Chiral Magnetic Effect with Wigner Functions

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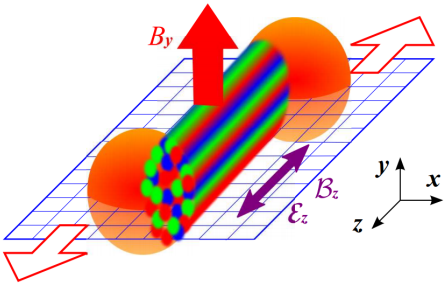
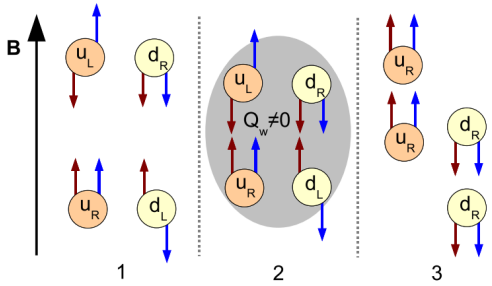
Chiral Magnetic Effect

Non-central collisions in heavy ion collisions create an unexpected electric current orthogonal to the collision plane:



- Background: very strong B field due to highly charged nuclei passing near each other.
- Gauge field: QCD gluons

Chiral Magnetic Effect



D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008).

1. Neutral mixture in very strong B field:
   particles can only move parallell to the magnetic field based on their chirality and spin.
2. Gauge interaction with the gluon fields change chirality.
3. Chirality separation leads to charge separation, that leads to electric current.

Chiral Magnetic Effect

- The problem can be simplified to a Quantum Electrodynamical (QED) model, where the E_z, B_z components are coming from the gluonic fields and where B_y is the classical magnetic field.

- Idea by Kenji Fukushima, Dmitri E. Kharzeev, and Harmen J. Warringa Phys. Rev. Lett. 104, 212001 2010.

- How to model such a QED system under the influence of strong fields?

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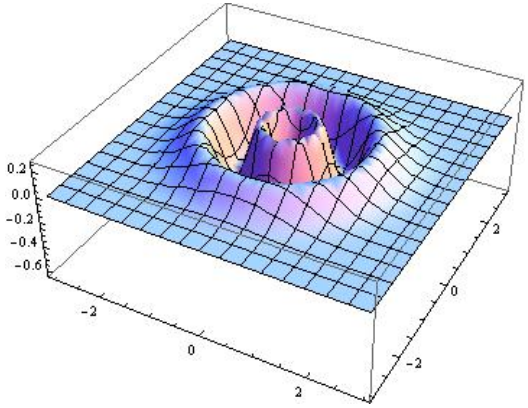
Wigner function

Tool of description: the Wigner function:

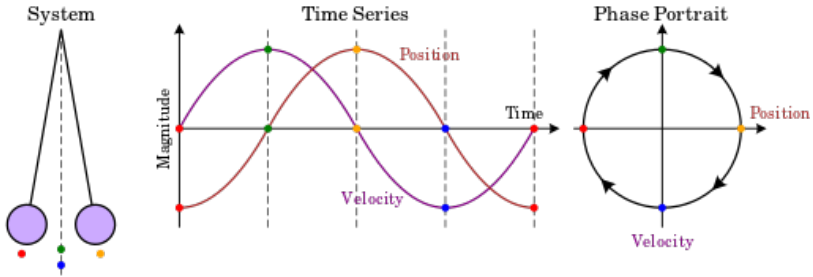
- Quantum analogue of the classical phase space distribution.



Wigner function of an n=3 Fock state.

Wigner function

What is a classical phase space?



- A space spanned by the possible positions and velocities of a particle

What is a classical phase space distribution?

- When we can define a probability distribution over the phase space

Wigner function

Now in quantum physics, the Heisenberg uncertainty makes it impossible to know the position and momentum of a particle at the same time.

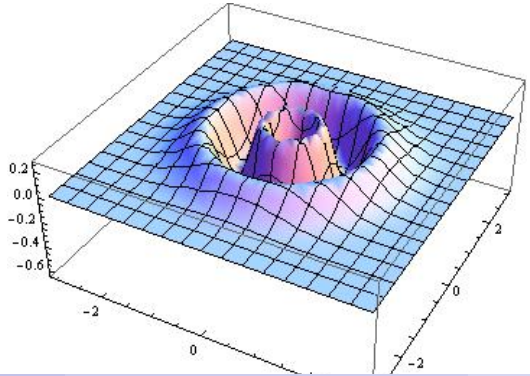
$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$

How to have a phase space then?

Wigner function

If we sacrafice strict non-negativity, we can have a quasi-probability distribution, describing the quantum phenomena, but can be projected to only spatial or only momentum space where we can have a proper probability measure.

This is the Wigner-function.



Wigner function

How it is defined?

- Take the equal time density matrix in terms of 'center of mass' coordinates:

$$\hat{\rho}(\vec{x},\vec{s},t) = e^{-ig\int_{-1/2}^{1/2}\vec{\mathcal{A}}(\vec{x}+\lambda\vec{s},t)\vec{s}\mathrm{d}\lambda}\left[\Psi(\vec{x}+\frac{\vec{s}}{2},t),\bar{\Psi}(\vec{x}-\frac{\vec{s}}{2},t)\right] \tag{2}$$

- Take the expectation value.
- Fourier transform it w.r.t the coordinate difference:

$$W(\vec{x},\vec{p},t) = -\frac{1}{2}\int e^{-i\vec{p}\vec{s}}\langle\Omega|\hat{\rho}(\vec{x},\vec{s},t)|\Omega\rangle\mathrm{d}^3s \tag{3}$$

Wigner function

When describing (relativistic) quantum systems under the influence of electromagnetic fields, the Wigner function has 16 components, some are intuitive:

- mass
- charge
- electric current
- axial charge (net chirality)
- axial current (chirality flow)

Wigner function

We are interested in the time evolution:

- Start with a known wigner function (e.g. vacuum)
- Apply external fields
- What is the resulting phase space distribution?

Equations of motion for the Wigner function

$$
\begin{aligned}
&D_t \mathbb{s} && && - && 2\vec{P}\cdot\vec{\mathbb{t}}_{\mathbb{1}} && = 0 && (4)\\
&D_t \mathbb{p} && && + && 2\vec{P}\cdot\vec{\mathbb{t}}_{2} && = 2m\mathbb{a}_0 && (5)\\
&D_t \mathbb{v}_0 && + && \vec{D}_{\vec{x}}\cdot\vec{\mathbb{v}} && && = 0 && (6)\\
&D_t \mathbb{a}_0 && + && \vec{D}_{\vec{x}}\cdot\vec{\mathbb{a}} && && = 2m\mathbb{p} && (7)\\
&D_t \vec{\mathbb{v}} && + && \vec{D}_{\vec{x}}\mathbb{v}_0 && + && 2\vec{P}\times\vec{\mathbb{a}} && = -2m\vec{\mathbb{t}}_{\mathbb{1}} && (8)\\
&D_t \vec{\mathbb{a}} && + && \vec{D}_{\vec{x}}\mathbb{a}_0 && + && 2\vec{P}\times\vec{\mathbb{v}} && = 0 && (9)\\
&D_t \vec{\mathbb{t}}_{\mathbb{1}} && + && \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_{2} && + && 2\vec{P}\mathbb{s} && = 2m\vec{\mathbb{v}} && (10)\\
&D_t \vec{\mathbb{t}}_{2} && - && \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_{\mathbb{1}} && - && 2\vec{P}\mathbb{p} && = 0 && (11)
\end{aligned}
$$

Wigner function

The equation has the following non-local differential operators:

$$D_t = \partial_t + g\vec{\mathcal{E}}(\vec{x},t)\vec{\nabla}_{\vec{p}} - \frac{g\hbar^2}{12}(\vec{\nabla}_{\vec{x}}\vec{\nabla}_{\vec{p}})^2\vec{\mathcal{E}}(\vec{x},t)\vec{\nabla}_{\vec{p}} + ... \tag{12}$$

$$\vec{D}_{\vec{x}} = \vec{\nabla}_{\vec{x}} + g\vec{\mathcal{B}}(\vec{x},t)\times\vec{\nabla}_{\vec{p}} - \frac{g\hbar^2}{12}(\vec{\nabla}_{\vec{x}}\vec{\nabla}_{\vec{p}})^2\vec{\mathcal{B}}(\vec{x},t)\times\vec{\nabla}_{\vec{p}} + ... \tag{13}$$

$$\vec{P} = \vec{p} + \frac{g\hbar}{12}(\vec{\nabla}_{\vec{x}}\vec{\nabla}_{\vec{p}})\vec{\mathcal{B}}(\vec{x},t)\times\vec{\nabla}_{\vec{p}} + ... \tag{14}$$

Vacuum initial conditions:

$$\mathbb{S} = -\frac{2m}{\omega}, \qquad \vec{\mathbb{V}} = -\frac{2\vec{p}}{\omega} \tag{15}$$

with $\omega = \sqrt{m^2+\vec{p}^2}$, and all other components are zero.

Simplification

We usually simplify things:

- Choose a special field configuration (e.g. position indepedent)
- Set some field components to zero
- Truncate the differential equation series
- Set mass to zero (only 8 components remain)

Equations of motion for the $m = 0$ spin-1/2 Wigner function

A system for 8 unknown real functions remains:

$$
\begin{aligned}
&D_t \mathbb{v}_0 && + && \vec{D}_{\vec{x}} \cdot \vec{\mathbb{v}} && && = 0 && (16)\\
&D_t \mathbb{a}_0 && + && \vec{D}_{\vec{x}} \cdot \vec{\mathbb{a}} && && = 0 && (17)\\
&D_t \vec{\mathbb{v}} && + && \vec{D}_{\vec{x}} \mathbb{v}_0 && + && 2\vec{P} \times \vec{\mathbb{a}} && = 0 && (18)\\
&D_t \vec{\mathbb{a}} && + && \vec{D}_{\vec{x}} \mathbb{a}_0 && + && 2\vec{P} \times \vec{\mathbb{v}} && = 0 && (19)
\end{aligned}
$$

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Numerical Solution

How to solve such a complicated differential equation system?

We know the unknown functions are analythic, so lets choose a basis set $\Phi_i(x)$ accordingly and expand any unknown functions over it:

$$f(x) \approx \sum_i a_i \Phi_i(x) \tag{20}$$

In the generic technique is known as pseudospectral collocation, we require the error to vanish at a finite number of points coming with the basis set:

$$f(x_j) \approx \sum_i a_i \Phi_i(x_j) = \sum_i a_i \Phi_{ij} \tag{21}$$

Thus we arrive at dense linear operations.

Numerical Solution

Dense linear operations?

- Elementwise additions, scalar multiplications
- Tensor transformation by matrices
- Modified transformations
- Large weighted Summations

Ideal for GPUs!
We can easily write a few kernels for them, e.g. in OpenCL.

Numerical Solution

Where does the dense linear operations come from?

Consider the expression, where f is the unknown function:

$$\alpha f(x) + \beta \partial_x f(x) + \gamma g(x) + \delta h(x) f(x) \tag{22}$$

After expansion $f(x_j) \approx \sum_i a_i \Phi_{ij}$:

$$\alpha \sum a_i \Phi_{ij} + \beta \sum a_i \Phi'_{ij} + \gamma \sum g(x_j) + \delta \sum a_i \Phi_{ij} g(x_j) \tag{23}$$

where $\Phi'_{ij} = \partial_x \Phi_i(x_j)$

Numerical Solution

In multiple dimensions the coefficients get larger:

$f(x, y) \approx \sum_i \sum_j a_{ij} \Phi_i(x) \Psi_j(y)$

After collocation x_k, y_l it turns into:

$$f_{kl} = \sum_i \sum_j a_i j \Phi_{ik} \Psi_{jl} \tag{24}$$

and so on...

C++ representation

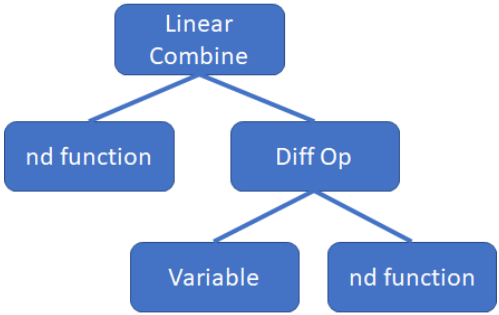
The expressions in the Wigner equations are quite complex, we don't want to write the equations manually and recalculate every formula when we change the fields or the simplifications...

Use expression trees in C++ to help!

Symbolic operations in the tree:

- Linear combination
- Summations
- Coordinate multiplications
- Differential operators

Leaf: n dimensional functions, variable symbols, constants



C++ representation

We also want to symbolicaly simplify things in the expressions (especially zeros), that can be easily done using some simple type tricks:

```
struct Null{};
```

C++ representation

We also want to symbolicaly simplify things in the expressions, that can be easily done using some simple type tricks:

```
struct Null{};

Null operator+ (Null const&, Null const&){ return Null(); }
Null operator- (Null const&, Null const&){ return Null(); }
Null operator* (Null const&, Null const&){ return Null(); }

template<typename T>
Null operator/ (Null const&, T const&){ return Null(); }
template<typename T>
Null operator* (Null const&, T const&){ return Null(); }
template<typename T>
Null operator* (T const&, Null const&){ return Null(); }
```

Numerical Solution

Next, we make all vectors and Wigner function states a tuple:

```
template<typename X, typename Y, typename Z>
struct Vector
{
        X x; Y y; Z z;

        template<typename X1, typename Y1, typename Z1>
        auto& operator= ( Vector<X1, Y1, Z1> const& v )
        {
                x = v.x; y = v.y; z = v.z;
                return *this;
        }
};
```

And write the usual operators accordingly.

Numerical Solution

At the end, we can write code like this:

```
double By, Bz, Ez;
Diff dx{'x', 1}; //Differential operator in x
Diff dy{'y', 1};
Diff dz{'z', 1};

//Momentum coordinate vector:
auto p0 = make_vector( Coord{'x'}, Coord{'y'}, Coord{'z'} );

//Vector crossed with the differential operator vector:
auto DB = make_vector(Null(), By, Bz) % nabla(dx, dy, dz);

auto p = p0 + make_vector(Null(), Null(), Ez);

wigner<...> w;

make_wigner( /*V0*/-1*(DB | w.v), /*A0*/-1*(DB | w.a),
             /*V*/ -2*(p  % w.a) - (DB * w.v0),
             /*A*/ -2*(p  % w.v) - (DB * w.a0) );
```

Numerical Solution

For the time stepping we can use a simple 4th order Runge-Kutta, and repeatedly evaluate the r.h.s. expressions containing the derivates.
From time to time, we integrate the quantities, and write out into a file.

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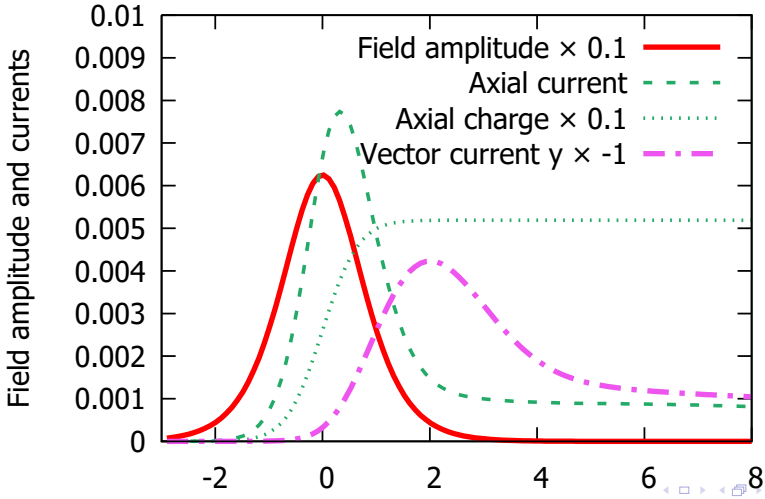
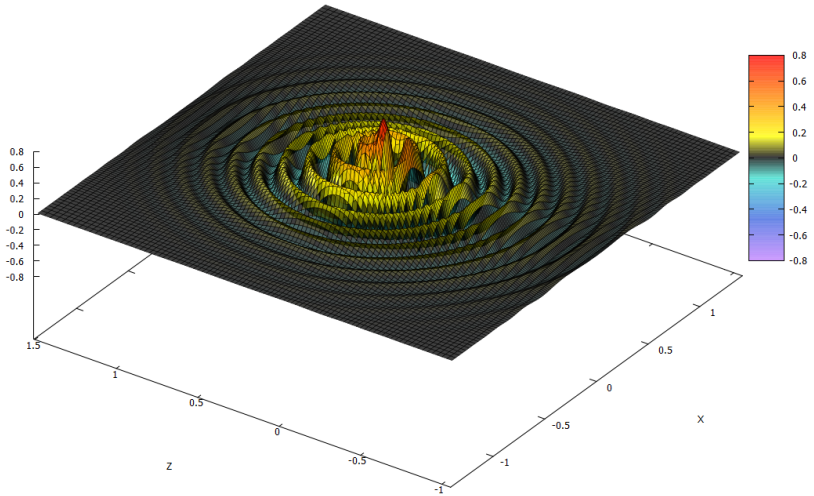
Results

Chiral Magnetic Current formation during the interaction:



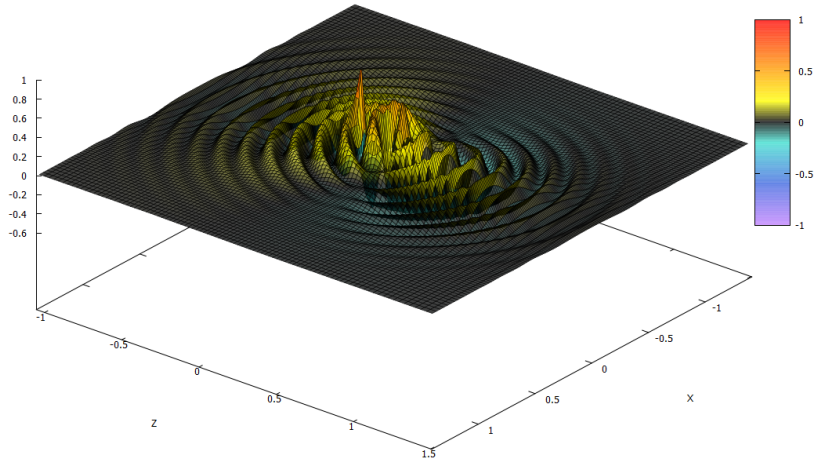
Results

The Anomalous component of the electric current:



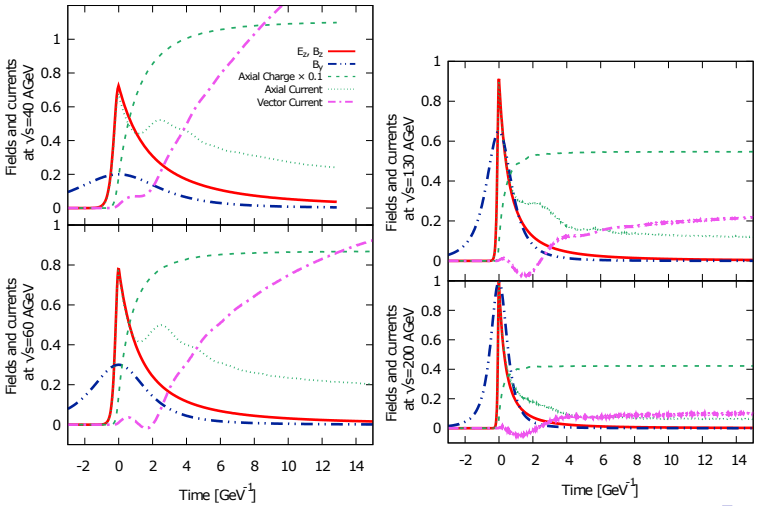
Results

The Axial current in X direction:



Results

Predictions for different collision energies:



Summary

- The Wigner function formalism can be used to model the Chiral Magnetic Effect in heavy-ion collisions.
- The complicated equation systems can be broken down into a few dense kernel operations, and even symbolic simplifications can be carried out by C++ type system.
- The resulting speed-up is around 30x w.r.t. a single threaded CPU version.

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