The modified Picard iteration and its applications

Renáta Pintér-Rajnai
Hungary, Eötvös University, Budapest

The Picard iteration

- A process to solve differential equations

Consider the following initial value problem:

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$

The general solution:

$$y(x) = y_0 + \int_{x_0}^{x} f(X, y)\, dX$$

The Picard iterative process:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(X, y_n)\, dX \quad y(x_0) = y_0$$

If “f” is analytical, the sequence of Taylor polynomials also converge to “y”

The modified Picard iteration

(The Parker-Sochacki method)

- The Picard iteration is not so practical - “…It is impossible to compute explicitly more than a few members of the sequence, so the limit function can only be found in rare cases…”

- G. E Parker & J. S. Sochacki - Implementing the Picard iteration

  $\Rightarrow$ generate or approximate the Taylor series solutions to any ODE, that has a polynomial generator

The Parker-Sochacki method

- Equations of motions:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = f(x, t)$$

- Assume that x and v can be expressed as a truncated Taylor series in time:

$$x := x_0 + x_1 t + x_2 t^2 + \ldots + x_n t^n = \sum_{j=0}^{n} x_n t^n$$

$$v := v_0 + v_1 t + v_2 t^2 + \ldots + v_n t^n = \sum_{j=0}^{n} v_n t^n$$

$$x_1 + 2 x_2 t + \ldots + n x_n t^{n-1} = v_0 + v_1 t + v_2 t^2 + \ldots + v_n t^n$$

$$\rightarrow \quad x_{n+1} = v_n / (n+1)$$

$$v_{n+1} = k_n(x) / (n+1)$$

The N-body problem

- Equations of motions:

$$\frac{dx_{il}}{dt}=v_{il}$$

$$\frac{dv_{il}}{dt}=\sum_{k=1}^{N} G m_k \frac{(x_{ik}-x_{il})}{\rho_{lk}^3}$$

$$\rho_{lk}=(\sum_{i=1}^{3}[x_{il}-x_{ik}]^2)^{1/2}$$

- The separation in the denominator makes it analytically unsolvable
- The key of the modified Picard iteration: to replace these factors with a polynomial approximation: $\xi_{lk}:=1/\rho_{lk}$
- The modified equations:

$$\frac{dx_{il}}{dt}=v_{il}$$

$$\frac{dv_{il}}{dt}=\sum_{k=1}^{N} G m_k (x_{ik}-x_{il})\xi_{lk}^3$$

$$\frac{d\xi_{lk}}{dt}=-\xi_{lk}^3[\sum_{i=1}^{3}(x_{il}-x_{ik})(v_{il}-v_{ik})]$$

The N-body problem

$$\frac{dx_{il}}{dt}=v_{il}$$
$$\frac{dv_{il}}{dt}=\sum_{k=1}^{N} G m_k (x_{ik}-x_{il})\xi_{lk}^3$$
$$\frac{d\xi_{lk}}{dt}=-\xi_{lk}^3\left[\sum_{i=1}^{3}(x_{il}-x_{ik})(v_{il}-v_{ik})\right]$$

- The method:
- Assuming that

$$x:=\sum_{j=0}^{p} x_p dt^p;\quad v:=\sum_{j=0}^{p} v_p dt^p;\quad \xi:=\sum_{j=0}^{p} \xi_p dt^p$$

$\Rightarrow$

$$\mathbf{x_{i,l,p+1}} = \mathbf{v_{ilp}}/(\mathbf{p}+\mathbf{1})$$

$$\mathbf{v_{i,l,p+1}} = \left[\sum_{k=1}^{N} \mathbf{G m_k} \sum_{j=0}^{p} (\mathbf{x_{ikj}}-\mathbf{x_{ilj}})(\xi_{lk}^3)_{p-j}\right]/(\mathbf{p}+\mathbf{1})$$

$$\xi_{l,k,p+1} - \left[\sum_{j=0}^{p} (\xi_{lk}^3)_j \alpha_{l,k,p-j}\right]/(\mathbf{p}+\mathbf{1})$$

ahol $\alpha_{lkp} = \sum_{j=0}^{p} \sum_{i=1}^{3} (\mathbf{x_{ilj}}-\mathbf{x_{ikj}})(\mathbf{v_{i,l,p-j}}-\mathbf{v_{i,k,p-j}})$

$$(\xi_{lk}^3)_p = \sum_{j=0}^{p} (\xi_{lk}^2)_j \xi_{l,k,p-j} \qquad (\xi_{lk}^2)_p = \sum_{j=0}^{p} \xi_{lkj}\xi_{l,k,p-j}$$

The motions of satellites - The Gravitational potential

- Equations of motions of satellites:

$$\mathbf{U} = \mathbf{GM}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[\frac{\bar{\mathbf{r}}_{\oplus}^{l}}{\mathbf{r}^{l+1}} \cdot \mathbf{P}_{l}^{m}(\sin\varphi) \cdot (\mathbf{C}_{lm}\cos(m\lambda) + \mathbf{S}_{lm}\sin(m\lambda))\right]$$

$$\ddot{\underline{\mathbf{r}}} = \frac{\mathbf{dU}}{\mathbf{d}\underline{\mathbf{r}}}$$

$$C_{lm} = \frac{1}{M\bar{r}_{\oplus}^{l}}\frac{2(l-m)!}{\delta_m(l+m)!}\int_V r'^{l} P_l^m(sin\varphi')cos(m\lambda')\varrho d\tau,$$

$$S_{lm} = \frac{1}{M\bar{r}_{\oplus}^{l}}\frac{2(l-m)!}{\delta_m(l+m)!}\int_V r'^{l} P_l^m(sin\varphi')sin(m\lambda')\varrho d\tau,$$

- Changing the coordinate system:

$$\mathbf{U} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\left[\frac{\beta_{lmk}}{(\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2)^{\frac{l+1}{2}}} \cdot \mathbf{P}_l^m\left(\frac{\mathbf{z}}{(\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2)^{1/2}}\right) \cdot \frac{\mathbf{x}^k\mathbf{y}^{m-k}}{(\mathbf{x}^2+\mathbf{y}^2)^{m/2}}\right]$$

$$\beta_{lmk} = \mathrm{GM}\bar{\mathrm{r}}_{\oplus}^{l}\binom{\mathrm{m}}{\mathrm{k}}\left[\mathrm{C}_{lm}\cos\left(\frac{1}{2}(\mathrm{m}-\mathrm{k})\cdot\pi\right) + \mathrm{S}_{lm}\sin\left(\frac{1}{2}(\mathrm{m}-\mathrm{k})\cdot\pi\right)\right]$$

The Gravitational potential

- Equations of motions:

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y \qquad \frac{dz}{dt} = v_z$$

$$\frac{dv_x}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m} \frac{\beta_{lmk}\cdot x^{k-1}y^{m-k}}{(x^2+y^2)^{\frac{m+2}{2}}(x^2+y^2+z^2)^{\frac{l+2}{2}}} \cdot \left[P_l^m\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)(x^2+y^2+z^2)^{1/2}\cdot[ky^2+x^2(k-m-l-1)] + P_{l+1}^m\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)x^2z(l-m+1)\right]$$

$$\frac{dv_y}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m} \frac{\beta_{lmk}\cdot x^{k}y^{m-k-1}}{(x^2+y^2)^{\frac{m+2}{2}}(x^2+y^2+z^2)^{\frac{l+2}{2}}} \cdot \left[-P_l^m\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)(x^2+y^2+z^2)^{1/2}\cdot[y^2(k+l+1)+x^2(k-m)] + P_{l+1}^m\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)y^2z(l-m+1)\right]$$

$$\frac{dv_z}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m} \frac{\beta_{lmk}\cdot x^{k}y^{m-k}}{(x^2+y^2)^{\frac{m+2}{2}}(x^2+y^2+z^2)^{\frac{l+2}{2}}} \cdot \left[P_{l+1}^m\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)(x^2+y^2)(l-m+1)\right]$$

- Using additional variables:

$$r := (x^2+y^2+z^2)^{1/2} \quad s := 1/r; \qquad u := (x^2+y^2)^{1/2} \quad w := 1/u;$$

The Gravitational potential

- Equations of motions:

$$\frac{dx}{dt} = v_x$$
$$\frac{dy}{dt} = v_y$$
$$\frac{dz}{dt} = v_z$$
$$\frac{dv_x}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}x^{k-1}y^{(m-k)}w^{(m+2)}s^{(l+2)}\left[P_l^m(s\cdot z)r\{ky^2+(k-m-l-1)x^2\}+P_{l+1}^m(s\cdot z)x^2z(l-m+1)\right]$$
$$\frac{dv_y}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}x^{k}y^{(m-k-1)}w^{(m+2)}s^{(l+2)}\left[-P_l^m(s\cdot z)r\{(k+l+1)y^2+(k-m)x^2\}+P_{l+1}^m(s\cdot z)y^2z(l-m+1)\right]$$
$$\frac{dv_z}{dt} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}x^{k}y^{(m-k)}w^{(m+2)}s^{(l+2)}\left[P_{l+1}^m(s\cdot z)(x^2+y^2)(l-m+1)\right]$$
$$\frac{dr}{dt} = s(xv_x+yv_y+zv_z)$$
$$\frac{ds}{dt} = -s^3(xv_x+yv_y+zv_z)$$
$$\frac{dw}{dt} = -w^3(xv_x+yv_y)$$

The Gravitational potential

$$(1)\quad \mathbf{x_{n+1} = v_{x,n}/(n+1)} \quad \rightarrow \quad (x^2)_n = \sum_{j=0}^{n}(x^2)_j x_{n-j}, \quad \ldots, \quad (x^{(k-1)})_n = \sum_{j=0}^{n}(x^{(k-2)})_j x_{n-j}, \quad (x^k)_n = \sum_{j=0}^{n}(x^{(k-1)})_j x_{n-j}$$

$$(2)\quad \mathbf{y_{n+1} = v_{y,n}/(n+1)} \quad \rightarrow \quad (y^2)_n = \sum_{j=0}^{n}(y^2)_j y_{n-j}, \quad \ldots, \quad (y^{(m-k-1)})_n = \sum_{j=0}^{n}(y^{(m-k-2)})_j y_{n-j}, \quad (y^{(m-k)})_n = \sum_{j=0}^{n}(y^{(m-k-1)})_j y_{n-j}$$

$$(3)\quad \mathbf{z_{n+1} = v_{z,n}/(n+1)}$$

$$(7)\quad \mathbf{r_{n+1} = \left[\sum_{j=0}^{n} s_j (r^\star)_{n-j}\right]/(n+1)} \quad \Leftarrow \quad (r^\star)_n = \sum_{j=0}^{n}(x_j v_{x,n-j} + y_j v_{y,n-j} + z_j v_{z,n-j})$$

$$(8)\quad \mathbf{s_{n+1} = -\left[\sum_{j=0}^{n}(s^3)_j (r^\star)_{n-j}\right]/(n+1)} \quad \Leftarrow \quad (s^2)_n = \sum_{j=0}^{n} s_j s_{n-j}, \quad (s^3)_n = \sum_{j=0}^{n}(s^2)_j s_{n-j}, \quad \ldots, \quad (s^{(l+2)})_n = \sum_{j=0}^{n}(s^{(l+1)})_j s_{n-j}$$

$$(9)\quad \mathbf{w_{n+1} = -\left[\sum_{j=0}^{n}(w^3)_j (w^\star)_{n-j}\right]/(n+1)} \quad \Leftarrow \quad (w^\star)_n = \sum_{j=0}^{n}(x_j v_{x,n-j} + y_j v_{y,n-j})$$

$$(w^2)_n = \sum_{j=0}^{n} w_j w_{n-j}, \quad (w^3)_n = \sum_{j=0}^{n}(w^2)_j w_{n-j}, \quad \ldots, \quad (w^{(m+2)})_n = \sum_{j=0}^{n}(w^{(m+1)})_j w_{n-j}$$

$$(4)\quad \mathbf{v_{x,n+1}\cdot(n+1) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}\sum_{j=0}^{n}(a_{lmk})_j\left[(k-m-l-1)(e_{lm})_{n-j} + k(f_{lm})_{n-j} + (l-m+1)(g_{lm})_{n-j}\right]}$$

$$(5)\quad \mathbf{v_{y,n+1}\cdot(n+1) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}\sum_{j=0}^{n}(b_{lmk})_j\left[(m-k)(e_{lm})_{n-j} - (k+l+1)(f_{lm})_{n-j} + (l-m+1)(h_{lm})_{n-j}\right]}$$

$$(6)\quad \mathbf{v_{z,n+1}\cdot(n+1) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\beta_{lmk}(l-m+1)\sum_{j=0}^{n}(c_{lmk})_j\left[(G_{lm})_{n-j} + (H_{lm})_{n-j}\right](l-m+1)}$$

$$\Uparrow$$

$$\mathbf{a}_{lmk} = A_{km}D_{lm} \quad \mathbf{b}_{lmk} = B_{km}D_{lm} \quad \mathbf{c}_{lmk} = C_{km}D_{lm} \quad \mathbf{f}_{lm} = rF_{lm} \quad \mathbf{e}_{lm} = rE_{lm} \quad \mathbf{g}_{lm} = zG_{lm} \quad \mathbf{h}_{lm} = zH_{lm}$$

$$A_{km} = x^{k-1}y^{m-k} \quad B_{km} = x^k y^{m-k-1} \quad C_{km} = x^k y^{m-k} \quad D_{lm} = w^{m+2}s^{l+2}$$

$$E_{lm} = P_l^m(s\cdot z)x^2 \quad F_{lm} = P_l^m(s\cdot z)y^2 \quad G_{lm} = P_{l+1}^m(s\cdot z)x^2 \quad H_{lm} = P_{l+1}^m(s\cdot z)y^2$$

The motions of satellites:

Additional forces:

Additional terms in the equations of motions

- Inertial Forces:

  Centrifugal force, Coriolis force, Euler force
- N-body problem (+ Moon, +Sun)
- Air Friction force