Beyond the Exponential Statistical Factor

Academia Europaea, Physics and Technology Section

T.S. Biró

H.A.S. Wigner Research Centre for Physics, Budapest

Budapest, Sep.3. 2017.

Abstract

I present my personal view on non-extensive thermodynamics, where generalized canonical statistical factors replace the exponential function. By generalizing the familiar factor, $\exp(-E/T)$, in counting the relative occurence frequency of states with energy E at temperature T one encounters primarily mathematical challenges. However, it is of equal importance to built up the formalism on physical phenomena, with an ample number of particular examples from the physical world. This motivates to show simple theoretical problems first and then gradually generalize. At the end alternative entropy formulas are presented.

Phase space and entropy

Boltzmann: $S \sim \log \Omega$; Einstein: $\Omega \sim e^S$

Phase-space volume with total energy, E, and n particles in some dimensions.

hypervolume of n-ball in L_p-norm with radius $R(E)$ and coordinates x_i:

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq R(E), \qquad \Omega_n^{(p)}(R) = \frac{\Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)} (2R)^n. \qquad (1)$$

abs $\Omega_n^{(1)}(E) = \frac{1}{n!}(2E)^n$ — 1-dim jet light particles

euc $\Omega_{2n}^{(2)}(\sqrt{2mE}) = \frac{1}{n!}(2\pi mE)^n$ — 2-dim massive particles

max $\Omega_n^{(\infty)}(E) = (2E)^n$ — n-dim hypercube

Phase space volume ratios

Basis of thermodynamics

Picking up 1 particle with energy ϵ. Statistical factor is based on the ratio

$$r_n = \frac{\Omega_1(\epsilon)\,\Omega_{n-1}(E-\epsilon)}{\Omega_n(E)}. \tag{2}$$

abs $r_n^{(1)}(E) = \frac{\epsilon}{E}\, n\left(1-\frac{\epsilon}{E}\right)^{n-1}$ 1-dim jet light particles

euc $r_{2n}^{(2)}(\sqrt{2mE}) = \frac{\epsilon}{E}\, n\left(1-\frac{\epsilon}{E}\right)^{n-1}$ 2-dim massive particles

max $r_n^{(\infty)}(E) = \frac{\epsilon}{E}\left(1-\frac{\epsilon}{E}\right)^{n-1}$ n-dim hypercube

Boltzmann–Gibbs statistical factor

the origin

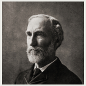
When picking up a single degree of freedom with energy ϵ, the phase space fraction for this partition in an 1-dim relativistic or 2-dim nonrelativistic ideal gas is given by

$$r_n^{(1)} = r_{2n}^{(2)} = -\epsilon \frac{\partial}{\partial \epsilon} \left(1 - \frac{\epsilon}{E}\right)^n . \tag{3}$$

Base statistical factor

$$\rho_{n,E}(\epsilon) = (1 - \epsilon/E)^n$$

In the textbook limit, $n \to \infty$ and $E \to \infty$ while $E/n = T$ constant it delivers the Boltzmann–Gibbs factor:



$$\overline{\rho}_{Gibbs} = \lim_{n \to \infty} \rho_{n,E}(\epsilon) = e^{-\epsilon/T} . \tag{4}$$

However

there are other ways for getting exponential

Consider a fluctuating number of dimensionality of the phase space at fixed total energy, with the probability P_n.

$$\overline{\rho} := \langle \rho_{n,E}(\epsilon) \rangle \;=\; \sum_{n=0}^{\infty} P_n \,(1 - \epsilon/E)^n \,. \tag{5}$$

No "thermodynamical limit" is taken.

Poissonian distribution leads to

$$\overline{\rho}_{\mathrm{POI}} := \sum_{n=0}^{\infty} \frac{\langle n \rangle^n}{n!} \mathrm{e}^{-\langle n \rangle} \,(1 - \epsilon/E)^n \;=\; \mathrm{e}^{-\langle n \rangle\, \epsilon/E} . \tag{6}$$

Boltzmann–Gibbs factor with $T = E/\langle n \rangle$.

valid even for $\langle n \rangle < 1$!

Negative Binomial Distribution

Beyond the exponential factor

$$P_n = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \tag{7}$$

In this case $\langle n \rangle = f(k+1)$ and we obtain

$$\overline{\rho}_{\text{NBD}} = \left(1 + \frac{\langle n \rangle}{k+1} \frac{\epsilon}{E}\right)^{-k-1}, \tag{8}$$

a **Tsallis–Pareto** distribution with the temperature parameter $T = E/\langle n \rangle$ (again) and the power law tail with the negative power $(k+1)$. Previous result: $k \to \infty$.

General P_n

Gibbs, Pareto, etc. are approximations

Demanding the approximate equality

$$\overline{\rho}(\epsilon) = \sum_{n=0}^{\infty} P_n \left(1-\epsilon/E\right)^n \approx \left(1+(q-1)\frac{\epsilon}{T}\right)^{-\frac{1}{q-1}}, \tag{9}$$

expand for $\epsilon \ll E$ both sides. Linear and quadratic terms deliver

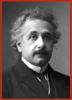
$$\frac{\langle n \rangle}{E} = \frac{1}{T}, \qquad \text{and} \qquad \frac{\langle n(n-1) \rangle}{E^2} = \frac{q}{T^2}. \tag{10}$$

the meaning of q: second scaled factorial moment ☺

$$q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}. \tag{11}$$

General Phase Space

$\Omega(E) = \mathrm{e}^{S(E)}$



$$\overline{\rho}(\epsilon) \;=\; \left\langle \mathrm{e}^{S(E-\epsilon)-S(E)} \right\rangle . \tag{12}$$

Expand and compare:

$$\left\langle 1 - \epsilon S'(E) + \frac{\epsilon^2}{2}\left(S''(E) + S'(E)^2\right) + \ldots \right\rangle = 1 - \frac{\epsilon}{T} + \frac{q\,\epsilon^2}{2T^2} + \ldots \tag{13}$$

Identify leading terms:

$$\frac{1}{T} \;=\; \langle S'(E) \rangle \qquad \text{and} \qquad q \;=\; \frac{\left\langle S''(E) + S'(E)^2 \right\rangle}{\langle S'(E) \rangle^2} . \tag{14}$$

$\boldsymbol{q \;=\; 1 - 1/C + \Delta\beta^2 / \langle \beta \rangle^2}$

Non-additivity $\rightarrow$ entropy formula

for general associative rules

The general composition rule

$$S_{12} = S_1 \oplus S_2 = h(S_1, S_2),$$

if associative,

$$K(S_{12}) \; = \; K(S_1) \, + \, K(S_2),$$

shows also the way to the deformation of the entropy.

deformed entropy is additive

$$K(S) \; := \; \sum_i p_i K(-\ln p_i). \tag{15}$$

Canonical factor using deformed entropy

$$\overline{\rho}_K(\epsilon) = \left\langle \mathrm{e}^{K(S(E-\epsilon))-K(S(E))} \right\rangle \tag{16}$$

From linear and quadratic term coefficients compared to the expansion of Pareto:

$$q_K = \left[1 + \frac{K''}{K'^2}\right]\left(1 + T^2 \Delta\beta^2\right) - \frac{1}{C}\,\frac{1}{K'}. \tag{17}$$

Requiring now $q_K = 1$, since we want to use exactly that $K(S)$ deformation which results in an additive system, we obtain a differential equation determining the sought $K(S)$.

Solution of $q_K = 1$

for S-independent C and $T^2\Delta\beta^2$

With $h_\alpha(x) := (e^{\alpha x} - 1)/\alpha$:

$$K(S) = h_\lambda^{-1}\left(h_\mu(S)\right) \quad \text{and} \quad K^{-1}(\sigma) = h_\mu^{-1}\left(h_\lambda(\sigma)\right). \qquad (18)$$

with $\lambda = \frac{\Delta\beta^2}{1/T^2+\Delta\beta^2}$ and $\mu = (1-\lambda)/C$.

additive entropy for non-additive $\ln p_i$

$$S^{\mathrm{add}}_{\lambda,\mu} = \frac{1}{\lambda}\sum_i p_i \ln\left[1 + \frac{\lambda}{\mu}\left(p_i^{-\mu} - 1\right)\right]. \qquad (19)$$

The Tsallis–Pareto power index is related to the parameters of the more general entropy formula as $q = \frac{1-\mu}{1-\lambda}$.

Particular entropy formulas

corresponding to different assumptions

1. $q = 1$: $\qquad (\Delta\beta / \langle\beta\rangle = 1/\sqrt{C})$

$$S_{\lambda=\mu} = -\sum_i p_i \ln p_i .$$

2. $q \leq 1$: $\qquad (\Delta\beta / \langle\beta\rangle \ll 1/\sqrt{C})$

$$S_{\lambda\ll\mu} = \frac{1}{1-q}\sum_i \left(p_i^q - p_i\right) .$$

3. $q \geq 1$: $\qquad (\Delta\beta / \langle\beta\rangle \gg 1/\sqrt{C})$

$$S_{\lambda\gg\mu} = \frac{1}{\lambda}\sum_i p_i \ln\left(1 - \lambda \ln p_i\right) .$$

BACKUP