Relativistic model for cold spherical interstellar gas clouds

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Aims:

- Generating an **exact solution** of the Einstein's equations for interstellar gas clouds that is **compatible** with and results of astronomical measurements
- Describe the distribution of pressure and density, the energy, speed, trajectory, and further relevant physical features of the cloud's particles

Further possibilities of study:

- Generalization of the model to **warm** and **ionized** matter by the consideration of GR thermodynamics and EM fields

Application:

- Provide a proper background in general relativity to investigate the **interaction of gravitational waves** and matter

Basic properties of the cloud

Properties of the spherically symmetric static cloudes:

- Spherically symmetry in the self-gravitating system
- The hydrostatic pressure is balanced by the cloud's self-gravitation
- Cold, dense and heavy *giant molecular clouds* $\Rightarrow$ rare and weak collisions between low-energy particles $\Rightarrow$ ideal gas equation of state: $p = c_s^2 \rho$

The following criteria must be met:

1. $M_R \leq M_{BE}$, where M_{BE} refers to the *Bonnor–Ebert mass* given by

$$M_{BE} = c_{BE} \frac{c_s^4}{\sqrt{p}}, \quad c_{BE} \simeq 1.18. \tag{1}$$

2. $\rho(r)$, $p(r) > 0$ and $d\rho/dr$, $dp/dr < 0$ everywhere in the cloud, the maximum of the density and pressure are in $r = 0$. Boundary conditions:

$$\lim_{r \to R} \rho(r) = \lim_{r \to R} p(r) = 0, \quad \lim_{r \to R} \frac{d\rho}{dr} = \lim_{r \to R} \frac{dp}{dr} = 0. \tag{2}$$

3. The speed of sound c_s in the medium must be less than the speed of light, that is

$$c_s^2 = \frac{dp}{d\rho} < 1. \tag{3}$$

Field equations for the compact gas cloud

In Einstein's equations we use:

- Schwarzschild metric
- Stress-energy tensor for perfect fluids
- Isotropic configuration of the system

Let's introduce a pair of new variables: $\alpha = -\lambda' e^{-\lambda}\beta^2$ and $\beta = \frac{r\nu'}{2} + 1$

$\Rightarrow$ The field equation reduces to a second order algebric equation in β:

$$\underbrace{r(r\nu' + 2)\frac{d}{dr}e^{-\lambda} + (2r^2\nu'' + r^2\nu'^2 - r\nu' - 4)e^{-\lambda} + 4 = 0}_{\textbf{ODE}} \Longrightarrow \underbrace{2(\alpha+1)\beta^2 + (r\alpha' + 8\alpha)\beta + 4\alpha = 0}_{\textbf{Quadratic algebric equation}}$$

For any function α the quadratic equation is solved by the real roots

$$\beta_\pm = 4^{-1}(\alpha+1)^{-1}\left(8\alpha - r\alpha' \pm \sqrt{(r\alpha' + 8\alpha)^2 - 32\alpha(\alpha+1)}\right)$$

$\Rightarrow$ We can calculate the metric functions distribution belonging to β!

$$\lambda = \ln\left(\frac{\beta^2}{\alpha}\right) \qquad \nu = \int_0^r \frac{2(\beta-1)}{r}dr + \nu_0$$

$$\rho = \frac{1-(r\alpha/\beta^2)'}{8\pi r^2} \qquad p = \frac{(2\beta-1)\alpha - \beta^2}{8\pi\beta^2 r^2}$$

Field equations for the compact gas cloud

Generating function: The simple, but still realistic choice for α is the ratio of two polynomials of the radial coordinate r:

$$\alpha = 1 + \frac{A^2 r^2}{1 + Br^2} \leftarrow \text{the lowest degree form which is physically valid}$$

- A and B are positive constants
- It is advisable to introduce $C^2 = 2B/A^2 - 2$ and use it in place of B

The centre gets into $\xi_c = \text{arcsinh}(2C/3)$, and the spatial infinity $\xi_\infty = \text{arcsinh}(C/2)$, and the new variable is restricted by $0 < \xi_\infty \leq \xi \leq \xi_c$.

$$\alpha = \frac{(C^2 - 4)\sinh\xi + 4C}{(C^2 + 2)\sinh\xi}, \quad \beta = \frac{(C\coth\frac{\xi}{2} - 2)\cosh\frac{\xi}{2}}{\cosh\frac{\xi}{2} + C\sinh\frac{\xi}{2}}$$

The **inner Schwartzchild metric** appears to be

$$ds^2 = -e^\nu dt^2 + e^\lambda \frac{C^2\cosh^2\xi d\xi^2}{4A^2(C^2+2)(2C - 3\sinh\xi)(2\sinh\xi - C)^3}$$
$$+ \frac{2C - 3\sinh\xi}{A^2(C^2+2)(2\sinh\xi - C)}(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- A corresponds to a constant conformal transformation of the metric

Field equations for the compact gas cloud

Restrictions on the constants:

1. Moving away from the centre of the cloud, $p \to 0$ on the boundary
2. Any choice of constants A and C satisfies the restriction on the speed of sound in the medium.

Assuming $B \ll 1 \ll C$, then one sees that p and ρ vanish simultaneously at $r = R$ if and only if $B = 4/R^2 \Rightarrow A = 8c_s^2$

Functions of state: by eliminating the variable ξ

$$\rho = \frac{(4Br^2 - 1)(8Br^2 - 3)}{4\pi C}, \quad p = \frac{(4Br^2 - 1)(2Br^2 - 1)}{4\pi C^2}$$

- If $B \ll 1 \ll C$, the equation of state is nearly linear for every $r \leq R$

$$\Rightarrow \frac{p}{\rho} = \frac{1}{C}\frac{2Br^2 - 1}{8Br^2 - 3} = c_s^2 < 1 \quad \text{(as we required)}$$

- This fixes the last unknown constant as

$$C = 4/c_s^2$$

Distributions and metric functions

Consequently, ρ and p differ from one another only by a constant factor, thus verifying the legitimacy of the isothermal equation of state.

$$e^{\nu} = \frac{c_s^2}{4}\left(1 + \frac{c_s^2}{4}\frac{r^2}{R^2}\right), \quad e^{\lambda} = \exp\left(-\frac{c_s^2}{2}\frac{r^2}{R^2}\right).$$

The pressure and density distribution correspond with the classical results of [Bohigas (1988)] and [Kritsuk et al (2011)], hence the metric functions consistent with the distributions must be valid.

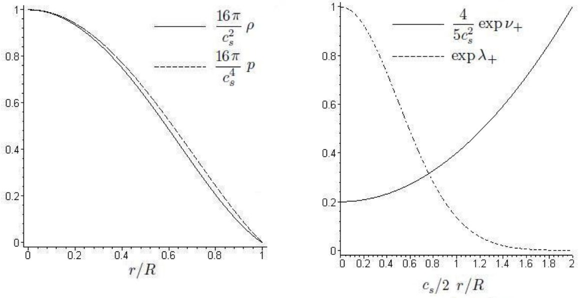

Figure: The normalized density and pressure profile. The evolution of the normalized metric functions $\exp\nu$ and $\exp\lambda$ within a fraction of distance R.

Lagrangian function and geodetics of the particles

The relativistic Lagrangian function of the particles in the investigated spacetime is

$$L = -e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)$$

Euler–Lagrange equations provide the **geodetic equations**:

$$\begin{aligned} e^{\nu}\dot{t} &= L_t \\ \ddot{r} + \frac{1}{2}\frac{d\lambda}{dr}\dot{r}^2 + \frac{1}{2}\frac{d\nu}{dr}e^{\nu-\lambda}\dot{t}^2 - re^{-\lambda}\dot{\varphi}^2 &= 0 \\ r^2\dot{\varphi} &= L_{\varphi} \end{aligned}$$

- The appearing constants L_t and L_{φ} proportional to the *total energy* and the *angular momentum* of the particles.

Energy equation:

Inside

$$\dot{r}^2 + \frac{L_{\varphi}^2}{r^2}e^{-\lambda} = \left(L_t^2 e^{-\nu} - 1\right)e^{-\lambda}$$

Outside

$$\dot{r}^2 + \underbrace{\frac{L_{\varphi}^2}{r^2}\left(1 - \frac{2M}{r}\right) - \frac{2M}{r}}_{\text{classical potential}} = \underbrace{L_t}_{\propto E}{}^2 - 1$$

Orbit for a particle in the equatorial plane $\vartheta = \pi/2$:

$$\left(\frac{d\tilde{r}}{d\varphi}\right)^2 + \tilde{r}^2 e^{-\lambda} = \frac{1}{L_{\varphi}^2}\left(L_t^2 e^{-\nu} - 1\right)e^{-\lambda}, \quad \tilde{r} \equiv 1/r$$

Circular motion on stabil orbits, velocity of the gas particles

In the equatorial plain for circular motion: $r =$ constant, and thus $\dot{r} = \ddot{r} = 0$. This restriction imposes $\tilde{r}' = \dot{\tilde{r}}/L_\varphi \tilde{r}^2 = 0$; consequently $\tilde{r}''$ is zero too. Utilizing the

1. Metric functions
2. Energy equations

one can identify the constants of motion as

$$L_t = \frac{c_s}{2}\left(1 + \frac{c_s^2 r^2}{4R^2}\right) \quad \text{and} \quad L_\varphi = \frac{c_s}{2}\frac{r^2}{R}.$$

The components of 4-velocity of a particle:

$$[u^\mu] = \left[\frac{2}{c_s}, 0, 0, c_s \frac{r}{2R} \sin\vartheta\right]$$

Bounded orbits:

For a bounded orbit $E < m_0$ is required, so as long as $L_t = 1$, the limits on r for the orbit to be bound are given by $1 = \frac{c_s}{2}\left(1 + \frac{c_s^2 r^2}{4R^2}\right)$ which is satisfied when $r = \frac{2R}{c_s}\sqrt{\frac{2}{c_s} - 1}$.

Conclusions:

1. The density and pressure distribution can expressed by a decreasing function of radius in terms of only the speed of sound in the medium and the size of the cloud. The profils correspond with astrophysical measurments.
2. All the circular orbits are stable, thus the cloud rotate rigidly and theoretically it remains stable permanently.
3. The value of the four-velocity of a particle slightly differs from the one observed in an ordinary Schwartzschild spacetime, but the angular velocity is inversely proportional to the radius.

Thank you for your attention!
Questions now or later.

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