal{N} = 4$ SYM theory is not QCD
- $1/4\pi$ limit is not stable against higher curvature/dilaton corrections.
- Counterexample: N species with the same interaction $\Rightarrow$ η is not changed, $s \sim \ln N$ (mixing entropy) $\Rightarrow$ $\dfrac{\eta}{s} \sim \dfrac{1}{\ln N}$

  (A. Cherman, T. D. Cohen, and P. M. Hohler, JHEP 02, 026 (2008), 0708.4201.)
- in QFT there is always a continuum – effect on η/s?

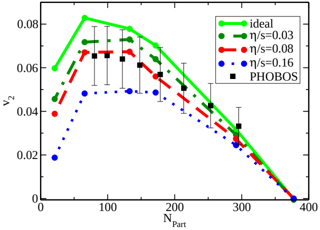


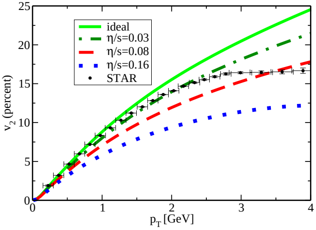
pure hadron gas vs. hadron gas with continuum $\Rightarrow$ considerable difference

(J. Noronha-Hostler, J. Noronha and C. Greiner, PRL 103, 172302 (2009), 0811.1571)

RHIC data

Non-central heavy ion collisions have initial anisotropy.
Time evolution of anisotropy: the larger the viscosity, the more extent the initial anisotropy is washed out

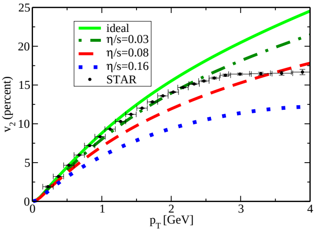




(P. Romatschke, U. Romatschke, Phys.Rev.Lett.99:172301,2007.)

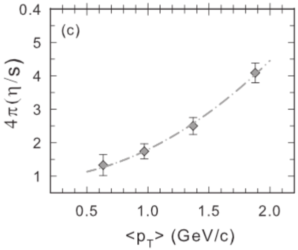
upper bound: $\frac{\eta}{s} \lesssim 0.16 \quad \Rightarrow$ is there a lower bound?

RHIC data vs. $1/4\pi$



P. Romatschke, U. Romatschke, PRL.99:172301,2007.

$\Rightarrow$



R.A. Lacey, A. Taranenko, R. Wei, arXiv:0905.4368

- Quadratic correction in p_T is expected (D.A. Teaney, arXiv:0905.2433)
- Statistically $\frac{\eta}{s} < \frac{1}{4\pi}$ is not excluded (favored: $\left.\frac{\eta}{s}\right|_{\langle p_T \rangle = 0} \approx 0.9 \pm 0.07$)

RHIC seriously challenges the conjectured lower bound!

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Idea: if we knew the exact energy levels

- $\Rightarrow$ calculate (or learn something about) η/s
- $\Rightarrow$ generic statements (e.g. for lower bound)
- $\Rightarrow$ effect of continuous spectrum.

QM-based description:

$$\sum_n |n\rangle\langle n| = V\sum_{\mathcal{Q}}\int\frac{d^4p}{(2\pi)^4}\,\varrho_{\mathcal{Q}}(p)\,|p,\mathcal{Q}\rangle\langle p,\mathcal{Q}| \equiv \int_Q |p,\mathcal{Q}\rangle\langle p,\mathcal{Q}|\,,$$

where ϱ is the spectral function (density of states, DoS), $\mathcal{Q}$ denotes conserved quantities (quantum channel), $p=(p_0,\mathbf{p})$ is the total energy-momentum of the state.

- use volume normalization to calculate densities
- DoS can depend on the temperature.

$$C_J(x) = \langle [J_i(x), J_i(0)] \rangle \quad \Rightarrow \quad \eta_J = \lim_{\omega \to 0} C_J(\omega)/\omega$$

generic transport coefficient.

- insert complete basis of energy-momentum eigenstates

$$C_J(x) \; = \frac{1}{Z} \sum_{n,m} \left[\langle n \,| e^{-\beta H} J(x) |\, m \rangle \langle m \,| J(0) |\, n \rangle - \{x \leftrightarrow 0\} \right]$$

- translation:
  $J(x) = e^{iPx} A(0) e^{-iPx} \quad \Rightarrow \quad \langle n \,| J(x) |\, m \rangle = e^{i(P_n - P_m)x} \langle n \,| J(0) |\, m \rangle$
- Fourier transformation, $p = (p_0, \mathbf{p})$

$$C_J(p) \; = \frac{1}{Z} \sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m} \right) (2\pi)^4 \delta(p + P_n - P_m) | \langle m \,| J(0) |\, n \rangle |^2$$

- introduce spectral densities

$$\eta_J = \beta \frac{V^2}{\mathcal{Z}} \sum_{\mathcal{Q}} \int \frac{d^4k}{(2\pi)^4}\, \varrho^2_{\mathcal{Q}}(k)\, e^{-\beta k_0} |\langle k, \mathcal{Q}|J_i|k, \mathcal{Q}\rangle|^2.$$

- current matrix element: $\langle k, \mathcal{Q}|J_i|k, \mathcal{Q}\rangle = \mathcal{J}_{\mathcal{Q}}(k)\frac{k_i}{Vk_0}$
  - $\sim k_i/k_0 \sim v_i$ since J_i is a current
  - in free case $\mathcal{J}_{\mathcal{Q}}(k)$ is the charge carried by the current: for electric current $\mathcal{J} = e$ charge, for viscosity $\mathcal{J} \sim k_j$ momentum
  - in nonperturbative case $\mathcal{J}_{\mathcal{Q}}(k)$ can be momentum dependent.
- angular averaging

Finally:

$$\eta_J = \beta \frac{1}{3\mathcal{Z}} \sum_{\mathcal{Q}} \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{k}^2}{k_0^2}\, e^{-\beta k_0} \left(\mathcal{J}_{\mathcal{Q}}(k)\varrho_{\mathcal{Q}}(k)\right)^2.$$

Outlines

free energy density $\mathcal{Z} = e^{-\beta F} = \mathrm{Tr}\, e^{-\beta H} = V \sum_{\mathcal{K}} \int \frac{d^4 k}{(2\pi)^4}\, \varrho_{\mathcal{K}}(k) e^{-\beta k_0}$

Volume dependence of the free energy:

- for small sizes it is arbitrary
- as $V \to \infty$ we recover linear volume dependence
- crossover at scale L
    - volume elements larger than L interact weakly (surface interaction)
    - coarse graining scale $\Rightarrow$ free energy density cen be defined at scales larger than L
    - L is also an effective IR cutoff for the interactions.

$\Rightarrow$ we choose a volume $V = L^3$ to define free energy density:

$$f = -\frac{T}{L^3} \ln \left(1 + L^3 \sum_{\mathcal{K}} \int \frac{d^4 k}{(2\pi)^4}\, \varrho_{\mathcal{K}}(k)\, e^{-\beta k_0} \right).$$

The eta/s ratio

with $s = -\frac{\partial f}{\partial T}$ and with angular averaging

$$\frac{\eta}{s} = \frac{\frac{\beta}{3\mathcal{Z}} \sum_{\mathcal{K}} \int \frac{d^4k}{(2\pi)^4} \, \frac{\mathbf{k}^2}{k_0^2} \, e^{-\beta k_0} \left(\mathcal{J}_{\mathcal{K}}(k) \varrho_{\mathcal{K}}(k)\right)^2}{-\frac{\partial}{\partial T} \frac{T}{L^3} \ln\left(1 + L^3 \sum_{\mathcal{K}} \int \frac{d^4k}{(2\pi)^4} \, \varrho_{\mathcal{K}}(k) \, e^{-\beta k_0}\right)}.$$

Is there a lower bound in this formula?

Generic structure

Structure of η/s for small ϱ is

$$\frac{\eta_J}{s} \sim \frac{\int f_1 \varrho^2}{\int f_2 \varrho} \xrightarrow{rescaling} \frac{\langle \varrho^2 \rangle}{\langle \varrho \rangle}.$$

$\langle \varrho^2 \rangle \geq \langle \varrho \rangle^2 \quad \Rightarrow \quad$ we expect $\eta \gtrsim s^2$ up to rescaling factors.

Quasiparticle vs. non-quasiparticle systems:

- large peak in $\varrho \quad \Rightarrow \quad \varrho^2$ even larger $\quad \Rightarrow \quad \eta/s$ large
- ϱ small everywhere $\quad \Rightarrow \quad \varrho^2$ even smaller $\quad \Rightarrow \quad \eta/s$ small

in non-quasiparticle systems η/s is naturally small!

Lower bound – mathematical approach

More exactly:

- need a sum rule in each energy channel $\int \frac{dk_0}{2\pi}\, \varrho_Q(k_0) = U_Q$
- minimize η by tuning ϱ with respecting the sum rules and keeping the entropy density constant.
- technically: Lagrange multiplicators
- two cases analytical: small/large s. The minimum values:

$$\left.\frac{\eta}{s}\right|_{min} \sim \frac{\mathcal{F}(L^3 s)}{N_Q (LT)^4}, \qquad \mathcal{F} = \begin{cases} x & \text{for small } x \\ e^x/x & \text{for large } x \end{cases}$$

N_Q: number of effective quantum channels (species).

$\Rightarrow$ There is a lower bound at finite s, but it is not universal.

Simplifications

For the concrete model calculations we use simplifications

- We use generalized quasiparticle systems:

$$f = T\int \frac{d^4k}{(2\pi)^4}\, \varrho_{QP}(k)\,(\mp)\ln\left(1 \pm e^{-\beta k_0}\right).$$

- we omit the effect of $\mathcal{J}_Q$ and define a "reduced" viscosity coefficient as

$$\bar{\eta} = \frac{\beta}{15}\int \frac{d^4k}{(2\pi)^4}\,\frac{(\mathbf{k}^2)^2}{k_0^2}\, e^{-\beta k_0}\, \varrho_{\mathcal{K}}^2(k).$$

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Assume that the lowest lying states can be approximated with Breit-Wigner form

$$\varrho(q) = \frac{2\Gamma}{(q_0 - \varepsilon_q)^2 + \Gamma^2}.$$

In the small width limit $\varrho(q)^2 \approx \frac{2}{\Gamma} 2\pi\delta(q_0 - \varepsilon_q)$. As a consequence

$$\frac{\bar{\eta}}{s} = \frac{T}{\Gamma} f(\frac{m}{T}),$$

$f(m/T)$ depends on ε_k

if $\varepsilon_k = k,\; f = 540/\pi^4$ for bosons, $f = 4320/(7\pi^4)$ for fermions; if $\varepsilon_k = m + \frac{k^2}{2m},\; f = 30\pi T/m, \dots$

- in conformal case $\Gamma \sim T \quad \Rightarrow \quad \eta/s \sim$ constant
  lower limit may come from infinte coupling, $1/4\pi$.
- massive case at low temperature: $\Gamma \sim e^{-M/T}$ (M: energy of scattering state) $\quad \Rightarrow \quad \eta/s \sim Te^{M/T} \stackrel{T\to 0}{\longrightarrow} \infty$.

$\Rightarrow$ in the small width quasiparticle case there is a lower bound

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For an opposite case consider a flat spectral function:

$$\varrho(k_0,k)=\frac{2\pi}{E_2-E_1}\Theta(E_1<k_0<E_2)$$

step function, where $E_{1,2}(k)=\sqrt{k^2+m_{1,2}^2}$. At small temperatures ($T<m_1$)

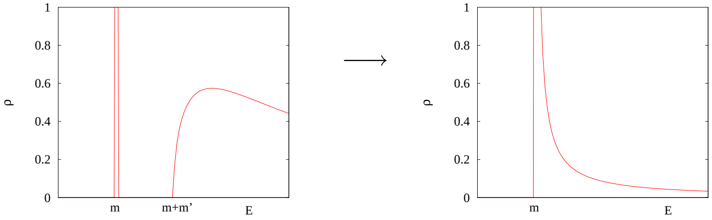
$$\frac{\eta}{s}=6\pi\frac{T}{m_2-m_1}$$

$\Rightarrow$ by broadening the distribution the viscosity to entropy density ratio has no lower bound

Outlines

Zero mass excitations

- If there are in the system zero mass particles, then $m' = 0$
  $\Rightarrow$ cut and the delta-peak melt together
- 1-loop threshold behavior linear $\Rightarrow$ $\varrho \sim 1/(E - E_{\rm thr})$



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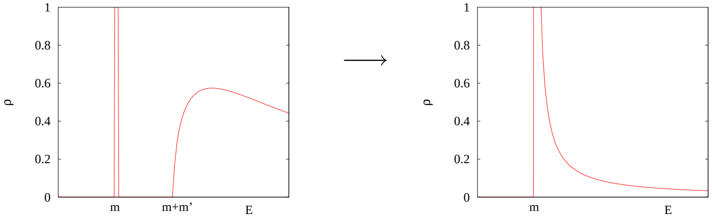
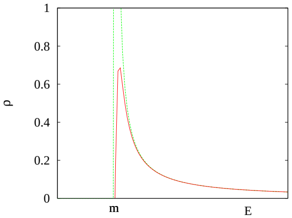


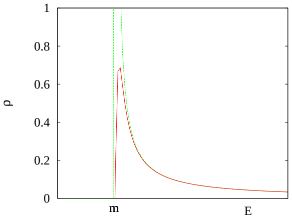
- This is not normalizable! $\Rightarrow$ IR divergences near the threshold, which smear out the $1/x$ singularity

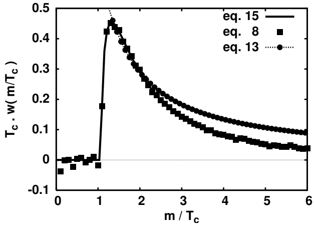
Zero mass excitations



- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons

Zero mass excitations





- Interpretation: no single charged particle (electron, quark), it is always surrounded by soft gauge bosons
- In QCD: from fitting to MC pressure data one obtains similar distribution of quasiparticle masses

(T.S.Biro, P.Levai, P.Van, J.Zimanyi, Phys.Rev.C75:034910,2007)

The η/s ratio at low temperatures

$\varrho^{\#} e^{-\beta q_0}$ enhances the lowest lying states $\Rightarrow$ power expand near the threshold:

$$\varrho(q) = \mathcal{C} q_0 \Theta(q - M)(q^2 - M^2)^w.$$

C is dimensionfull: $[\mathcal{C}] = [E]^{-2(1+w)}$
$\eta_J \sim \mathcal{C}^2$ and $f \sim \mathcal{C}$ $\Rightarrow$ $\mathcal{C}$ remains in the ratio.
In the massive and massless case we find

$$\frac{\eta_J}{s} \sim \mathcal{C} M^w T^{2+w} \quad \text{and} \quad \mathcal{C} T^{2(1+w)} \xrightarrow{T\to 0} 0$$

for an integrable threshold $(w > -1)$

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Conclusions

- $1/4\pi$ lower bound for η/s is true only for quasiparticle and conformal theories
- in general the lower bound in a given environment depend on several factors;for small s

$$\frac{\eta}{s} \gtrsim \frac{s}{N_Q L T^4}$$

- model constructions with $\eta/s < 1/4\pi$:
  - step function spectral function
  - zero mass excitation with integrable threshold
- in QCD any of these effects can be important

Hydrodynamics

System with local collective flow: QM description?
Construction of the system: at $t = -\infty$ equilibrium in rest ($\bar{u} = (1,0,0,0)$), then a modified time evolution corresponding to the flow u – denote $\Delta u = u - \bar{u}$:

$$H^{(0)} = \bar{u}_\mu P^{(0)\mu} = u_\mu P^\mu = H + \Delta u_\mu T^{0\mu} \quad \Rightarrow \quad \delta H = -\int d^3x\, \Delta u_\mu T^{0\mu}$$

$$\delta H = \int_{-\infty}^{t} dt'\, \partial_0 \delta H = -\int_{-\infty}^{t} dt' d^3x\, \left[\partial_0 u_\mu T^{0\mu} + \Delta u_\mu \partial_0 T^{0\mu}\right]$$

With energy-momentum conservation $\partial_0 T^{0\mu} = -\partial_i T^{i\mu}$ and partial integration

$$\delta H = -\int_{-\infty}^{t} dt' d^3\mathbf{x}\, \partial_\mu u_\nu T^{\mu\nu}$$

- Linear response theory (→ the flowing and the original systems are not too far ⇒ nonrelativistic):
$$\delta\left\langle X(t)\right\rangle = i\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' d^3\mathbf{x}'' \left\langle [X(t), T^{\mu\nu}(x'')]\right\rangle \partial_\mu u_\nu .$$
- Hydrodynamical approximation: $\partial u \approx$ const. ⇒ $\delta\left\langle X\right\rangle$ time independent.
- spatial rotational symmmetry of the ground state ⇒ $\delta\left\langle \pi_{ij}\right\rangle \equiv \delta\left\langle T_{ij} - \frac{1}{3}\delta_{ij} T^{k}_{.k}\right\rangle = \frac{\eta}{2}\left[\partial_k v_\ell + \partial_\ell v_k - \frac{2}{3}\delta_{k\ell}\partial v\right]$,

the coefficient from above (denote $C(x) = \left\langle [T_{12}(0), T_{12}(x)]\right\rangle$

$$\eta = i\int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' d^3\mathbf{x}'\, C(x') = \lim_{\omega\to 0}\frac{C(\omega,\mathbf{k}=0)}{\omega}$$

Kubo formula

Zero temperature limit of viscosity

The viscosity η and the entropy have a common form

$$F_{n,m} = \frac{3a^5}{2\pi^3 T}\int \frac{d^4k}{(2\pi)^4}\Theta(k_0)\Theta(k^2-\sigma^2)e^{-k_0/T}(ak_0)^n\varrho^m(k),$$

since

$$\eta = N^2\Delta^2 F_{0,2}, \qquad s = 2F_{1,1}.$$

After reducing the integrals

$$a^3F_{n,m} = C(a\sigma)^n\,(a^2\sigma T)^{3/2}e^{-\sigma/T}\int_0^\infty dz\,e^{-z}\left(\frac{2(wz)^{5/2}}{1+(wz)^5}\right)^m,$$

where $w = 2\Delta^{-2/5}T/\sigma$ rescaled temperature.
BOTH η and s goes to zero at zero temperature