ODE/IM correspondence and the Argyres-Douglas Theory

Katsushi Ito

Tokyo Institute of Technology

December 9, 2016@Wigner Research Centre for Physics

in collaboration with Christopher Locke and Hongfei Shu
arXiv: 1312.6759,1502.00906, 1605.04668+ work in progress

Introduction

- The ODE/IM correspondence is a relation between spectral analysis approach of ordinary differential equation (ODE), and the "functional relations" approach to 2d quantum integrable model (IM). [Dorey-Tateo 1998]
- This is an example of non-trivial correspondence between classical and quantum integrable models

- Dorey-Tateo (1998) studied the spectral determinant of the second order differential equation

$$\left(-\frac{d^2}{dx^2}+x^{2M}-E\right)\psi(x,E)=0$$

and its relation to the A_{2M-1}-type Thermodynamic Bethe-ansatz (TBA) equations.

- Lukyanov-Zamolodchikov (2010) studied the relation for the linear problem associated with the sinh-Gordon equation

$$\varphi_{tt}-\varphi_{xx}+\sinh\varphi=0$$

and the quantum XXZ model. In the conformal limit, this relation reduces to the above ODE/IM correspondence.

- This suggests existence of Lie algebraic structure behind this mysterious correspondence.

Motivation

- Understand mathematical structure and physical meaning of the ODE/IM correspondence
  making (complete) dictionaries
  relations among integrable models
- Application to the AdS/CFT correspondence
  Gluon scattering amplitudesat strong coupling [Alday-Maldacena, Alday-Maldacena-Sever-Vieira, Hatsuda-KI-Sakai-Satoh]
- Application to $\mathcal{N} = 2$ SUSY gauge theories
  Omega-background and quantum spectral curve
  [Nekrasov-Shatashvili]

We will

- introduce some basic concepts of the ODE/IM correspondence
- and discuss its application to the Argyres-Douglas theory

ODE/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

- ODE

$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y(x,E,\ell)=0$$

$x\in\mathbf{C}$, E complex, ℓ: real, $M>0$

- large, real positive x : we have two (divergent and convergent) solutions. subdominant (small) solution

$$y(x,E,\ell)\sim\frac{x^{-\frac{M}{2}}}{\sqrt{2i}}\exp\left(-\frac{x^{M+1}}{M+1}\right)\quad(x\to\infty)$$

is defined uniquely in the sector $|\arg x|<\frac{\pi}{2M+2}$

- small x asymptotics:

$$y(x,E,\ell)\sim x^{\ell+1},x^{-\ell}$$

- The ODE is invariant under the rotation $x \to ax$, $E \to a^{2M}E$:

$$\left[\frac{1}{a^2}\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}\right)+a^{2M}(x^{2M}-E)\right]y(ax, a^{2M}E, \ell)=0$$

  for $a=\omega=\exp(\frac{2\pi i}{2M+2})$.

- Symanzik rotation of $y(x,E,\ell)$:

$$y_k(x,E,\ell)=\omega^{\frac{k}{2}}y(\omega^{-k}x,\omega^{2k}E,\ell)$$

  is also the solution of the ODE. ($k \in \mathbf{Z}$)

- y_k is subdominant in the sector $\mathcal{S}_k=\left\{x \middle| |\arg x-\frac{2k\pi}{2M+2}|<\frac{\pi}{2M+2}\right\}$

- $\{y_k, y_{k+1}\}$ forms a basis of the solutions.
- The Wronskian $W[f,g] := fg' - f'g$
  - If f, g are the solutions, then $W[f,g]$ is a constant, independent of x.
  - $W[f,g] = -W[g,f]$
  - $W_{k_1,k_2}(E,\ell) := W[y_{k_1}, y_{k_2}]$
- $W_{0,1} = 1$ (evaluated by asymptotic behaviour of y_0 and y_1)
- Periodicity (Symanzik rotation)

$$W_{k_1+1,k_2+1}(E,\ell) = W_{k_1,k_2}(\omega^2 E,\ell)$$

-

$$W_{k,k+1}(E,\ell) = 1$$

- $\{y_0, y_1\}$ are chosen as a fixed basis. We have the Stokes relation

$$y_k = -\frac{W_{1,k}}{W_{0,1}} y_0 + \frac{W_{0,k}}{W_{0,1}} y_1 = -T_{k-2}^{[k+1]} y_0 + T_{k-1}^{[k]} y_1$$

- T-function

$$T_s(E,\ell) := \left(\frac{W_{0,s+1}(E,\ell)}{W_{0,1}} \right)^{[-(s+1)]}, \quad (s \in \mathbf{Z})$$

$$f(E,\ell)^{[m]} := f(\omega^m E, \ell)$$

- The Plücker relation

$$W[y_{k_1}, y_{k_2}] W[y_{k_3}, y_{k_4}] = W[y_{k_1}, y_{k_4}] W[y_{k_3}, y_{k_2}] + W[y_{k_3}, y_{k_1}] W[y_{k_4}, y_{k_2}]$$

for $(k_1, k_2, k_3, k_4) = (1, s+2, 0, s+1)$ leads to the functional relation called the T-system

$$T_s^{[+1]} T_s^{[-1]} = T_{s-1} T_{s+1} + 1 \quad (s \in \mathbf{Z})$$

- The Y-functions: $Y_s = T_{s-1} T_{s+1}$ define the Y-system:

$$Y_s^{[+1]} Y_s^{[-1]} = (1 + Y_{s+1})(1 + Y_{s-1})$$

The boundary conditions for the T-system

- generic M and ℓ ($T_s \propto W_{0,s+1}$)

$$T_{-1} = 0, \quad T_0 = 1$$

  T_s ($s \geq 0$) are non-zero.

- $\ell = 0$, $2M + 2 = n \geq 4$: integer
  no singularity (monodromy) at $x = 0$

$$y_n(x) \propto y_0(e^{-2\pi i}x) = y_0(x)$$

  $T_{n-1} = 0 \Longrightarrow A_{n-2}$-type T-system

- $\ell \neq 0$, $2M + 2 = n \geq 4$: integer
  There is a monodromy around $z = 0$. $\Longrightarrow D_{n-2}$-type T-system

Baxter's T-Q relation

The basis of the ODE around $x = 0$

$$\psi_+(x,E,\ell) := x^{\ell+1} + \cdots, \quad \psi_-(x,E,\ell) := x^{-\ell} + \cdots$$

- Monodromy around the origin: $\psi_+(e^{2\pi i}x) = e^{2\pi i(\ell+1)}\psi_+(x)$
- Q-function: $Q_\pm(E,\ell) = W[y_0,\psi_\pm](E,\ell)$

$$W[y_k,\psi_\pm](E,\ell) = \omega^{\pm(\ell+\frac{1}{2})}W[y_0,\psi_\pm](\omega^{2k}E,\ell)$$

Baxter's T-Q relation: $-T_{-3}y_0 = y_{-1} + y_1$

$$(-T_{-3})Q_\pm(E,\ell) = \omega^{\mp(\ell+\frac{1}{2})}Q_\pm(\omega^{-2}E,\ell) + \omega^{\pm(\ell+\frac{1}{2})}Q_\pm(\omega^{2}E,\ell)$$

- $E = E_n$ such that $T(E_n,\ell) = 0$ (y_0, y_1 become linearly dependent)

$$\frac{Q_\pm(\omega^{-2}E_n,\ell)}{Q_\pm(\omega^{2}E_n,\ell)} = -\omega^{\pm(2\ell+1)}$$

Bethe ansatz equation of the twisted six-vertex model

Y-system and TBA equation

- [Zamoldchikov] The Y-system can be transformed into the non-linear integral equations (the Thermodynamic Bethe-ansatz (TBA) equation). $\epsilon_a(\theta) = \log Y_a(\theta)$: pseudo-energy $(E = \exp(2M\theta/(M+1)))$

$$\epsilon_a(\theta) = m_a L e^{\theta} - \sum_b \int_{-\infty}^{+\infty} \phi_{ab}(\theta - \theta') \log(1 + e^{-\epsilon_b(\theta')}) d\theta'$$

- free energy

$$F(L) = -\frac{1}{4\pi} \sum_a \int_{-\infty}^{+\infty} m_a e^{\theta} \log(1 + e^{-\epsilon_a(\theta)}) d\theta = -\frac{\pi c_{eff}}{6L}$$

- The UV effective central charge becomes

$$c_{eff}^{UV} = 1 - \frac{6\left(\ell + \frac{1}{2}\right)^2}{M+1}$$

which agrees with the one of the twisted six-vertex model.

Affine Toda Field Equations and ODE/IM correspondence

- two-dimensional affine Toda field Theory based on $\hat{\mathfrak{g}}$
- r-component scalar fields: $\phi(z,\bar{z}) = (\phi^1, \dots, \phi^r)$
- complex coordinates: $z = \frac{1}{2}(x^0 + ix^1), \quad \bar{z} = \frac{1}{2}(x^0 - ix^1)$ $(z = \rho e^{i\theta})$
- Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - \left(\frac{m}{\beta}\right)^2 \sum_{i=0}^{r} n_i \left[\exp(\beta\alpha_i \cdot \phi) - 1\right],$$

- $\alpha_1, \cdots, \alpha_r$: simple roots of $\mathfrak{g}$
  $\alpha_0 = -\theta = -\sum_{i=1}^r n_i\alpha_i$: highest root, $n_0 \equiv 1$
- affine Toda field equation:

$$\partial_z \partial_{\bar{z}} \phi + \left(\frac{m^2}{\beta}\right) \sum_{i=0}^{r} n_i \alpha_i \exp(\beta\alpha_i \phi) = 0\,.$$

modified affine Toda field equation

- without the potential term $e^{\beta\alpha_0\phi}$, the theory is conformally invariant (e.g. Liouville theory)
- with the potential term $e^{\beta\alpha_0\phi}$, it becomes massive theory. The equation of motion changes under the conformal transformation.

conformal transformation ($\rho^\vee$: co-Weyl vector)

$$z \to \tilde{z} = f(z)\,, \quad \phi \to \tilde{\phi} = \phi - \frac{1}{\beta}\rho^\vee \log(\partial f \bar{\partial}\bar{f})\,,$$

modified affine Toda equations:

$$\partial\bar{\partial}\phi + \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r} n_i\alpha_i \exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z}) n_0\alpha_0 \exp(\beta\alpha_0\phi)\right] = 0,$$

$$p(z) = (\partial f)^h, \quad \bar{p}(\bar{z}) = (\bar{\partial}\bar{f})^h.$$

Lax formalism

- The modified affine Toda equation can be expressed as the zero-curvature condition: $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \frac{\beta}{2}\partial\phi \cdot H + me^{\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + p(z)\sqrt{n_0^{\vee}}E_{\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\},$$

$$\bar{A} = -\frac{\beta}{2}\bar{\partial}\phi \cdot H - me^{-\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{-\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + \bar{p}(\bar{z})\sqrt{n_0^{\vee}}E_{-\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\}$$

λ: spectral parameter

- linear problem: $(\partial + A)\Psi = 0$ and $(\bar{\partial} + \bar{A})\Psi = 0$.
- gauge transformation: $\Psi \to U\Psi$, $A \to UAU^{-1} + U\partial U^{-1}$

Example: modified sinh-Gordon equation: $A_1^{(1)}$

modified Sinh-Gordon equation [Lukyanov-Zamolodchikov 1003.5333]

$$\partial_z \partial_{\bar{z}} \phi - e^{2\phi} + p(z)\bar{p}(\bar{z}) e^{-2\phi} = 0, \quad p(z) = z^{2M} - s^{2M}$$

- zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \begin{pmatrix} \frac{1}{2}\partial\phi & -e^{\lambda}e^{\phi} \\ p(z)e^{\lambda}e^{\phi} & -\frac{1}{2}\partial\phi \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{1}{2}\bar{\partial}\phi & -e^{-\lambda}e^{\phi} \\ \bar{p}(\bar{z})e^{-\lambda}e^{\phi} & \frac{1}{2}\bar{\partial}\phi \end{pmatrix}$$

- asymptotic behavior of $\phi(z, \bar{z})$ at $\rho \to 0, \infty$ $(z = \rho e^{i\theta})$
  - $\phi(\rho, \theta) \to M \log \rho$ $(\rho \to \infty)$
  - $\phi(\rho, \theta) \to \ell \log \rho$ $(\rho \to 0)$
    We can introduce a new parameter ℓ for the boundary condition at $\rho = 0$.

linear problem and its asymptotic solutions

- linear problem $(\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0$
- linear problem is invariant under
  Symanzik rotation Ω: $\theta \to \theta + \frac{\pi}{M}$, $\lambda \to \lambda - \frac{i\pi}{M}$
- At $\rho \to \infty$, the subdominant solution is

$$\Psi \sim \begin{pmatrix} e^{\frac{iM\theta}{2}} \\ e^{-\frac{iM\theta}{2}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1}\cosh(\lambda + i(M+1)\theta)\right)$$

- $\rho \to 0$ basis $\Psi_+(\rho,\theta|\lambda) \to \begin{pmatrix} 0 \\ e^{(i\theta+\lambda)\ell} \end{pmatrix}$, $\Psi_-(\rho,\theta|\lambda) \to \begin{pmatrix} e^{(i\theta+\lambda)\ell} \\ 0 \end{pmatrix}$
-

$$\Psi = Q_-(\lambda)\Psi_+ + Q_+(\lambda)\Psi_-$$

  $Q_\pm(\lambda)$ defines the Q-function satisfying the Bethe ansatz equation.
- T-functions and the Y-functions are also defined. They satisfy the D-type Y-system.

From MShG to ODE

- Take the light-cone limit $\bar{z} \to 0$. The linear system reduced to a holomorphic differential equation. $(\partial + A_z)\Psi = 0$.
- Under the gauge transformation by $U = diag(e^{\phi}, e^{-\phi})$, it becomes

$$(\partial_z + \tilde{A}_z)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \tilde{A}_z = \begin{pmatrix} \partial\phi & e^{\lambda} \\ p(z)e^{\lambda} & -\partial\phi \end{pmatrix}$$

- linear system $\Longrightarrow$ ODE (Miura transformation)

$$\Big[(\partial_z - \partial_z\phi)(\partial_z + \partial_z\phi) - e^{2\lambda}p(z)\Big]\psi_1(z) = 0$$

- conformal limit:
  $z \to 0$, $\lambda \to \infty$ with fixed $x = ze^{\frac{\lambda}{M+1}}, E = s^{2M}e^{\frac{2\lambda M}{1+M}}$, $\phi \sim \ell \log x$

$$\left[-(\partial_x - \frac{\ell}{x})(\partial_x + \frac{\ell}{x}) + x^{2M}\right]\psi = \left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2M}\right]\psi = E\psi$$

  This is the ODE of [Dorey-Tateo, BLZ]

ODE/IM correspondence for $\hat{\mathfrak{g}}^\vee$ modified affine Toda field equations

ODE	$\Longleftrightarrow$ ODE/IM	BAE $U_q(\hat{\mathfrak{g}})$	$\Longleftrightarrow$	CFT
$\Uparrow$ Conformal limit		$\Uparrow$ UV limit		$\Uparrow$
Linear problem of modified affine Toda equation $\hat{\mathfrak{g}}^\vee$ with $p(z) = z^{hM} - s^{hM}$	$\Longleftrightarrow$ massive ODE/IM	BAE $U_q(\hat{\mathfrak{g}})$	$\Longleftrightarrow$	massive IM

Langlands Duality: [Masoero-Raimondo-Valeri, KI-Locke] The modified affine Toda equation for the Langlands dual $\hat{\mathfrak{g}}^\vee$ corresponds to the $\mathfrak{g}$-type Bethe ansatz equation [Reshetikhin-Wiegmann, Kuniba-Suzuki].

- $\hat{\mathfrak{g}}^\vee = \hat{\mathfrak{g}}$ for $\hat{\mathfrak{g}} = A_r^{(1)}, D_r^{(1)}, E_r^{(1)},\ A_{2r}^{(2)}$
- $(B_r^{(1)})^\vee = A_{2r-1}^{(2)},\ (C_r^{(1)})^\vee = D_{r+1}^{(2)},\ (F_4^{(1)})^\vee = E_6^{(2)},\ (G_2^{(1)})^\vee = D_4^{(3)}$

ODE for affine Lie algebras

[KI-Locke,1312.6759] (ABCD type:
[Dorey-Dunning-Masoero-Suzuki-Tateo 2006])

$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^{\lambda})^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^{\lambda})^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r}(me^{\lambda})^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^{\lambda})^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^{\lambda})^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^{\lambda})^{2h} p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^{r}\sqrt{2}(me^{\lambda})^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^{\lambda})^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^{\lambda})^4 D(\mathbf{h}^\dagger)p(z)$ $-(\omega+1)2\sqrt{3}(me^{\lambda})^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^{\lambda})^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1)$ $+(\omega-1)^3 12(me^{\lambda})^8 p\partial^{-1}p\}\psi = 0$

$D(h) := \partial + \beta h\cdot\partial\phi$
$D(\mathbf{h}) = D(h_r)\cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1)\cdots D(-h_r)$ for $\mathbf{h} = (h_r,\cdots,h_1)$ set of weight vectors for the fundamental representation of $\mathfrak{g}$.

ODE/IM correspondence and the Argyres-Douglas Theory

The Argyres-Douglas Theory
[Argyres-Douglas 1995, Argyres-Plesser-Seiberg-Witten, Eguchi-Hori-KI-Yang]

- strongly coupled N=2 SCFT in four dimensions
- mutually non-local monopole and dyons are both massless
- no microscopic Lagrangian description
- The curve of the AD theory is realized by degeneration of the SW curve
  ex. $SU(3)$ SW curve: genus two Riemann surface $\Longrightarrow$small+big torus

$G = ADE$ type SW theory:

- The SW curve is the spectral curve of periodic affine Toda lattice based on $(G^{(1)})^\vee$. [Gorsky et al. , Martinec-Warner]

$$z + \frac{\mu^2}{z} = W_G(x, u_1, \cdots, u_r), \quad \lambda_{SW} = x\frac{dz}{z}$$

$W_{A_r} = x^{r+1} - u_2 x^{r-1} - \ldots - u_{r+1}$

- The AD point of the theory is realized at

$$u_1 = \ldots = u_{r-1} = 0, \quad u_r = \pm 2\mu$$

- rescaling the variables (q_i: exponents of G), $\epsilon \to 0$

$$u_i = \epsilon^{q_i}\rho_i (i = 1, \ldots, r-1), \quad u_r = +2\mu + \epsilon^{q_r}$$

The SW curve of the AD theory (also rescaling of x, z)

$$\xi^2 = W(x, \rho_1, \ldots, \rho_{r-1}, 1), \quad \lambda_{SW} = \xi dx$$

$\xi = \sqrt{z} + \frac{\mu}{\sqrt{z}}$

Classification of AD theories

- periodic Toda lattice (A_1, G):
  The SW curve : $x^2 = W_G(z, u_i)$
- hypersurface singularity in the type IIB setup.
  - (G, G') AD theory [Cecotti-Neitzke-Vafa 1006.3435]

$$f_G(x_1, x_2) + f_{G'}(x_3, x_4) = 0$$

$$f_{A_r}(x, y) = x^2 + y^{r+1},\ f_{D_r} = x^{r-1} + xy^2,\ f_{E_6} = x^3 + y^4$$

  - Duality: $(G, G') \sim (G', G)$, $(A_1, E_6) \sim (A_2, A_3)$, $(A_1, E_8) \sim (A_2, A_4)$
- Hitchin system with irregular singularity $\mathfrak{g}^{(b)}[k]$ [Xie, Wang-Xie]
  For $b = h$ (Coxeter number)

$$\mathfrak{g}^{(h)}[k] = (\mathfrak{g}, A_{k-1})$$

2d/4d correspondence

- 4d N=2 central charge↔ central charge of 2d chiral algebra [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees 1312.5344]

$$c_{4d} = -\frac{1}{12} c_{2d}$$

- Schur limit of 4d superconformal index =vacuum character of 2d chiral algebra
[Cordova-Shao 1506.00265]
- For the AD theory $\mathfrak{g}^{(b)}[k]$ the corresponding 2d theory is

$$\mathcal{A} = \frac{\mathfrak{g}_\ell \times \mathfrak{g}_1}{\mathfrak{g}_{\ell+1}}, \quad \ell = -\frac{kh-b}{k}$$

which is the $W\mathfrak{g}(p', p) = W\mathfrak{g}(b+k, b)$ minimal model. [Xie 1204.2270, Wang-Xie 1509.00847, Xie-Yan-Yau, 1604.02155]

SW theory in the NS limit of the Ω-bakground

- Let us consider $\mathcal{N} = 2$ theory in the Nekrasov-Sahashvili limit ($\epsilon_2 \to 0$) of the Ω-background. ($\epsilon_1 =: \epsilon$)
- the SW differential $\lambda = x\frac{dz}{z} = x d\xi$ ($\xi = \log z$) defines the symplectic structure

$$d\lambda = dx \wedge d\xi$$

In the NS background, the ϵ induces the quantization condition:

$$\{x, \xi\} = 1 \quad \Longrightarrow [\hat{x}, \hat{\xi}] = i\epsilon$$

- The quantum spectral curve [Mironov-Morozov, ...]

$$x^2 - u - z - \frac{1}{z} = 0 \Longrightarrow (-\epsilon^2 \partial_\xi^2 - u - 2\cosh\xi)\psi(\xi) = 0,$$

$\log\psi(\xi) = \frac{1}{\epsilon}\int^{\xi}\lambda + \cdots$: deformed period $\to$ Nekrasov partition function in the NS limit

AD theory in the NS limit of the Ω-bakground

- quantum spectral curve for AD theories

$$\xi^2 = W_G(x,\rho), \quad \xi \to \epsilon\partial_x$$

- quantization: the SW differential $\lambda = \xi dx$ defines the symplectic structure

$$d\lambda = d\xi \wedge dx$$

$$\{\xi, x\} = 1 \quad \Longrightarrow [\hat{\xi}, \hat{x}] = i\epsilon$$

- quantum SW curve

$$\xi^2 = x^{r+1} + \cdots \to (-\epsilon^2\partial_x^2 + x^{r+1} + \cdots)\psi(x) = 0$$

ODE/IM correspondence and AD theory

- We consider the simplest example. The AD_2 curve

$$x^2 = z^2 + 2a, \quad \lambda = xdz$$

the periods $(Z_e, Z_m) = (\int_{\gamma_e} \lambda, \int_{\gamma_m} \lambda) = 2\pi i(a, a_D)$.

- We compactify the theory on S^1 with radius R. Then its moduli space has a hyper-Kähler structure parametrized by $\zeta \in CP^1$. The coordinates of the moduli space are

$$(X_e, X_m) = (\exp(\frac{RZ_e}{\zeta} + R\bar{Z}_e\zeta) + \cdots, \exp(\frac{RZ_m}{\zeta} + R\bar{Z}_m\zeta) + \cdots)$$

- The dual period X_m jumps along the positive or negative a/ζ-axis. Its dicontinuity is captured by the A_1-type TBA equations [Gaiotto-Moore-Neitzke]

- The conformal limit $R \to 0, \zeta \to 0$ with fixed $\epsilon = \frac{R}{\zeta}$
  $X_m = \exp(\frac{2\pi i Z_m}{\epsilon})$ satisfies the massless A_1-type TBA equation.

$$\log X_m = \frac{Z_m}{\epsilon} + \frac{\epsilon}{\pi i} \int_{\ell_{-\gamma_e}} \frac{d\epsilon'}{(\epsilon')^2 - \epsilon^2} \log(1 + e^{\frac{-2\pi i a}{\epsilon'}})$$

- Gaitto [1403.6137] has shown that the TBA system for AD_2 model can be obtained from the oper

$$x^2 = z^2 + 2a \xrightarrow[x \to \epsilon \partial_z]{} (-\epsilon^2 \partial_z^2 + z^2 + 2a)\psi(z) = 0$$

- $z = \sqrt{\epsilon} x$ and $2a = -\epsilon E$

$$(-\partial_x^2 + x^2 - E)\psi(E) = 0$$

- the dual coordinate X_m is identified with the T-function

$$X_m \sim T(E)$$

- [Cecotti-del Zotto-Vafa, 1006.4708] [Cecotti-del Zotto 1403.7613] From the BPS spectrum analysis of the AD theory, one finds the the ADE type Y-system [Zamoldchikov] appears.
- Q: What is the quantum spectral curve for the ADE-type AD theory?
- The light-cone limit of the linear problem associated with the modified affine Toda field equation for $\hat{\mathfrak{g}} = ADE$ gives the first order linear-differential equation:

$$[\partial_z + A_z]\psi(z) = 0$$

We propose that this can be regarded as the quantum spectral curve of the AD theory of $(A_1, \mathfrak{g})$ type for $p(z) = z^2 - E$.

For $G = A_r$-type, we have the ODE

$$(\partial^{r+1} - z^2 + E)\psi_1(z) = 0$$

- Using the ODE/IM correspondence, we can derive the T and Y-system of the A_r-type [CZV]. One can compute the effective central charge from the TBA equations [CS].
- The BAE leads to the NLIE equations [Dorey-Tateo]. We can compute the central charge.
- Spectral duality $(A_1, A_r) \leftrightarrow (A_r, A_1)$ can be seen from the Fourier transformation.

$$((ik)^{r+1} - \frac{\partial^2}{\partial k^2} + E)\tilde{\psi}_1(k) = 0$$

- Dorey-Dunning-Tateo [0712.2010] argued that he ODE

$$(-\partial_x^2 + x^{2M} - E)\psi(x, E) = 0$$

corresponds to the non-unitary minimal model $M_{2,2M+2}$ with central charge

$$c = 1 - \frac{6M^2}{M+1}$$

perturbed by the operator $\phi_{1,M}$ with conformal dimension

$$\Delta_{1,M} = \frac{1-4M^2}{16(M+1)}$$

Note that for general ℓ, the effective central charge becomes $c_{eff} = c - 24\Delta = 1 - \frac{6(\ell+\frac{1}{2})^2}{M+1}$

- The $M_{2,M}$ is realized by the fractional coset CFT:

$$\frac{su(2)_L \times su(2)_1}{su(2)_{L+1}}, \quad L = \frac{1}{M} - 2$$

For the ODE and the BAE, Dorey et al. suggest that the ODE for $\mathfrak{g} = ABCD$ with

$$p(x) = x^{h^\vee M} - E$$

corresponds to the coset model

$$\frac{\hat{\mathfrak{g}}_L \times \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{L+1}} = W\mathfrak{g}(h^\vee M + h^\vee, h^\vee), \quad L = \frac{1}{M} - h^\vee$$

perturbed by the certain relevant operator.
The central charges agree with those predicted by the 2d/4d correspondence. [Cordova-Shao, Xie-Yan-Yau 1604.02155]

Conclusions and Outlook

- The ODE/IM correspondence between affine Toda field equations $\hat{\mathfrak{g}}^\vee$ and the $\hat{\mathfrak{g}}$ integrable models. (Langlands duality)
- The ODE/IM correspondence describes the relation between 4d SCFT (quantum spectral curve) and the related 2d conformal field theories (non-unitary W-minimal model).
- D_r and E_r spectral curve and the AD theory work in progress
- The ODE/IM cooorespondence would be useful to compute the non-perturbative correction to the deformed prepotential at strong coupling from the integrable model point of view.