Oscillator models of the Solar Cycle and the Waldmeier-effect

Melinda Nagy
MSc Astronomy
Department of Astronomy, ELTE

Supervisor
Dr. Kristóf Petrovay
Department of Astronomy, ELTE

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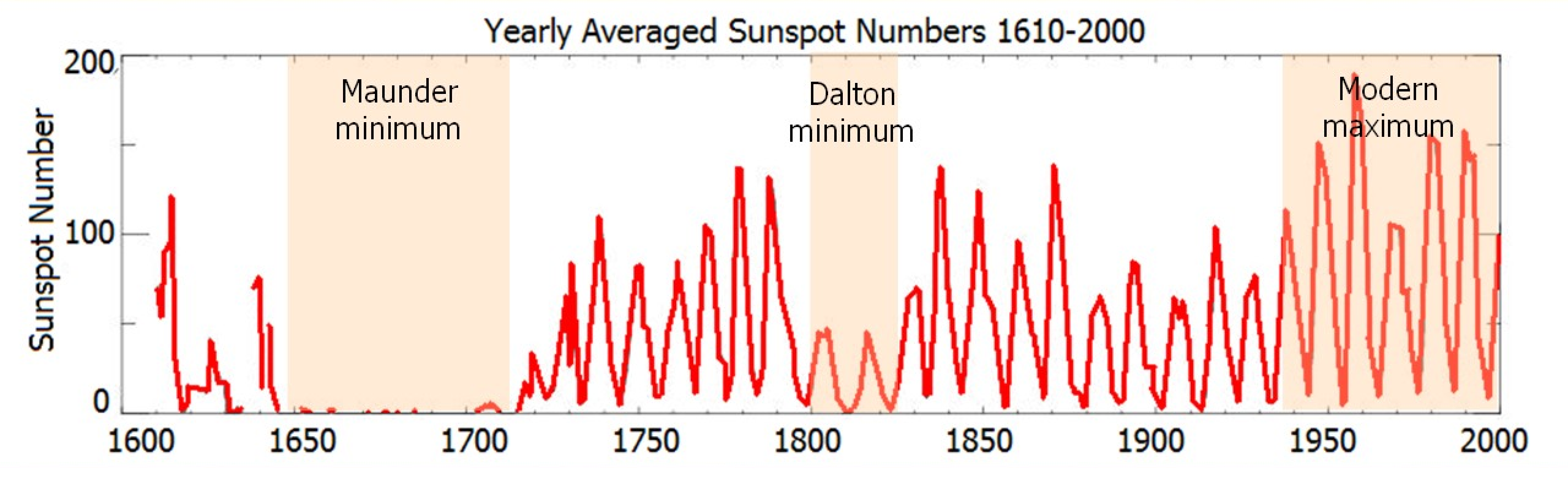
Our objectives were…

By varying the parameters of the van der Pol oscillator:

- To reproduce the Waldmeier-effect
- To approximate other known properties of the Solar Cycle

To extend the model to van der Pol-Duffing oscillator and vary the parameters together

The Solar Cycle



– Sunspot Number, introduced by Wolf (1859): asymmetric, cyclic behaviour

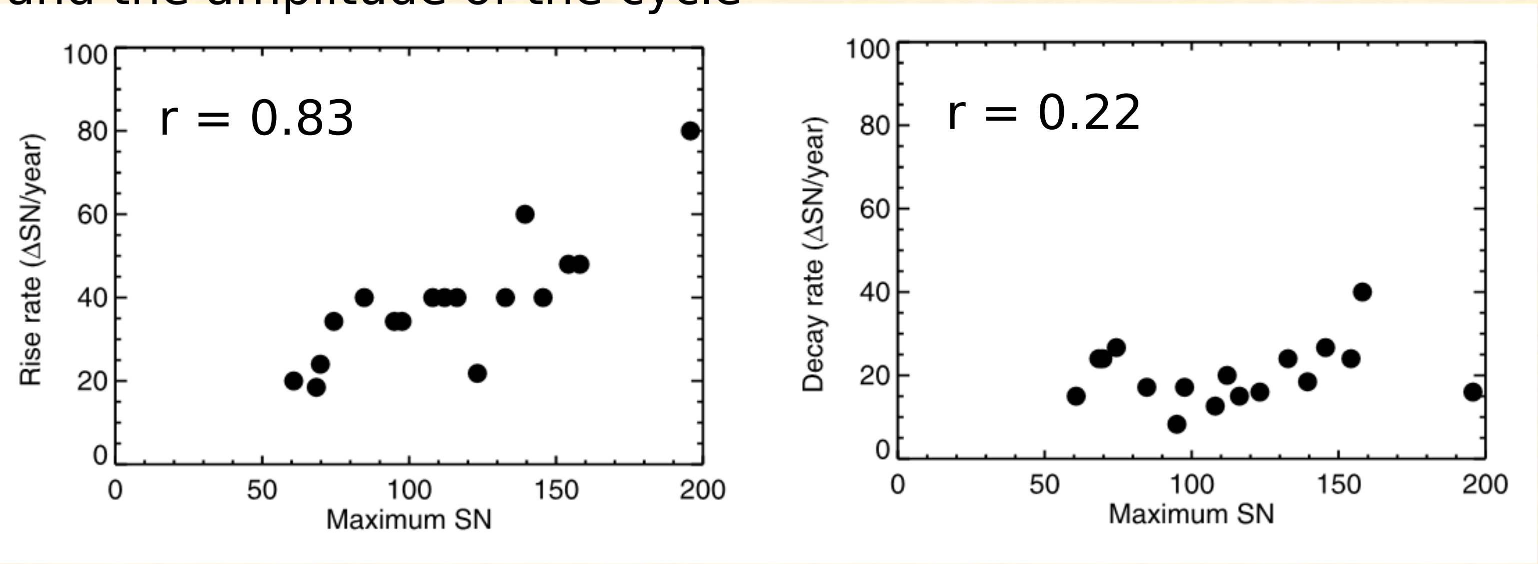
– From minimum to maximum: 3-4 years

– From maximum to minimum: 7-8 years

– The degree of the asymmetry correlates to the amplitude of the cycle

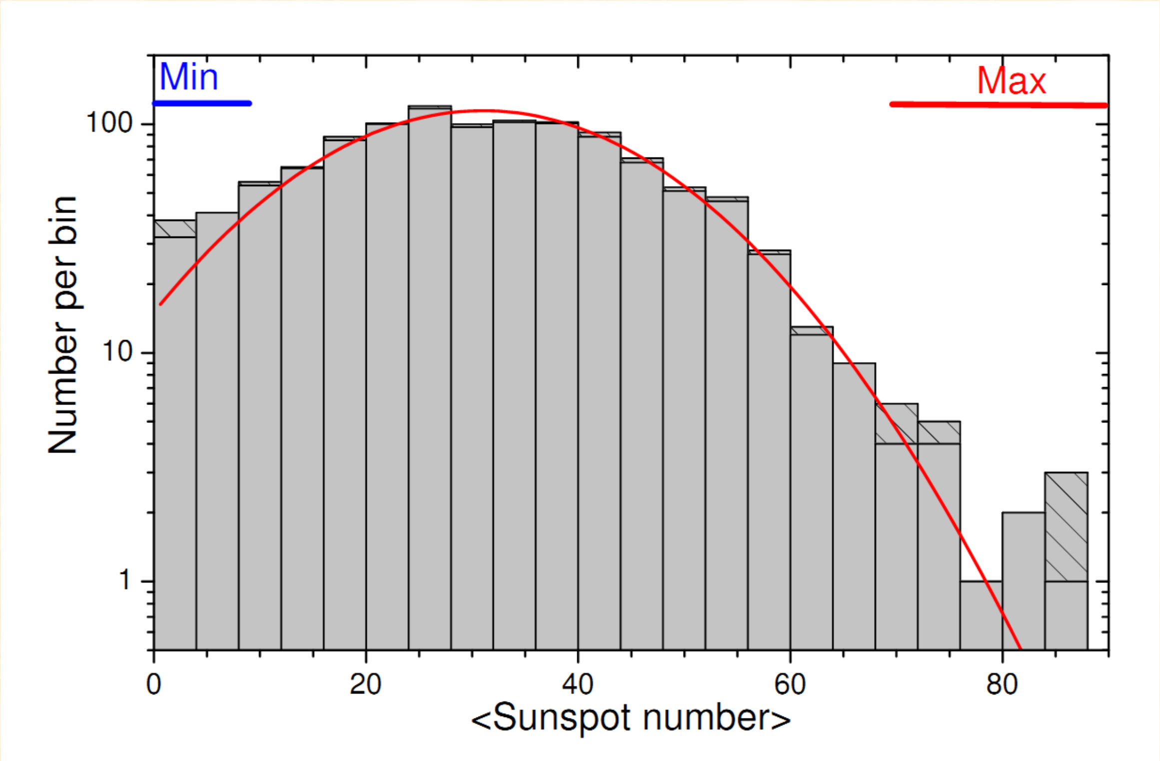
The Solar Cycle

- The degree of the asymmetry correlates to the amplitude of the cycle
  - first formulated by Waldmeier (1935) as an inverse correlation between rise time and the cycle maximum
  - the effect is clearer if we use the **rise rate** instead of rise time
- Correlation coefficient in the case of **rise rate** (left panel) and **decay rate** (right panel): no significant link between this and the amplitude of the cycle



Cameron & Schüssler (2008)

The Solar Cycle



Usoskin (2007):
sunspot number reconstruction by ^{14}C data for 11000 years.
The distribution can be characterized with Gaussian curve (σ=30, $\bar{R}$=31)

Oscillator model of the Solar Cycle

$$\ddot{x} = -\omega^2 x - \mu(3\xi x^2 - 1)\dot{x} + \lambda x^3$$

Lopes & Passos (2008) fitted van der Pol-Duffing oscillator on each magnetic cycle – λ was neglected. The main value of these fitted parameters:

$$\bar{\omega} = 0.3141 \qquad \bar{\mu} = 0.1866 \qquad \bar{\xi} = 0.0154$$

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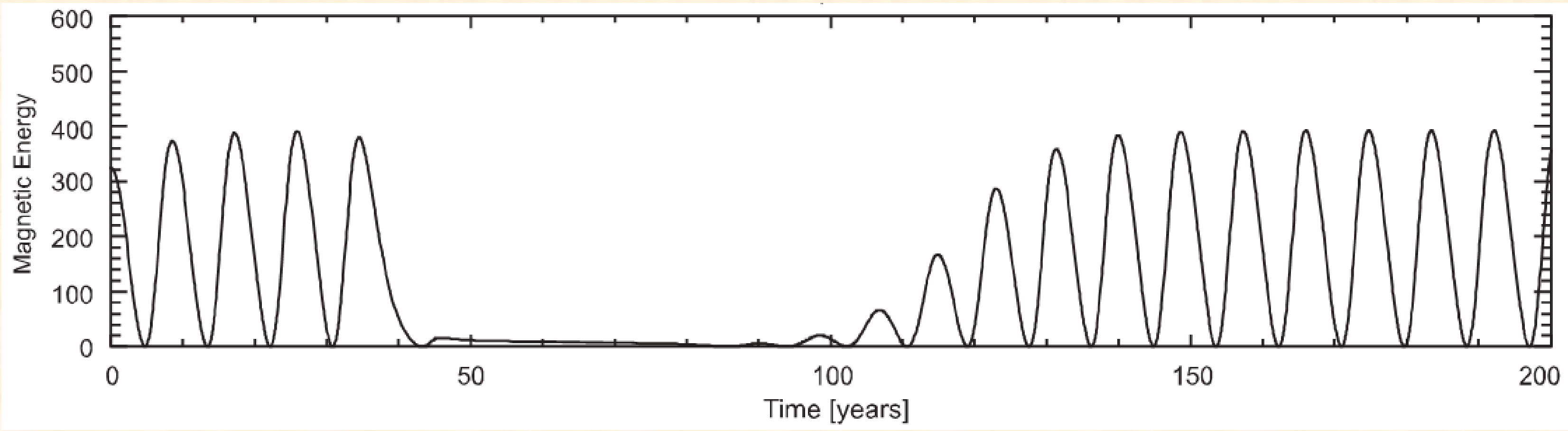
To produce grand minima, they also used van der Pol -Duffing oscillator, Lopes & Passos (2011). They kept constant the parameters (ω, μ, λ), and ξ was increased temporary – it caused grand minima like behaviour:



The method of our study

$$\ddot{x} = -\omega_0^2 x - \mu(t)\left[\xi(t)x^2 - 1\right]\dot{x} + \cancel{\lambda(t)x^3}$$

- The **behaviour of the van der Pol oscillator** was analyzed by making its parameters time dependent :
  - μ, dumping parameter and ξ, nonlinearity
- The requirements were:
  - **Waldmeier effect**: correlation coefficient > 0.8

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  - **Waldmeier-effect**: correlation coefficient > 0.8
  - the absolute value of correlation between **decay rate** and the amplitude < 0.5
  - the deviation of the **cycle length** and that of the **amplitude** should be more than 10%

- Amplitude-time functions were calculated for an interval of **2000 years**
- We used Lopes & Passos (2008) fitted parameters as **constants**

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In the case of correlated noise, *K* is the relaxation parameter

$$\mathrm{d}\mu(t) = \mathrm{d}\mu(t-\mathrm{d}t) + [-K_\mu \cdot \mathrm{d}\mu(t-\mathrm{d}t)] \cdot \mathrm{d}t + \Delta\mu \cdot R_\mu \cdot \sqrt{\mathrm{d}t}$$

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– constant value for a defined time interval (correlation time)

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We used these methods in two different way to create *μ(t)* and *ξ(t)*, in addition, we restricted the values to keep the oscillator stable.

- additive noise

$$\mu(t) = \mu_0 + \mathrm{d}\mu(t)$$

- multiplicative noise

$$\mu(t) \;=\; \mu_0 \cdot e^{\mathrm{d}\mu(t)}$$

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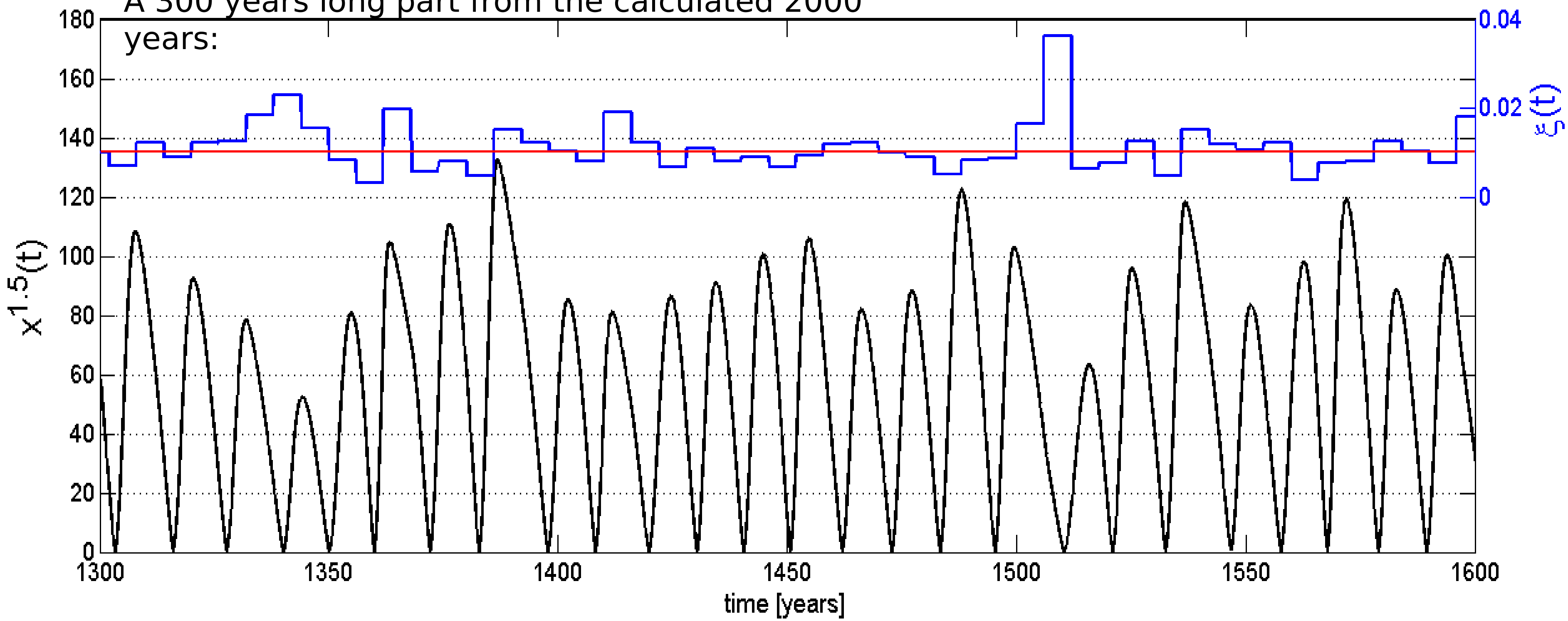
$$\mu(t) = \mu_0 \cdot e^{\mathrm{d}\mu(t)} \qquad 0.1\mu_0 < \mu(t) < 10\mu_0$$

Results I.

Constant variation for a defined correlation time: $\mathrm{d}\xi(t) = \Delta\xi \cdot R_\xi$ $\quad \xi(t) = \xi_0 \cdot e^{\mathrm{d}\xi(t)}$ $\quad \boxed{\xi_0 = 0.01}$

A 300 years long part from the calculated 2000 years:



Attributes of:	$Corr_{rise}$	$Corr_{decay}$	$rms_T[\%]$	$rms_A[\%]$	$\Delta\xi$	T_{corr} [yr]
The oscillator (2000 years)	0.87	-0.45	11.90	20.54	0.44	6
The Solar Cycle	0.83	< 0.35	~11 %	~30 %		

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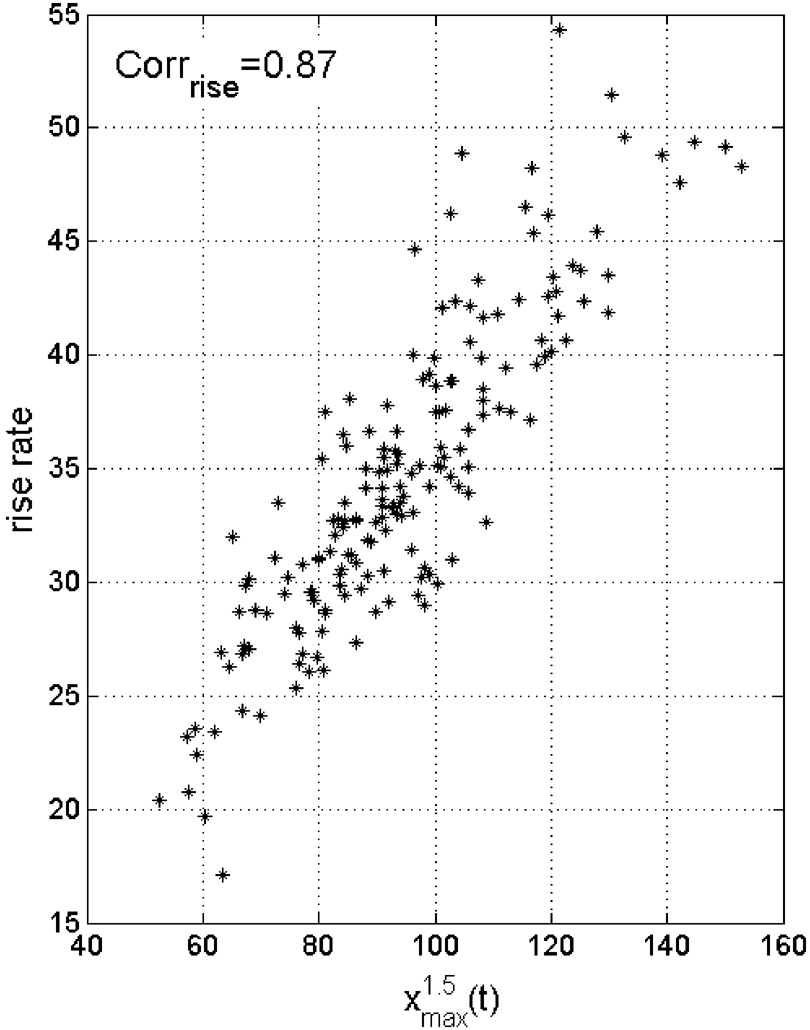
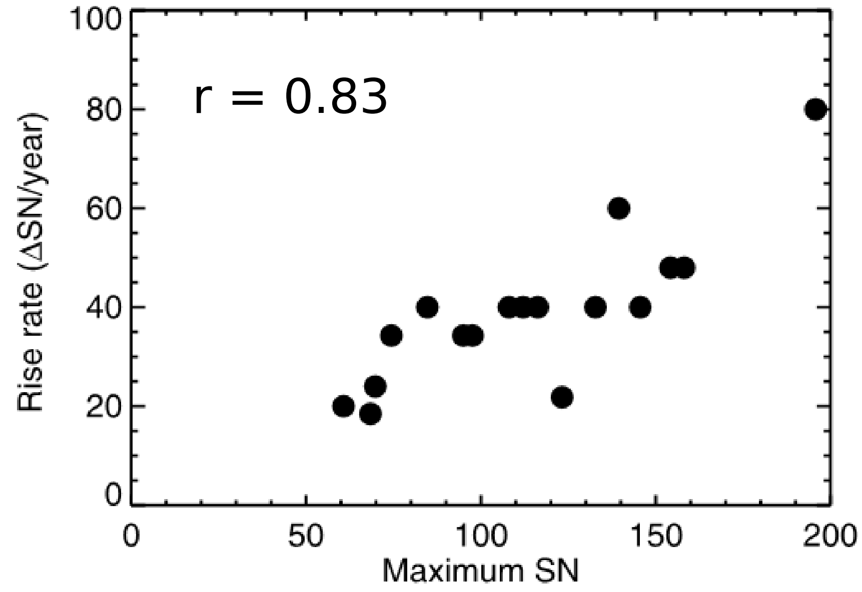
van der Pol oscillator



Solar Cycle



Cameron & Schüssler (2008)

Waldmeier-effect

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The extended model

Object: create grand minima like behaviour without extremely long periods.

$$\ddot{x} \quad = \quad -\omega_0^2 x - \mu(t)\left[\xi(t)x^2 - 1\right]\dot{x} + \lambda(t)x^3$$

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Van der Pol-Duffing oscillator with time dependent parameters, and defined connection between them (C_ξ and C_λ are constants):

$$\mu(t) \;=\; \mu_0 \cdot e^{\mathrm{d}\mu(t)} \rightarrow \;\xi(t) \;=\; \frac{C_\xi}{\mu(t)}, \quad \lambda(t) \;=\; C_\lambda \frac{\mu(t)}{2}$$

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Important: proper limits of the parameters.

Shape of the cycle: $0.1\mu_0 < \mu(t) < 3\mu_0$

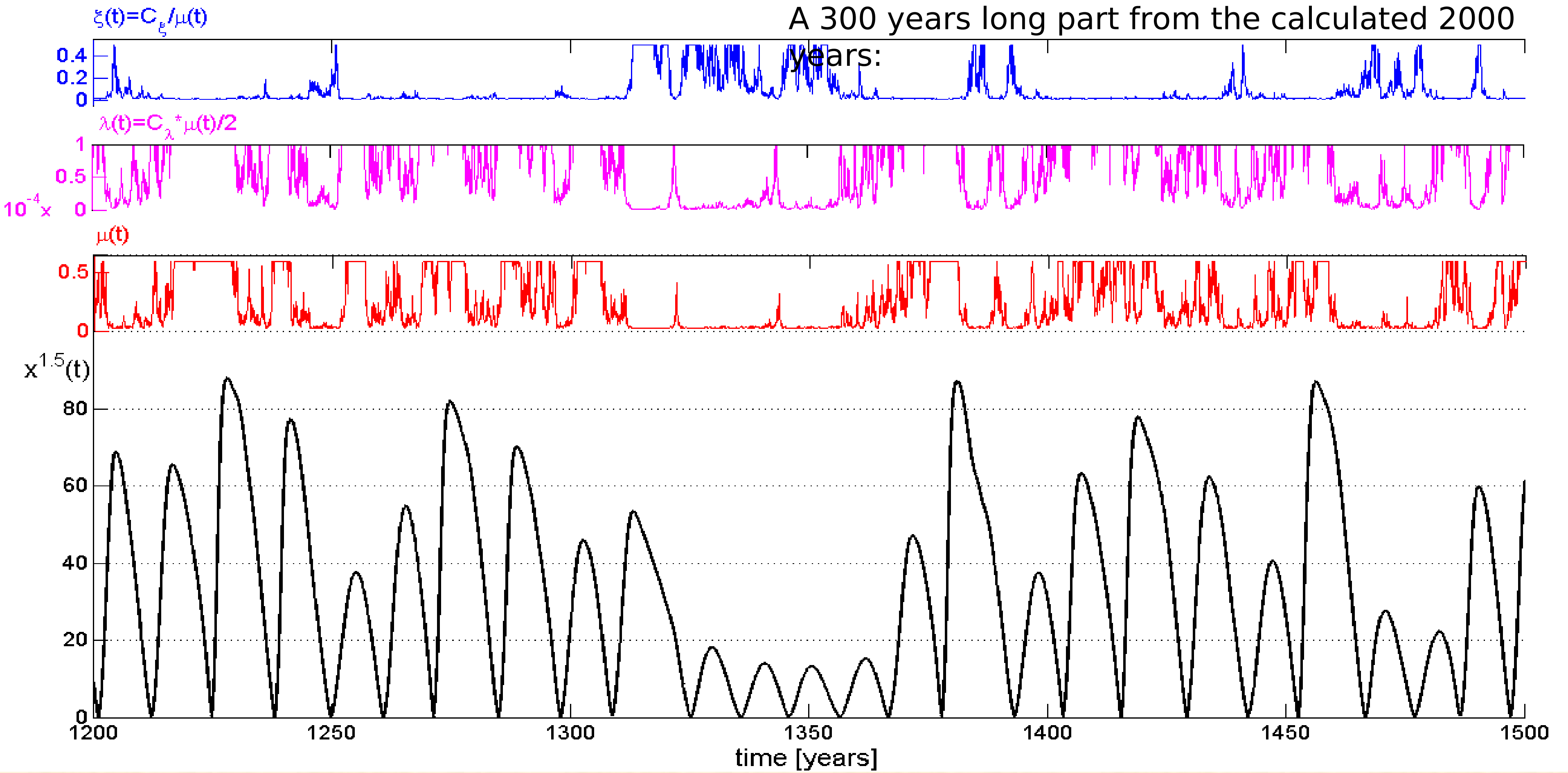
Stable cycle, and proper damping: $\xi_0 < \xi(t) < 50\xi_0$

Stable cycle: $0 < \lambda(t) < \lambda_0$

$$\omega_0 = 0.3 \quad \mu_0 = 0.2 \quad \xi_0 = 0.01 \quad \lambda_0 = 0.0001$$

Results II.

A 300 years long part from the calculated 2000 years:



Attributes of:	$Corr_{rise}$	$Corr_{decay}$	$rms_T[\%]$	$rms_A[\%]$	$\Delta\mu$	K_μ
The oscillator (2000 years)	0.94	-0.45	19.38	36.05	1.4	0.2
The Solar Cycle	0.83	≤ 0.35	~11 %	~30 %		

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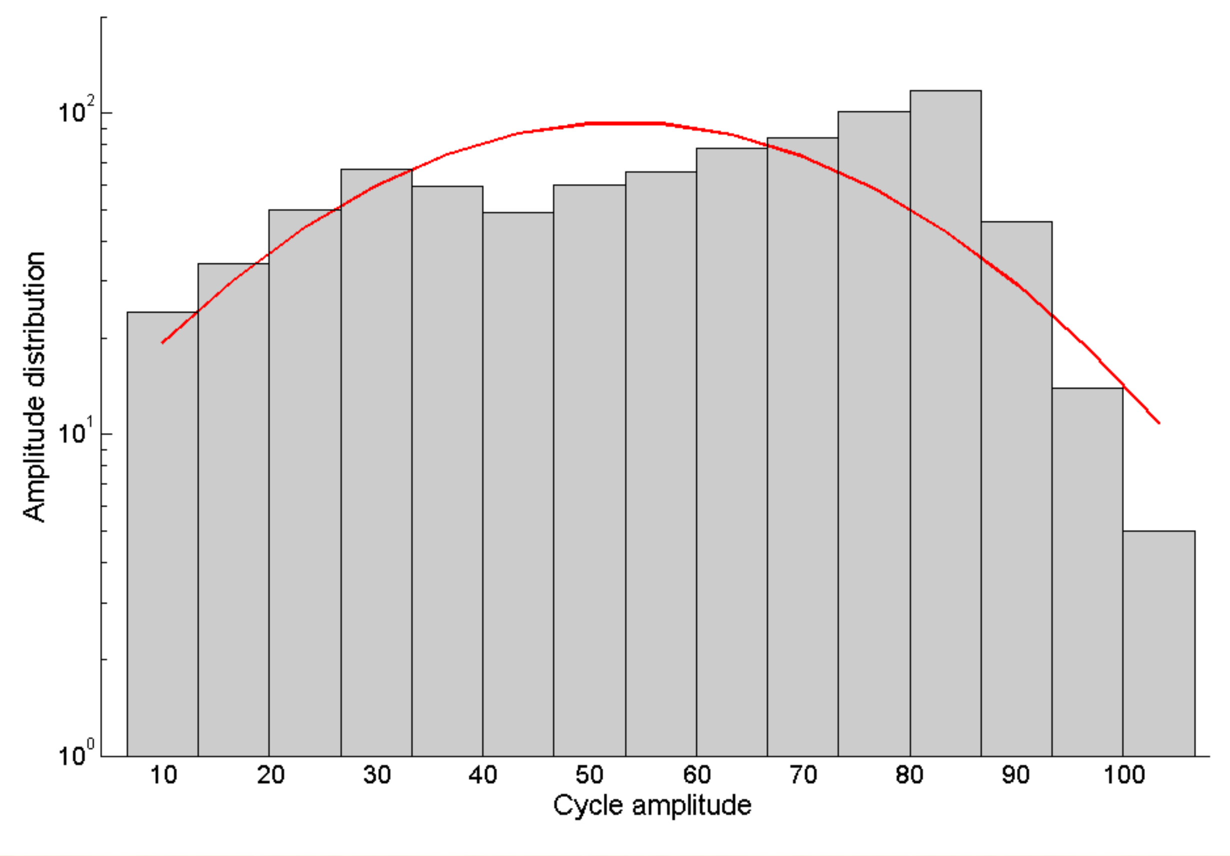
11 000 years long simulation:



Parameters of the fitted Gaussian curve:

Mean value: 52.96

Standard deviation: 24.24

In the case of the Sun:

Usoskin (2007), σ=30, $\overline{R}$=31

$\Delta\mu$	K_μ
1.4	0.2

Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

$$\ddot{x} \;=\; -\omega_0^2 x - \mu(t)\left[\xi(t)x^2 - 1\right]\dot{x} + \lambda(t)x^3$$

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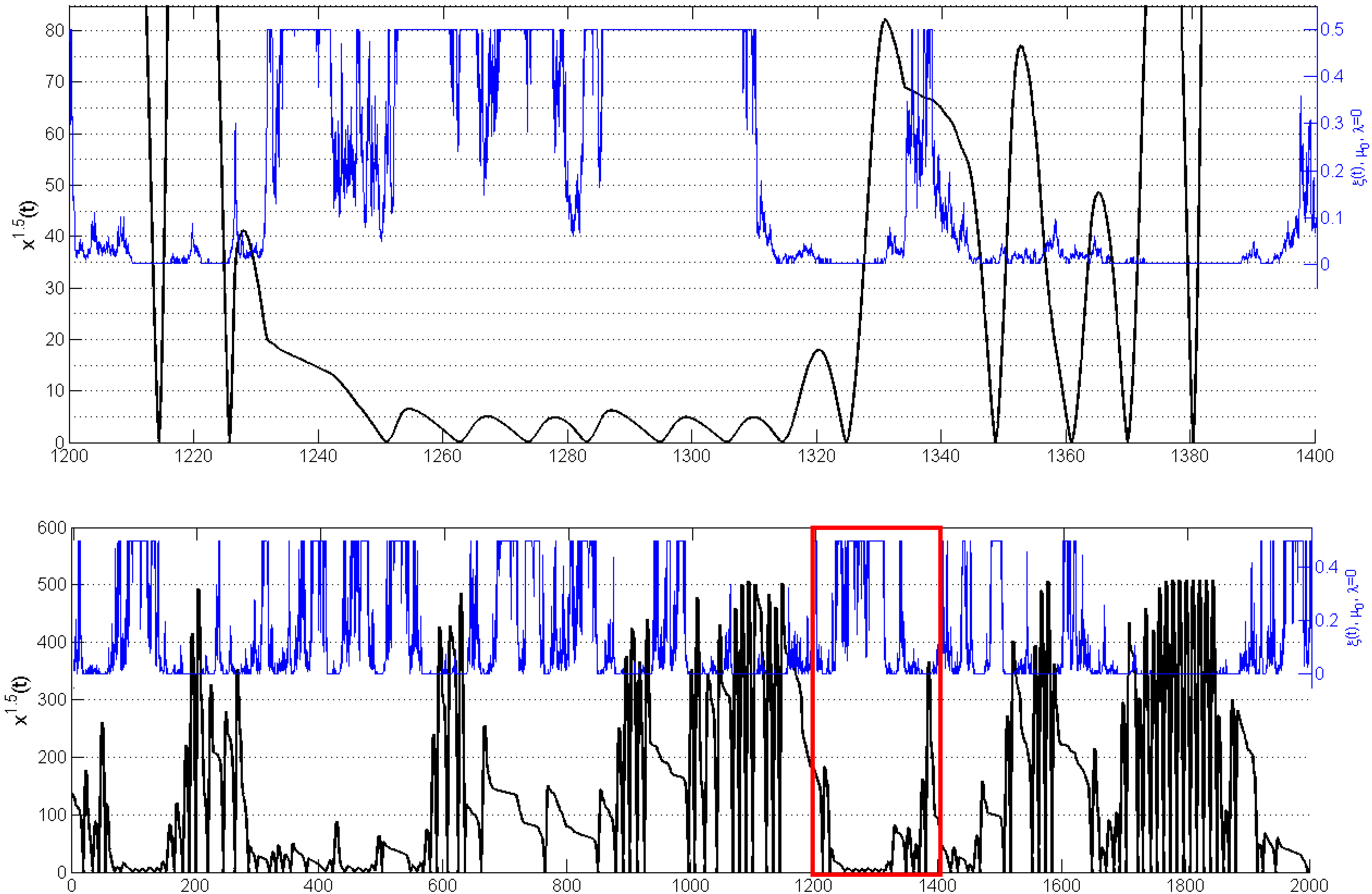


$x^{1.5}(t)$
$\xi(t), \mu_0, \lambda=0$
1200 1220 1240 1260 1280 1300 1320 1340 1360 1380 1400
0 10 20 30 40 50 60 70 80
0 0.1 0.2 0.3 0.4 0.5
$x^{1.5}(t)$
$\xi(t), \mu_0, \lambda=0$
0 200 400 600 800 1000 1200 1400 1600 1800 2000
0 100 200 300 400 500 600
0 0.2 0.4

11 000 years long simulation:

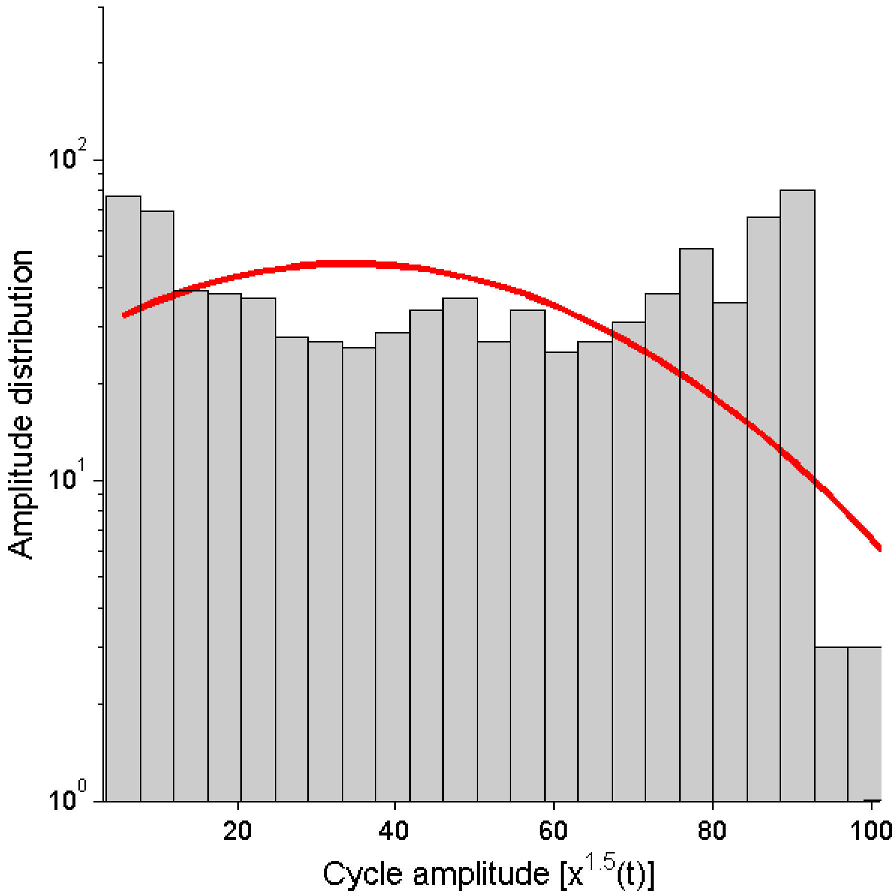
Parameters of the fitted Gaussian curve:

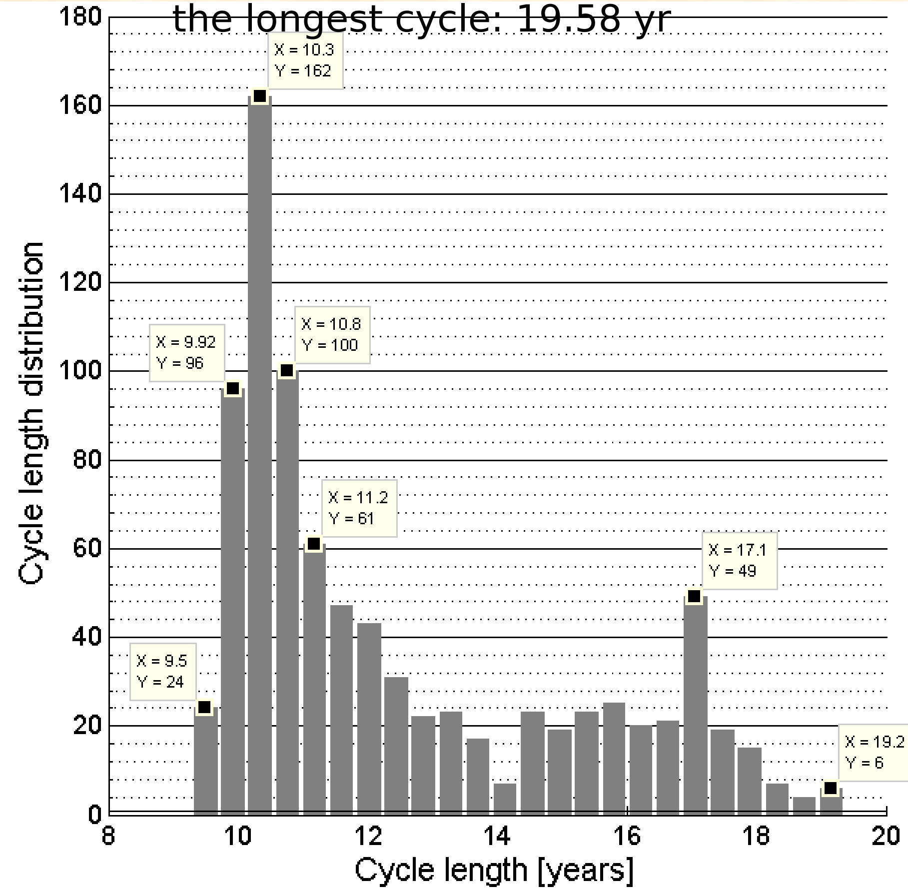
mean value: 34.25

standard deviation: 33.06

the average cycle length: 12.68 yr

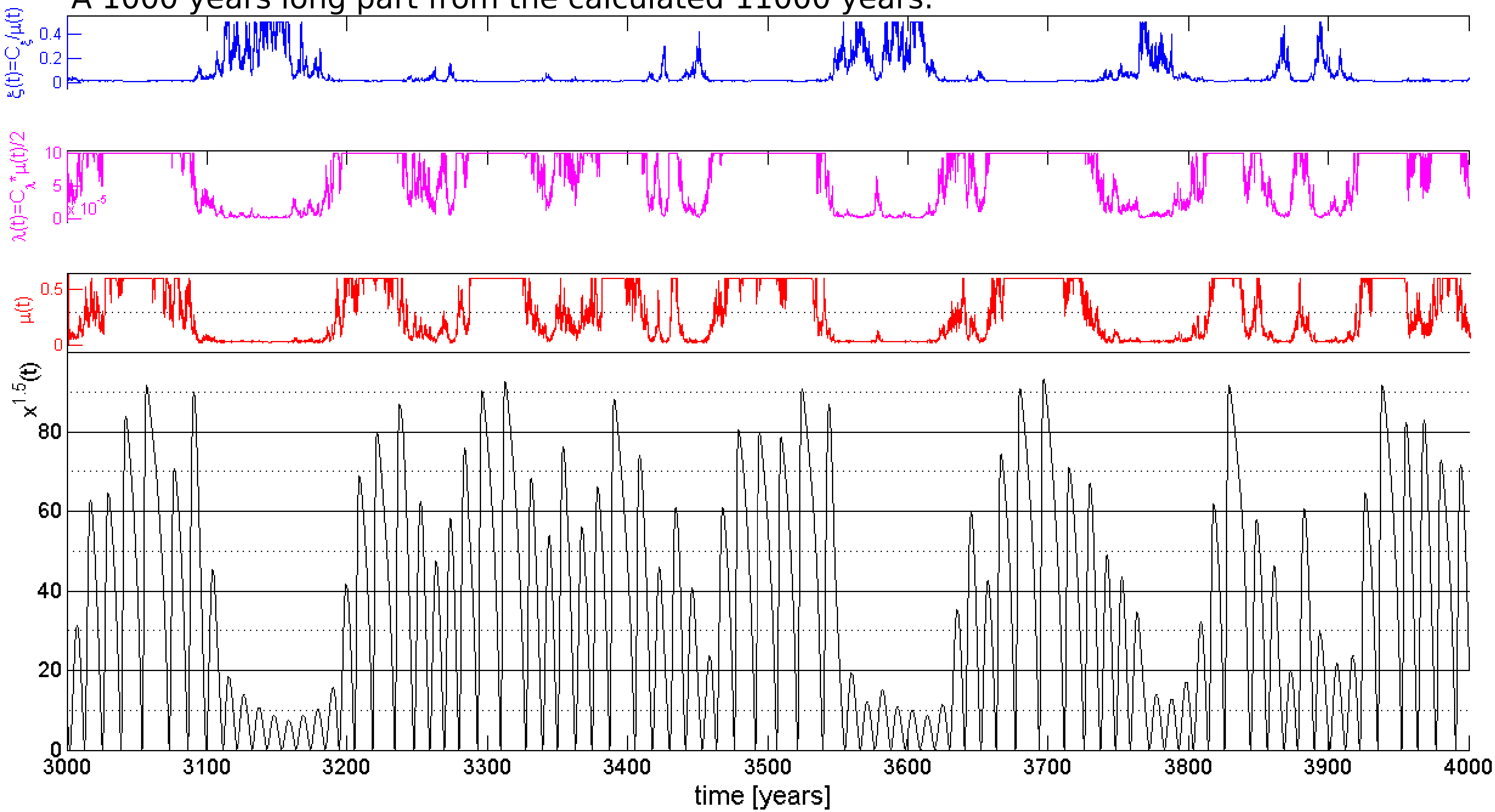
the longest cycle: 19.58 yr





Attributes of:	$Corr_{rise}$	$Corr_{decay}$	rms_T[%]	rms_A[%]	$\Delta\mu$	K_μ
The oscillator (2000 years)	0.94	-0.63	21.38	48.78	0.6	0.025
The Solar Cycle	0.83	< 0.35	~11 %	~30 %		

A 1000 years long part from the calculated 11000 years:



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The oscillator (2000 years)	0.94	-0.63	21.38	48.78	0.6	0.025
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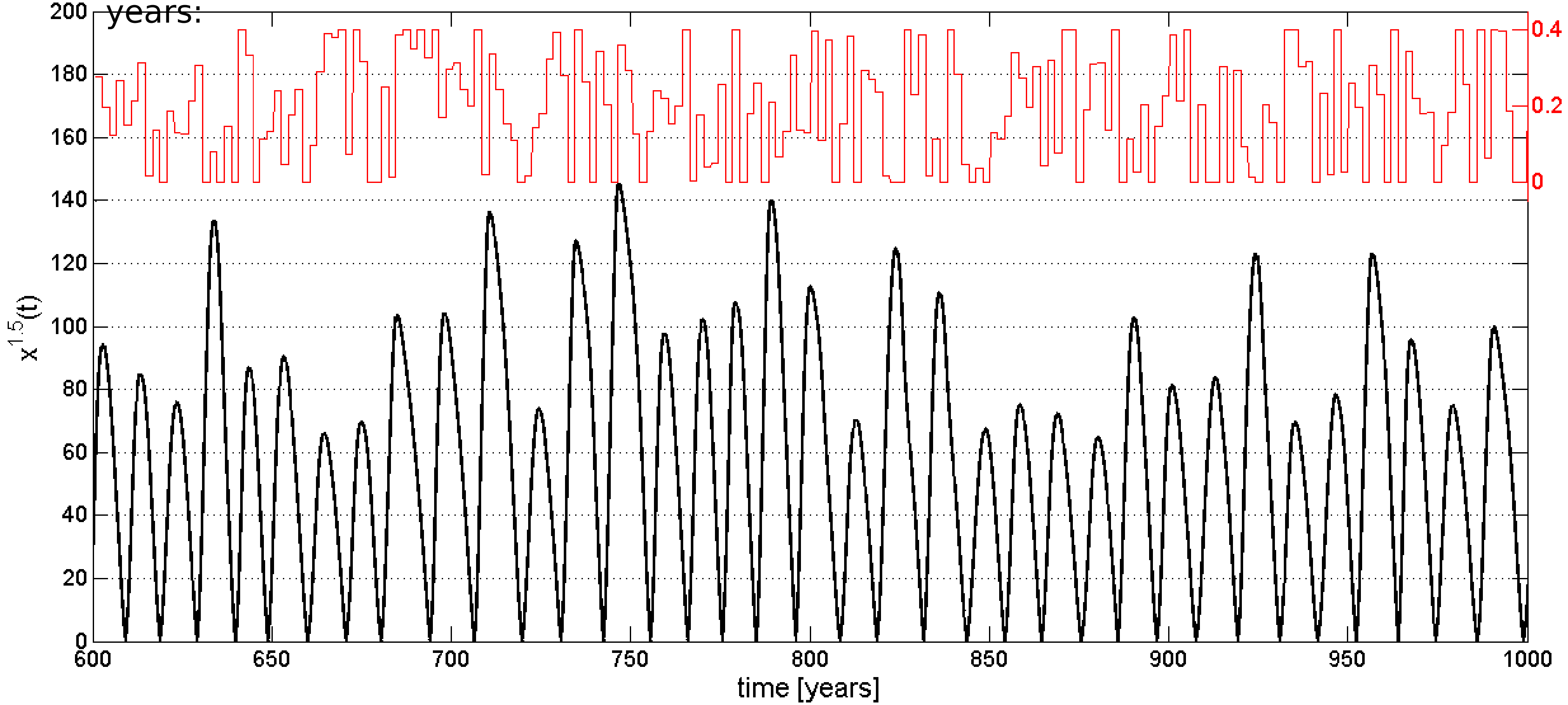
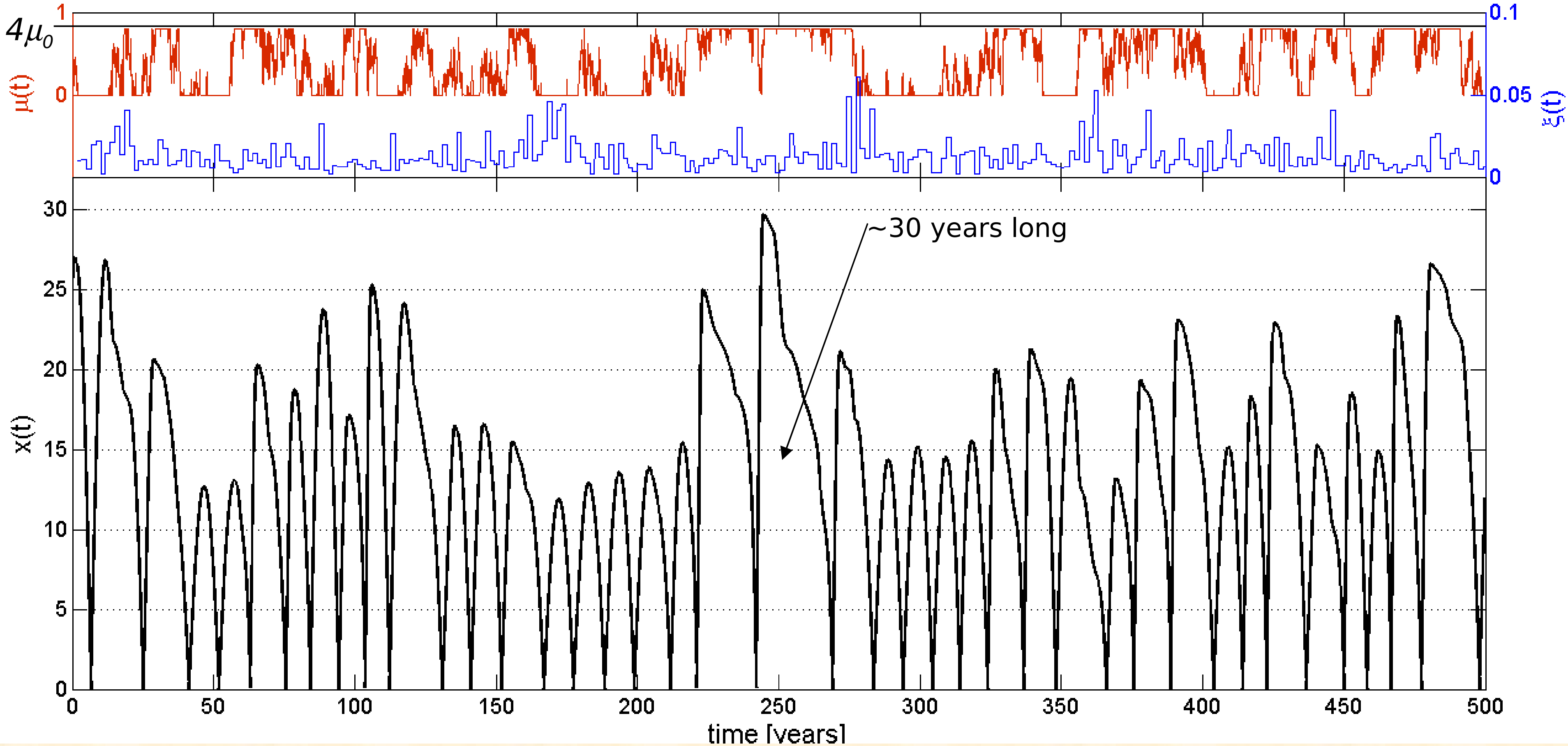
A 400 years long part from the calculated 2000 years:



Attributes of:	$Corr_{rise}$	$Corr_{decay}$	rms_T[%]	rms_A[%]	$\Delta\mu$	T_{corr} [yr]
The oscillator (2000 years)	0.78	-0.43	13.52	24.33	0.2	2
The Solar Cycle	0.83	< 0.35	~11 %	~30 %		

Constant variation for a defined correlation time: $$\mathrm{d}\mu(t) = \Delta\mu \cdot R_\mu$$ $$\mu(t) = \mu_0 + \mathrm{d}\mu(t)$$

With additive 'submethod':

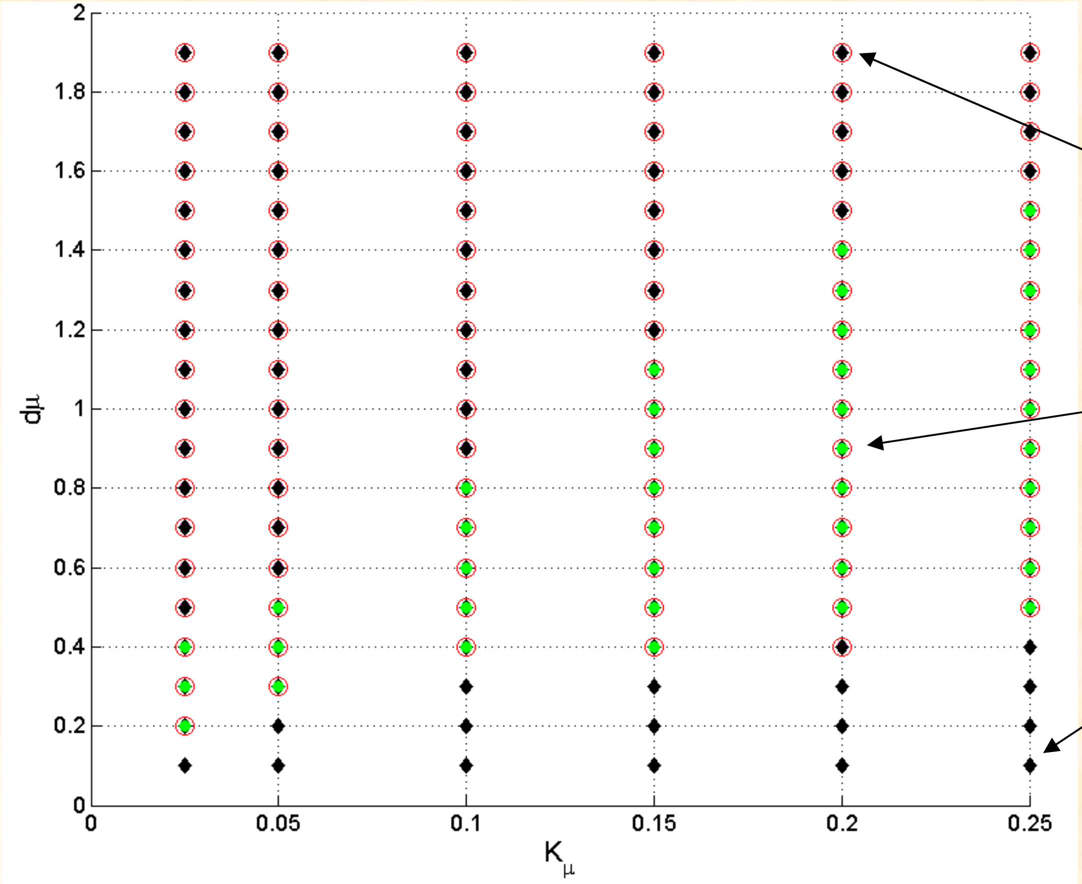


Attributes of:	$Corr_{rise}$	$Corr_{decay}$	$rms_T[\%]$	$rms_A[\%]$	$\Delta\xi$	T_{corr} [yr]	$\Delta\mu$	K_μ
The oscillator (500 years)	0.81	0.17	31.98	24.67	0.7	2	0.6	0.025
The Solar Cycle	0.83	< 0.35	~11 %	~30 %				

$$\mathrm{d}\xi(t) = \Delta\xi \cdot R_\xi \qquad \xi(t) = \xi_0 \cdot e^{\mathrm{d}\xi(t)} \qquad \xi_0 = 0.01$$

$\mathrm{d}\mu(t)$: deltacorr. noise $\qquad \mu(t) = \mu_0 + \mathrm{d}\mu(t) \qquad \mu_0 = 0.2$

$$\mathrm{d}\mu(t) = \mathrm{d}\mu(t-\mathrm{d}t) + [-K_\mu \cdot \mathrm{d}\mu(t-\mathrm{d}t)] \cdot \mathrm{d}t + \Delta\mu \cdot R_\mu \cdot \sqrt{\mathrm{d}t}$$

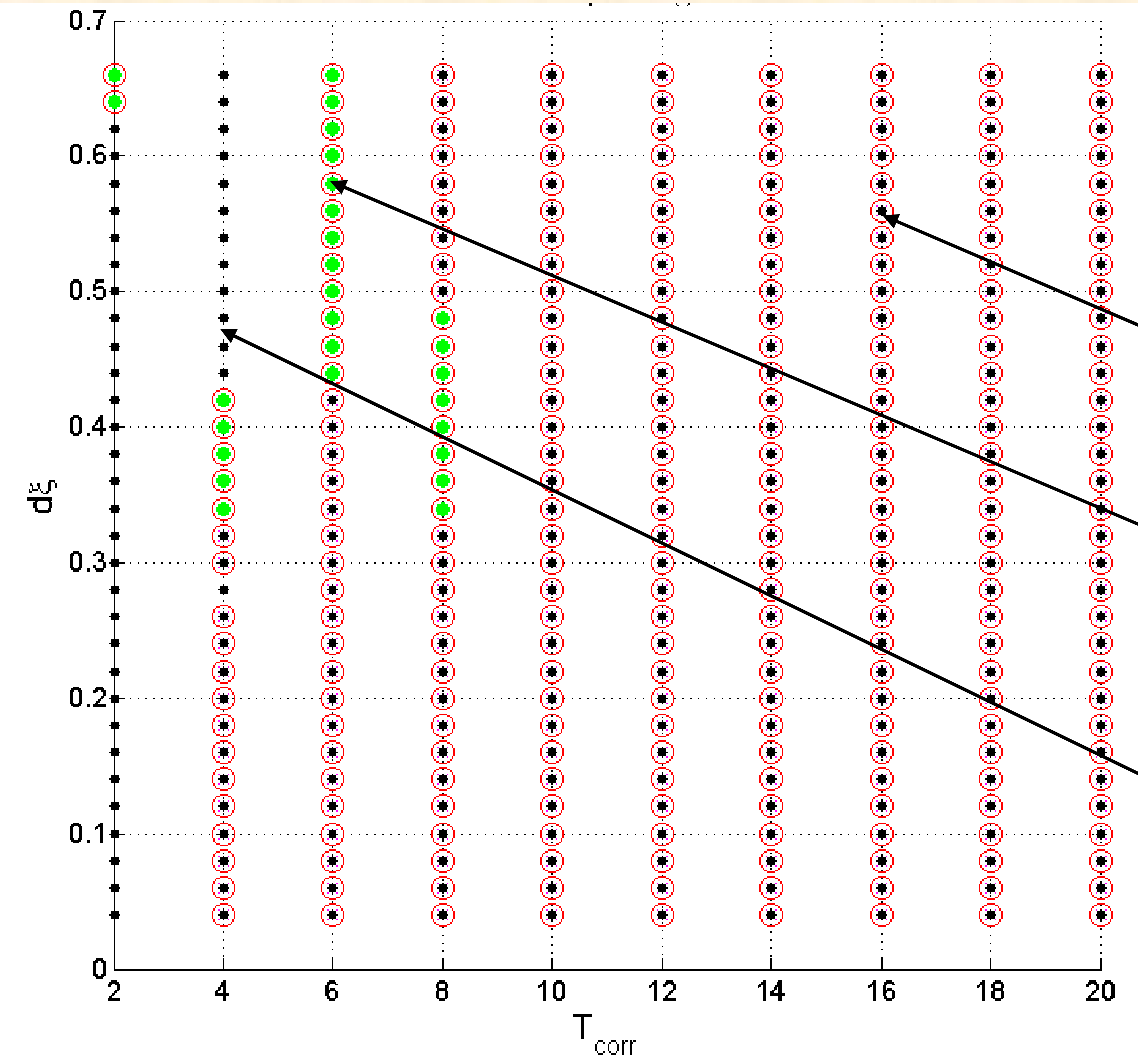


The Waldmeier effect is shown

All the required statistical parameter is shown

Correlation between the rise rate and the amplitude of the cycles

- **van der Pol-Duffing oscillator**, $x^{1.5}(t)$, varied parameters: μ, ξ and λ
- μ is defined as delta-correlated noise, with multiplicative submethod
- connection is defined between the parameters



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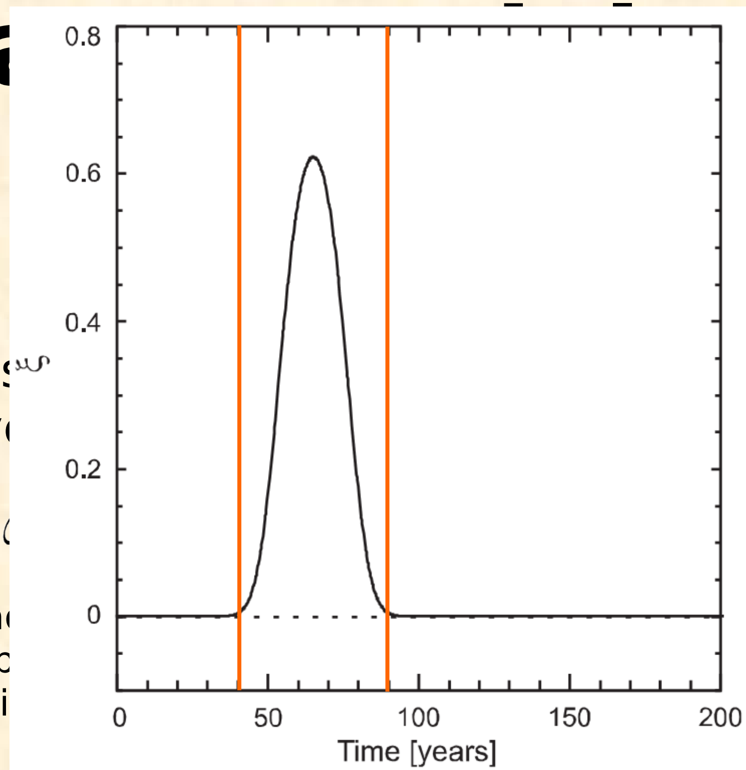
Correlation between the rise rate and the amplitude of the cycles

van der Pol oscillator, $x^{1.5}(t)$

varied parameter: ξ

constant for the interval of the correlation time, used multiplicatively

Oscilla… f the Solar



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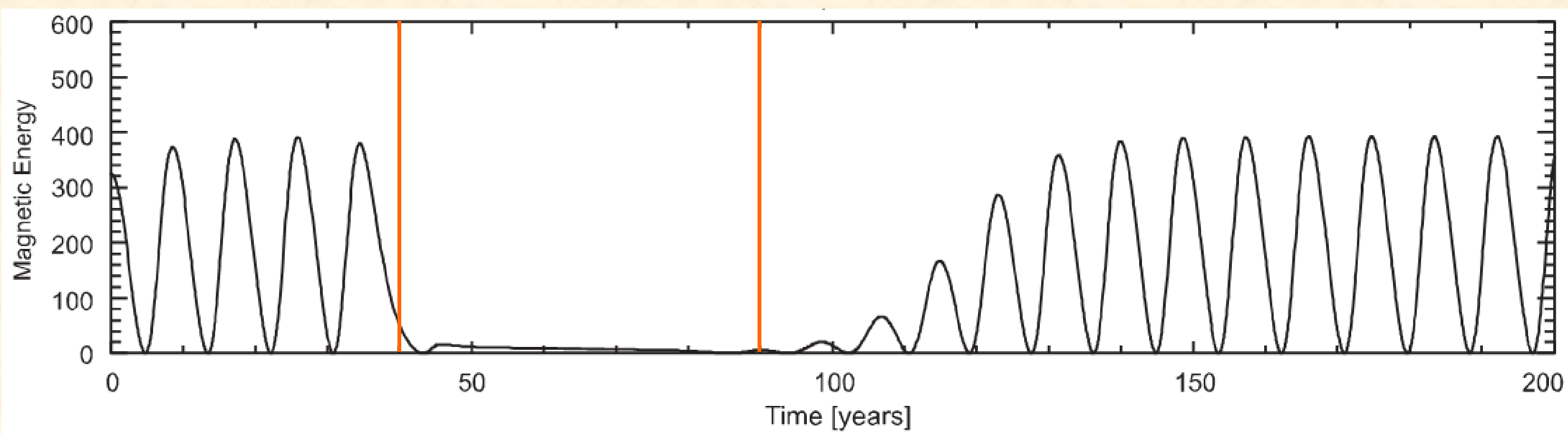

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = -\omega_0^2 x - \mu(t)\left[\xi(t)x^2 - 1\right]\frac{\mathrm{d}}{\mathrm{d}t}x + \lambda(t)x^3$$