Numerical Explorations of the Lattice Loop Equations

Peter Anderson
with Martin Kruczenski

Purdue University

ander324@purdue.edu

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Overview

1. Lattice Gauge Theory
2. Strong Coupling Expansion
3. The Lattice Loop Equations
4. Solutions of QCD_2
5. Numerical methods in 4d
6. SUSY on the Lattice

Lattice Gauge Theory

Recall: The Pure Yang-Mills Lagrangian

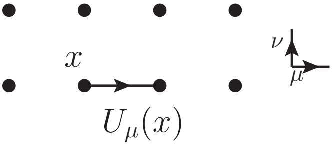
$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$

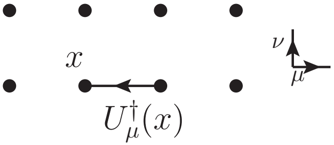
Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$

and $A_\mu(x) \xrightarrow{g.t.} A'_\mu(x) = \Omega(x)A_\mu(x)\Omega^\dagger(x) + i\Omega(x)\partial_\mu\Omega^\dagger(x)$

$$U_\mu(x) = \mathcal{P}e^{-i\int_x^{x+a\hat{\mu}} A\cdot dx}$$

$$U_\mu(x) \xrightarrow{g.t.} \Omega(x)U_\mu(x)\Omega^\dagger(x+a\hat{\mu})$$





Lattice Gauge Theory

- The Gauge Fields live on the links of the Lattice in $SU(N)$
- Continuum limit ($a \to 0$) reproduces the Pure Gauge Action
- Methods of calculation include strong coupling expansion, perturbation theory, and Monte Carlo Simulations

$$S = -\frac{N}{2\lambda} \sum_{\mu \neq \nu, x} \mathrm{Tr}\left(U_{\mu\nu}(x)\right)$$

Where, $U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x)$ and $\lambda = g^2 N$ is the 't Hooft coupling

$$\langle \mathcal{W} \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\mu} dU_\mu(x) \frac{1}{N} \mathrm{Tr}(U_{\mu_1} \ldots U_{\mu_n}) \exp(-S)$$

$$\mathcal{Z} = \int \prod_{x,\mu} dU_\mu(x) \exp(-S)$$

Strong Coupling Expansion Wilson 1974

- This is in the limit of $\frac{N}{\lambda} \to 0$ and $g^2 \to \infty$ which allows us to expand about the exponential in the path integral.

$$e^{\frac{N}{2\lambda}\sum_{p,x} \mathrm{Tr}\, U_p(x)} = 1 + \frac{N}{2\lambda}\sum_{p,x} \mathrm{Tr}\, U_p(x) + \frac{1}{2}\left(\frac{N}{2\lambda}\sum_{p,x} \mathrm{Tr}\, U_p(x)\right)^2 + \dots$$

- Calculating the value of Wilson Loops then just becomes a matter of combinatorics.
- Unfortunately the strong coupling limit does not correspond with the continuum physics that we are interested in

Solution of the plaquette

The Haar measure is normalized such that

$$\int dU = 1$$

$$\int dU U_{ij} U^{\dagger}_{kl} = \frac{1}{N}\delta_{il}\delta_{kj}$$

We then find that to first order that

$$\begin{aligned}\langle \mathcal{W}_{\Box} \rangle &= \frac{N}{2\lambda}\int dU_{1\ldots4}\frac{1}{N}U^{(1)}_{i_1 i_2}U^{(2)}_{i_2 i_3}U^{(3)\dagger}_{i_3 i_4}U^{(4)\dagger}_{i_4 i_1}U^{(4)}_{j_1 j_2}U^{(3)}_{j_2 j_3}U^{(2)\dagger}_{j_3 j_4}U^{(1)\dagger}_{j_4 j_1}\\ &= \frac{1}{2\lambda}\frac{1}{N^4}\delta_{i_1 j_1}\delta_{i_2 j_4}\delta_{i_2 j_4}\delta_{i_3 j_3}\delta_{i_3 j_3}\delta_{i_4 j_2}\delta_{i_4 j_2}\delta_{i_1 j_1}\\ &= \frac{1}{2\lambda}\end{aligned}$$

The Lattice Loop Equations

The dynamics of Yang-Mills theories are described by the Schwinger-Dyson equations

$$-\nabla_{\mu}^{ab} F_{\mu\nu}^{b}(x) \overset{w.s.}{=} \hbar \frac{\delta}{\delta A_{\nu}^{a}(x)}$$

The lattice loop equations are derived by performing a change of variables on a single link

$$U_{\mu}(x) \to (1 + i\epsilon_{\mu}(x))U_{\mu}(x)$$
$$U_{\mu}^{\dagger}(x) \to U_{\mu}^{\dagger}(x)(1 - i\epsilon_{\mu}(x))$$

Where, ϵ_{μ} is a traceless, hermitian matrix. This results in the lattice version of the Schwinger-Dyson equations. (Migdal-Makeenko 1979)

$$\langle -\delta_{\epsilon} S \mathcal{W}_{x}^{ab} + \delta_{\epsilon} \mathcal{W}_{x}^{ab} \rangle = 0$$

Derivation of Loop Equations

Varying the link $U_\alpha(x)$

$$\Big\langle \frac{iN}{2\lambda}\,\mathrm{Tr}\Big(\sum_{\nu\neq\alpha}\epsilon_\alpha\big\{U_\alpha U_\nu U_\alpha^\dagger U_\nu^\dagger + U_\alpha U_\nu^\dagger U_\alpha^\dagger U_\nu - U_\nu U_\alpha U_\nu^\dagger U_\alpha^\dagger - U_\nu^\dagger U_\alpha U_\nu U_\alpha^\dagger\big\}\Big)\mathcal{W}^{ij} + i\epsilon_\alpha^{ik}\mathcal{W}^{kj} + \sum_n \mathcal{W}^{il}\epsilon_\alpha^{lk}\mathcal{W}^{kj} - \sum_{\bar{n}} \mathcal{W}^{il}\epsilon_\alpha^{lk}\mathcal{W}^{kj}\Big\rangle = 0$$

Isolating ϵ gives $\epsilon_\alpha^{lk}A_{ij}^{kl} = 0$, resulting in

$$A_{ij}^{kl} - \frac{a_{ij}}{N}\delta^{kl} = 0$$

The Lattice Loop Equations

Lattice Loop Equations

$$\frac{1}{2\lambda}\langle \mathcal{W}_{\alpha\mu}\rangle + \left(1 - \frac{1 + n_m - \bar{n}_m}{N^2}\right)\langle \mathcal{W}\rangle + \sum_m \tau_m \langle \mathcal{W}_{nm}\mathcal{W}_{mn}\rangle - \frac{1}{2\lambda}\langle U_{\alpha\mu}\mathcal{W}\rangle = 0$$

Where $\tau_m = \pm 1$ depending if the link is parallel or anti-parallel

Note: There was no restriction of the action or the choice of gauge throughout the derivation

Large-N limit of QCD

In the large-N limit the expectation value of the product of Wilson loops will factorize in the following manner. (Migdal 1980)

$$\langle \mathcal{W}_1 \mathcal{W}_2 \rangle \rightarrow \langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle + \mathcal{O}\left(\frac{1}{N^2}\right)$$

Lattice Loop Equation

$$\frac{1}{2\lambda}\langle \mathcal{W}_{\alpha\mu} \rangle + \langle \mathcal{W} \rangle + \sum_m \tau_m \langle \mathcal{W}_{nm} \rangle \langle \mathcal{W}_{mn} \rangle = 0$$

Where $\tau_m = \pm 1$ depending if the link is parallel or anti-parallel

Gross-Witten Solution of QCD_2

Choosing Axial Gauge

$$U_1 = 1$$

The partition function factorizes

$$\mathcal{Z}_{2d} \to \mathcal{Z}_{1p}^{N_p}$$

Then all observables are

$$W_n = \langle \frac{1}{N} \operatorname{Tr} U_p^n \rangle_{1p}$$

Where n is the number of times the plaquette is wrapped around.

QCD_2 loop equation

$$W_{n+1} - W_{n-1} + 2\lambda W_n + 2\lambda \sum_{p=1}^{n-1} W_p W_{n-p} = 0$$

Gross-Witten Solution

By choosing a holomorphic generating function $f = \sum_{n=0} z^n W_n$, where z is within the unit circle, one finds the exact solution

$$W_0 = 1$$

$$W_1 = \frac{1}{2\lambda}, \lambda \geq 1$$

$$W_1 = 1 - \frac{\lambda}{2}, \lambda < 1$$

$$W_{n+1} = W_{n-1} - 2\lambda W_n - 2\lambda \sum_{p=1}^{n-1} W_p W_{n-p}$$

At $\lambda = 1$ a third order phase transition was discovered by Gross and Witten, which is unusual in statistical physics. Where the phase transitions usually occur in the infinite volume limit.

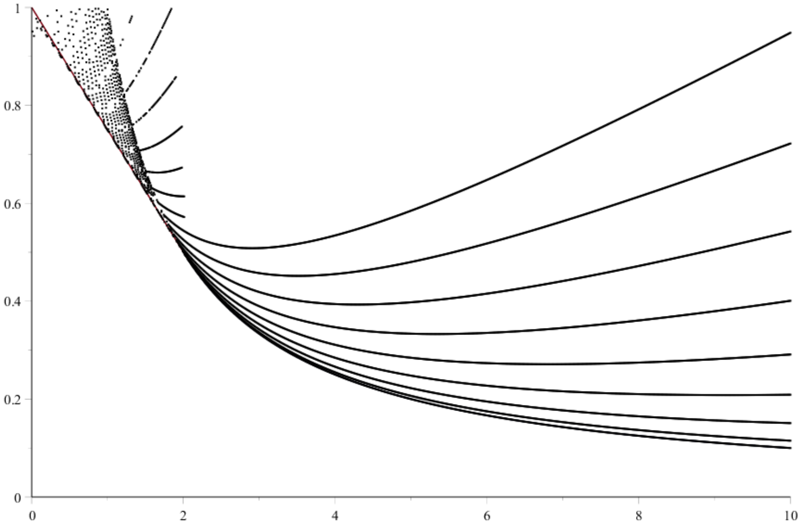
A Numerical Approach (Marchesini 1984)

- $W_0 = 1$ (from Unitarity) and W_1 is desired
- $W_{n+1} = P(W_1)$, where $P(W_1)$ is a polynomial derived from the loop equation
- $W_T = 0$, Truncate loops larger than certain length
- Roots of the polynomial correspond with the upperbound of the value of plaquette

Loop Equation, QCD_2

$$W_{n+1} = W_{n-1} - 2\lambda W_n - 2\lambda \sum_{p=1}^{n-1} W_p W_{n-p}$$

Marchesini's Method



The Large Loop Cutoff QCD_4

Consider the loop equation in the following form

$$K_{i\to j}W_j + W_i + C_{i\to jk}W_jW_k = \frac{1}{2\lambda}\delta_{i1}$$

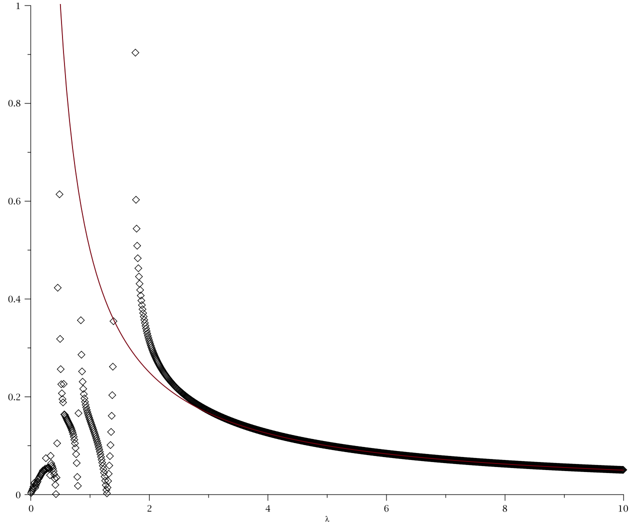
The equation for $i = 1$

$$\frac{1}{2\lambda}K_{i\to j}W_j + W_1 = \frac{1}{2\lambda}$$

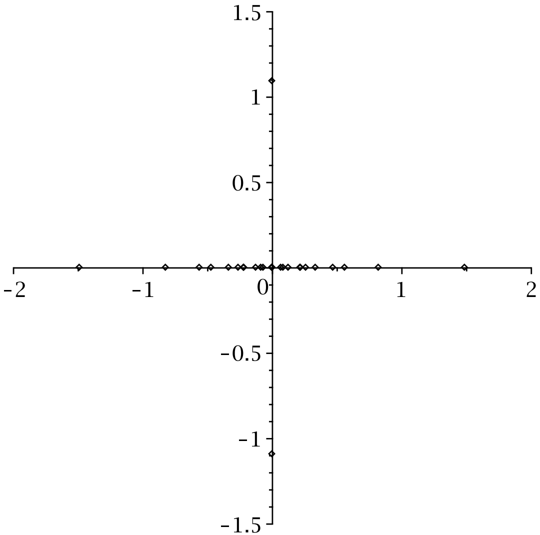
Truncate up to $L = 8$ (33 Wilson loops), taking $i, j > 1$

$$\frac{1}{2\lambda}K_{i\to j}W_j + W_i = -\frac{1}{2\lambda}K_{i\to 1}W_1 - C_{i\to 11}W_1^2$$
$$\Rightarrow W_j = -(\mathbb{K} + 2\lambda)^{-1}(K_{i\to 1}W_1 + 2\lambda C_{i\to 11}W_1^2)$$

Solution of W_1



Solution Behavior



Marchesini's Iterative Approach

- To avoid the poles of the $\mathbb{K}^{-1}$ matrix one can iteratively apply $\mathbb{K}$ and then truncate loops that grow too big

$$W_i = \frac{1}{2\lambda}\delta_{i1} - K_{i\to j}W_j - C_{i\to jk}W_jW_k$$

$$W_i^{(p)} = -K_{i\to j}W_j^{(p-1)} - C_{i\to jk}\sum_{l=1}^{p-1} W_j^{(l)}W_k^{(l-p)}$$

$$W_i^{(1)} = \frac{1}{2\lambda}\delta_{i1}$$

- Every iteration grows or shrinks the loops, eventually to a plaquette

$$W_i = \left(-\frac{1}{2\lambda}\right)^n K_{i\to j_1}K_{j_1\to j_2}\dots K_{j_{n-1}\to j_n}\delta_{j_n 1}$$

- This is equivalent to strong coupling

The Generalized Method

Consider the following basis

$$A = \sum_{0 \to x} c_i U(\mathcal{C}_i)$$

$$\text{Tr}(A^\dagger A) \geq 0$$

$$\sum_{i,j} c_i \, \text{Tr}\left(U^\dagger(\mathcal{C}_i) U(\mathcal{C}_j)\right) c_j \geq 0$$

We can now identify $H_{ij} = \text{Tr}\left(U^\dagger(\mathcal{C}_i) U(\mathcal{C}_j)\right)$ which is by definition a positive semi-definite matrix, which is populated with Wilson loops.

The Generalized Method (cont)

$$H_{ij} = \text{Tr}\left(U^{\dagger}(\mathcal{C}_i)U(\mathcal{C}_j)\right) \succeq 0$$

- $\mathcal{C}_i$ is the path from $0 \to x$
- All eigenvalues are 0 or positive
- Determinant and leading principal minors are 0 or positive
- All parameters of H_{ij} (Wilson loops) are constrained to a hyper-cone
- When combined with the loop equations the values of the Wilson loops become severely restricted

Applying to QCD_2

Let

$$A = \sum_{i=1} c_i U_p^i$$

This results in H_{ij} being a Toeplitz matrix with the basis $W_0, W_1, W_2, \ldots, W_n$.

$$H_{ij} = \begin{pmatrix} W_0 & W_1 & W_2 & \cdots & W_n \\ W_1 & W_0 & W_1 & \ddots & \vdots \\ W_2 & W_1 & W_0 & \ddots & W_2 \\ \vdots & \ddots & \ddots & \ddots & W_1 \\ W_n & \cdots & W_2 & W_1 & W_0 \end{pmatrix}$$

Example, $n = 2$

The H_{ij} matrix results in the following constraints on the Wilson loops

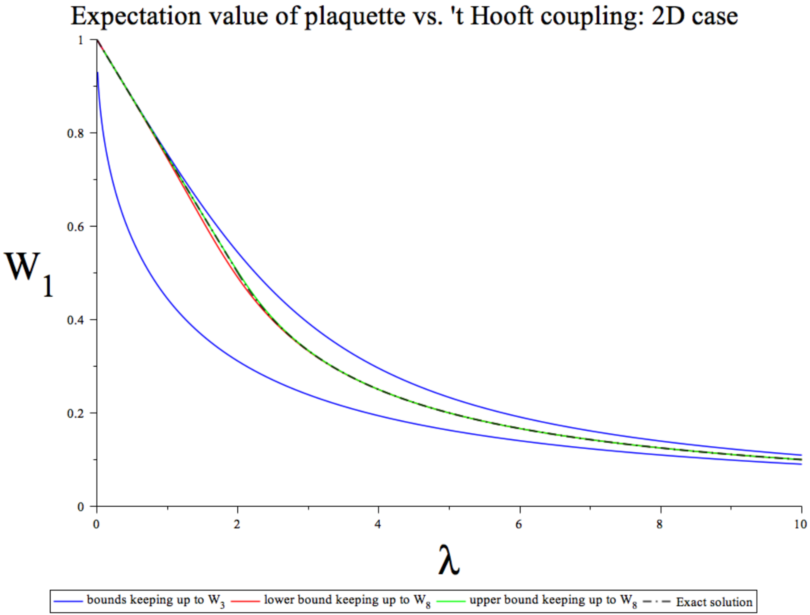
$$\left\{ W_1 \leq 1, W_2 \leq 1, W_2 \geq 2W_1^2 - 1 \right\}$$

While the loop equation tells us

$$W_2 - W_0 + 2\lambda W_1 = 0$$

When combined it results in the following

$$W_1 \leq -\frac{\lambda}{2} + \frac{1}{2}\sqrt{4 + \lambda^2}$$
$$W_1 \leq 1 - \frac{\lambda}{2}, \quad \lambda \to 0$$
$$W_1 \leq \frac{1}{\lambda}, \quad \lambda \to \infty$$
$$W_1 \geq 0$$


Expectation value of plaquette vs. 't Hooft coupling: 2D case
W1
λ
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10
bounds keeping up to W3
lower bound keeping up to W8
upper bound keeping up to W8
Exact solution

Numerical Solutions to QCD_4

- The number of Wilson loops are rather large now and grows greatly with the length of the loops
  - $L = 4 \to 2$
  - $L = 6 \to 5$
  - $L = 8 \to 33$
  - $L = 10 \to 421$
  - $L = 12 \to 9803$
  - $L = 14 \to 300000$
- The method of calculating loops is similar. However, they are now computationally much more expensive.
- The basis is also non-trivial and must be formed in such a way to include all loops included in loop equations

Semidefinite Programming

Minimize

$$\mathrm{Tr}(CX)$$

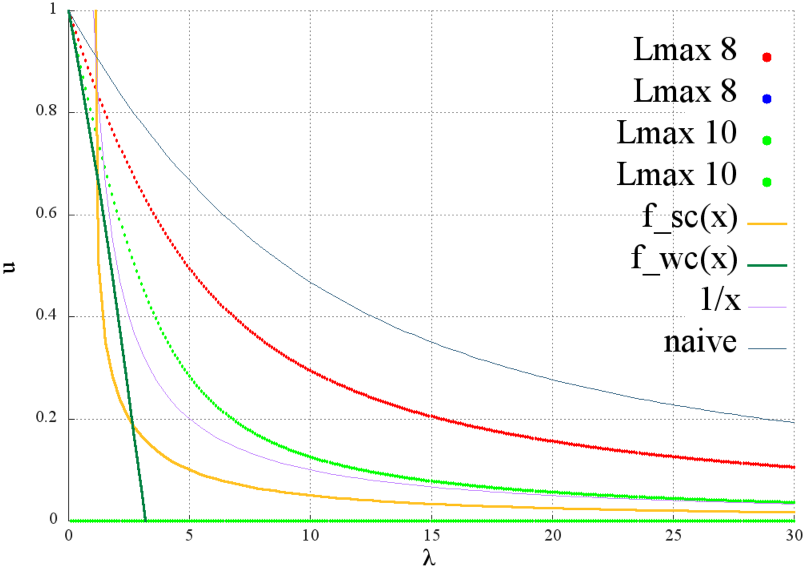
such that

$$\sum_i \mathrm{Tr}(A_i X) = b_i$$

$$X \succeq 0$$

There are a lot of robust solvers that currently exist and are implemented using convex optimization. However, it can only handle linear problems. Fails when the loops self-intersect.

Single plaquette, upper and lower bounds



Optimizing Calculation

- SDP is extremely efficient at handling linear problems
- Additional loop equations (Loop Bianchi Identity) were included
- The first self-intersecting loop equations (non-linear) enter in at $L = 12$. SDP can provide bounds to the loops, which will simplify the numerical calculations
- Selecting the basis is non-trivial and reduce the complexity of the problem greatly

The $\mathcal{N} = 4$ SYM Lattice Action

$$S = \mathcal{Q}\Lambda + S_{\text{closed}}$$
$$\Lambda = \sum_x a^4 \,\text{Tr}(\chi_{ab}\mathcal{F}_{ab} + \eta\bar{\mathcal{D}}_a^{(-)}\mathcal{U}_a - \frac{1}{2}\eta d)$$
$$S_{\text{closed}} = -\frac{1}{4}\sum_x a^4 \epsilon_{abcde}\chi_{de}\bar{\mathcal{D}}_c^{(-)}\chi_{ab}$$

The SUSY variations are

$$\mathcal{Q}\mathcal{U}_a = \psi_a$$
$$\mathcal{Q}\psi_a = 0$$
$$\mathcal{Q}\bar{\mathcal{U}}_a = 0$$
$$\mathcal{Q}\chi_{ab} = -\bar{\mathcal{F}}_{ab}$$
$$\mathcal{Q}\eta = d$$
$$\mathcal{Q}d = 0$$

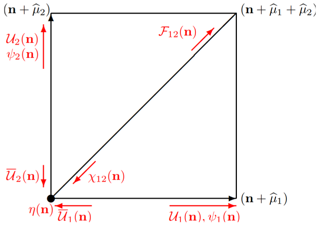


Figure : An example of the sitelinks and fermions in the hypercubic formulation (Catterall).

Applying to SYM $\mathcal{N} = 4$

- The lattice action is now much larger and includes fermions
- Computationally much harder to calculate
- The Ward identities from the exactly preserved SUSY charge will enter into the problem.
- These are able to produce a subset of purely bosonic loop equations

Summary

- The numerical methods presented here are able to put tight bounds on the analytic solution in QCD_2
- Results obtained so far in QCD_4 suggest that they can be expanded to the non-linear regime which should tighten the lower bound significantly
- For now focus will be on selecting a proper basis that encapsulates all of the necessary constraints
- Future work includes developing the lattice loop equations for lattice $\mathcal{N} = 4$ SYM that incorparate the Ward identities and investigating finite N

Questions?

Thank you for your attention