COMPACTIFIED TRIGONOMETRIC RUIJSENAARS-SCHNEIDER SYSTEMS

A joint work with Martin Hallnäs

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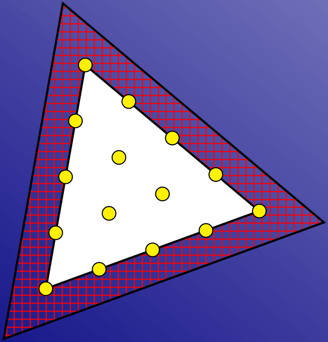
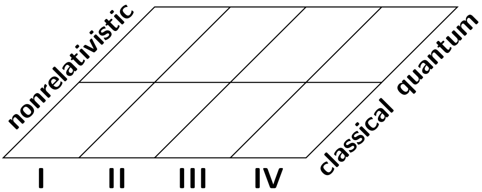
University of Szeged



December 15, 2017
Theoretical Physics Seminar
Wigner RCP, Budapest


nonrelativistic
quantum
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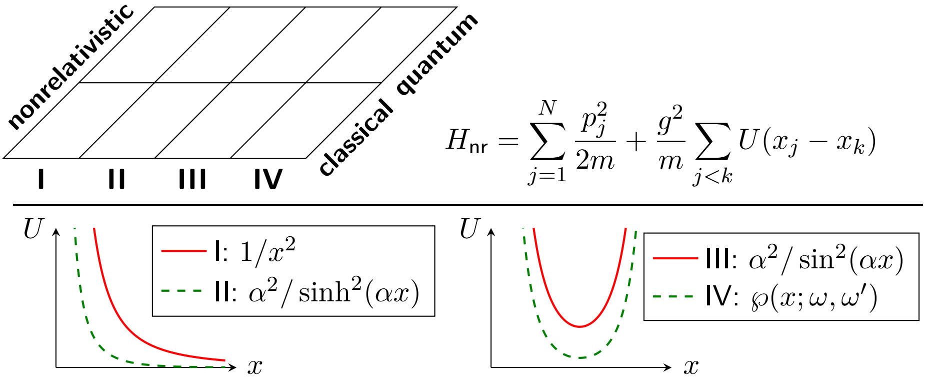
IV


nonrelativistic
classical
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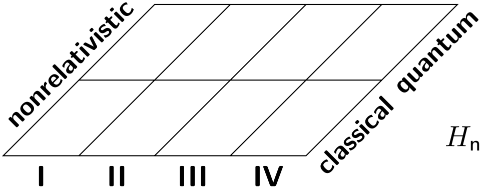
I
II
III
IV
$H_{\mathsf{nr}} = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \frac{g^2}{m} \sum_{j<k} U(x_j - x_k)$
U
x
I: $1/x^2$
II: $\alpha^2/\sinh^2(\alpha x)$
U
x
III: $\alpha^2/\sin^2(\alpha x)$
IV: $\wp(x;\omega,\omega')$



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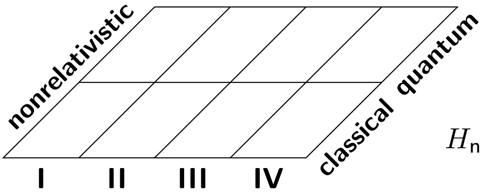

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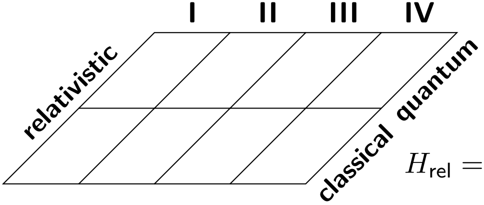
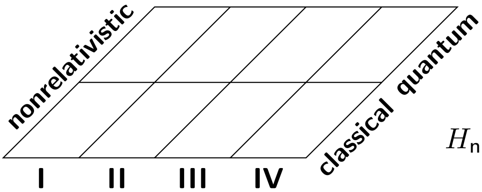


$$H_{\mathsf{rel}} = mc^2 \sum_{j=1}^{N} \cosh\left(\tfrac{p_j}{mc}\right) \sqrt{\prod_{k \neq j} f(x_j - x_k)}$$



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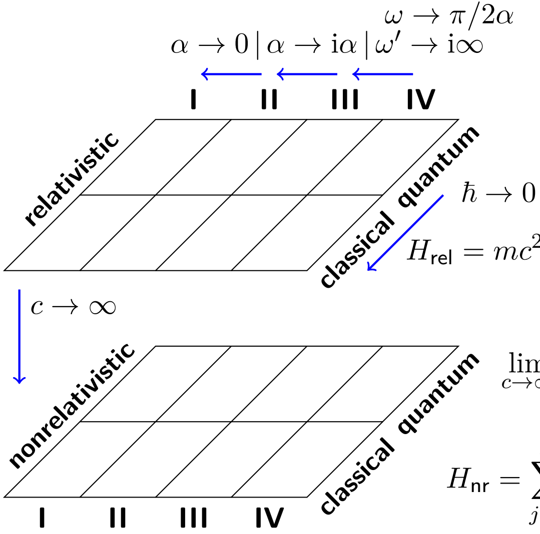

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$$\lim_{c \to \infty} (H_{\mathsf{rel}} - Nmc^2) = H_{\mathsf{nr}}$$

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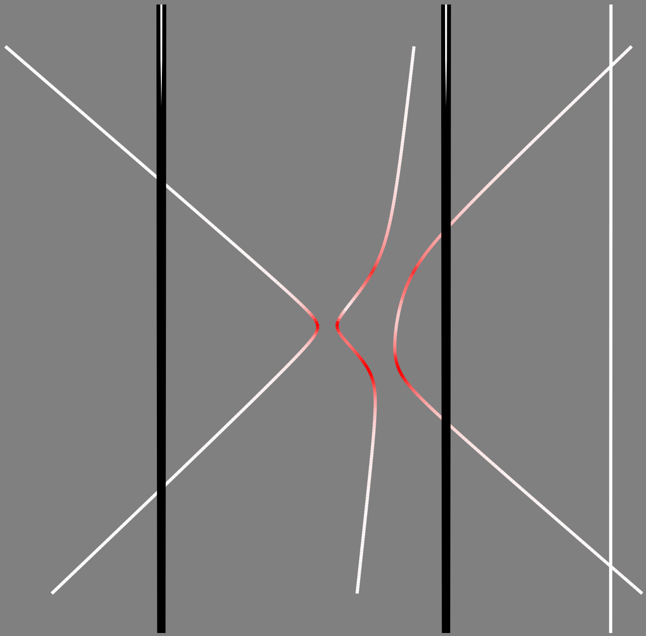
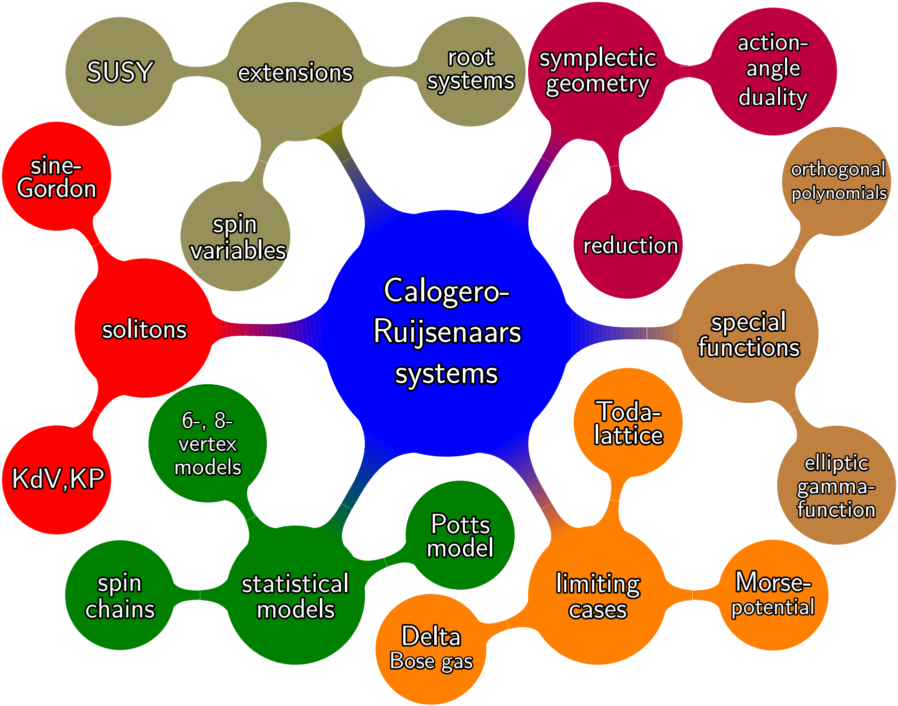
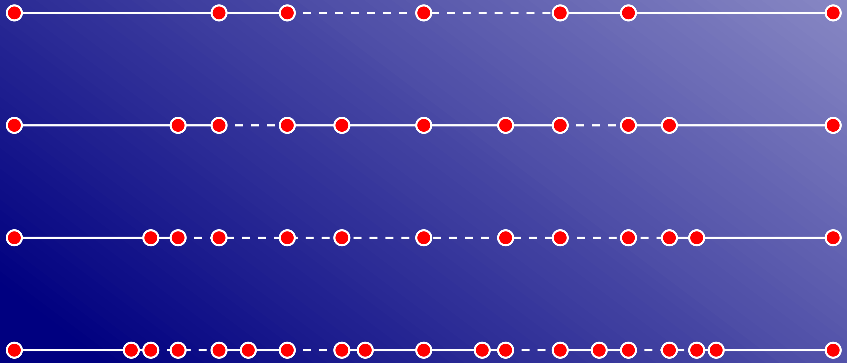




Calogero-Ruijsenaars systems
extensions
SUSY
root systems
spin variables
symplectic geometry
action-angle duality
reduction
solitons
sine-Gordon
KdV,KP
special functions
orthogonal polynomials
elliptic gamma-function
statistical models
6-, 8-vertex models
Potts model
spin chains
limiting cases
Toda-lattice
Morse-potential
Delta Bose gas

Compactified Trigonometric RS Model

Classical Case

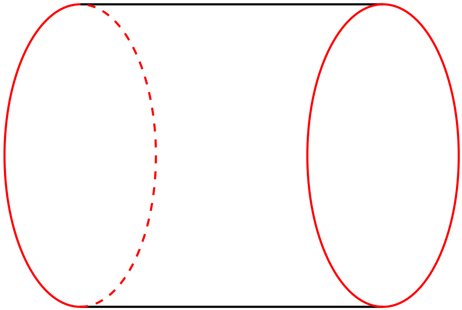


Obtained from the traditional trigonometric RS model via $\beta \to \mathrm{i}\beta$.
The 2-particle Hamiltonian reads

$$H(x_1, x_2, p_1, p_2) = (\cos(\beta p_1) + \cos(\beta p_2))\sqrt{1 - \frac{\sin^2(\frac{\alpha\beta g}{2})}{\sin^2 \frac{\alpha}{2}(x_1 - x_2)}}.$$

The center-of-mass phase space is the cylinder $(\beta g, \frac{2\pi}{\alpha} - \beta g) \times \mathbb{S}^1$ locally.

What is the ***completed*** phase space?



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What is the ***completed*** phase space?

The N-particle Hamiltonian can be written in terms of particle-positions $\boldsymbol{x} = (x_1, \dots, x_N)$ and momenta $\boldsymbol{p} = (p_1, \dots, p_N)$ as

$$H(\boldsymbol{x}, \boldsymbol{p}) = \sum_{j=1}^{N} \cos(\beta p_j) \sqrt{\prod_{k \neq j} \left(1 - \frac{\sin^2\left(\frac{\alpha\beta g}{2}\right)}{\sin^2 \frac{\alpha}{2}(x_j - x_k)}\right)}$$

with scale (α), deformation (β), and coupling (g) parameters subject to

$$\alpha > 0, \qquad \beta > 0, \qquad 0 < g < \frac{2\pi}{\alpha\beta}.$$

Integrability: A complete set of independent first integrals in involution

$$H_r(\boldsymbol{x}, \boldsymbol{p}) = \sum_{\substack{J \subset \{1,\dots,N\} \\ |J|=r}} \cos(\beta \textstyle\sum_{j\in J} p_j) \sqrt{\displaystyle\prod_{\substack{j \in J \\ k \notin J}} \left(1 - \frac{\sin^2\left(\frac{\alpha\beta g}{2}\right)}{\sin^2 \frac{\alpha}{2}(x_j - x_k)}\right)}.$$

The model was introduced by Ruijsenaars ('90), who imposed that $0 < g < 2\pi/\alpha\beta N$, henceforth referred to as the ***standard case***, and considered the thick-walled Weyl alcove

$$\Sigma_g = \{\boldsymbol{x} \in E \mid x_j - x_{j+1} > \beta g \ (j = 1, \dots, N-1),\ x_1 - x_N < 2\pi/\alpha - \beta g\},$$

sitting in the center-of-mass hyperplane $E : x_1 + \cdots + x_N = 0$, as configuration space. Quantised and solved for $N = 2$ by Ruijsenaars ('90), $N > 2$ by van Diejen-Vinet ('98).

Fehér and Kluck ('13) showed that the center-of-mass configuration space has drastically different shapes depending on the value of the parameter g.

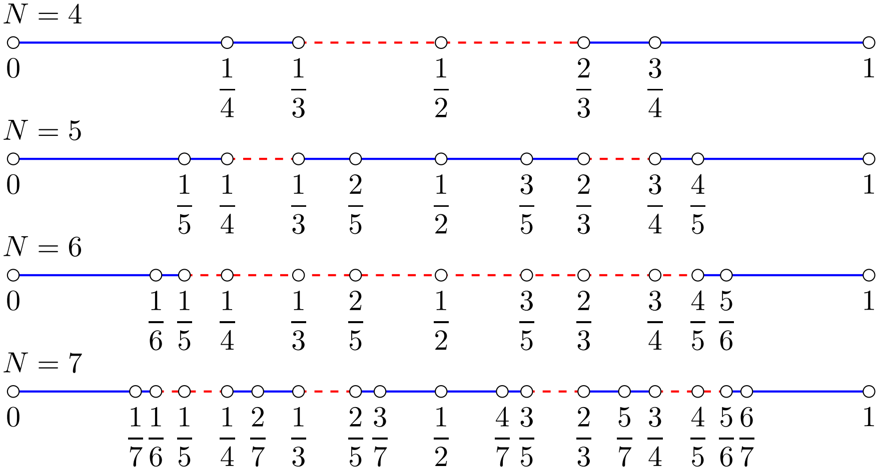

Figure: The range of $\alpha\beta g/\pi$ for $N = 4, 5, 6, 7$. The numbers displayed are excluded. Admissible values of g form intervals of *type (i)* (solid) and *type (ii)* (dashed) couplings.

Here we consider the case of type (i) couplings. They form punctured intervals around the points $2\pi p/\alpha\beta N$, labelled by the coprimes $p \in \{1, \ldots, N\}$ of N. The parameter

$$M = \frac{2\pi}{\alpha} p - \beta N g$$

helps to distinguish between couplings less/greater than $2\pi p/\alpha\beta N$.

The standard interval $0 < g < 2\pi/\alpha\beta N$ becomes the special case $p = 1$, $M > 0$.

Figure: The 3-particle configuration space Σ_g for $p = 1$.

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The configuration space $\Sigma_{g,p}$

For any fixed $p \in \{1, \ldots, N\}$ with $\gcd(N, p) = 1$ the local configuration space is a simplex determined by lower/upper bounds on p-nearest neighbour distances:

$$\Sigma_{g,p} = \{\boldsymbol{x} \in E \mid \operatorname{sgn}(M)(x_j - x_{j+p} - \beta g) > 0,\ j = 1, \ldots, N\}.$$

Here we extended the indices in a periodic manner: $x_{N+k} = x_k - 2\pi/\alpha$.

Figure: Possible configurations for $N = 3, p = 1$.

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Root system notation

Consider the standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\} \subset \mathbb{R}^N$ and the usual inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^N$, i.e. $\langle \boldsymbol{e}_j, \boldsymbol{e}_k \rangle = \delta_{jk}$. Let us focus on the root system

$$A_{N-1} = \{\boldsymbol{e}_j - \boldsymbol{e}_k \mid j, k = 1, \ldots, N,\ j \neq k\} \subset E.$$

For any $p \in \{1, \ldots, N\}$ relative prime to N we introduce the p-dependent base $\{\boldsymbol{a}_{1,p}, \ldots, \boldsymbol{a}_{N-1,p}\}$ of A_{N-1} consisting of the simple roots

$$\boldsymbol{a}_{j,p} = \boldsymbol{e}_j - \boldsymbol{e}_{j+p}, \quad j = 1, \ldots, N-1,$$

where we employ the periodicity convention $\boldsymbol{e}_{j+N} = \boldsymbol{e}_j$. Let $\{\boldsymbol{\omega}_{1,p}, \ldots, \boldsymbol{\omega}_{N-1,p}\}$ denote the corresponding fundamental weights defined via

$$\langle \boldsymbol{a}_{j,p}, \boldsymbol{\omega}_{k,p} \rangle = \delta_{jk}, \quad j, k = 1, \ldots, N-1.$$

We drop the subscript p in the $p = 1$ case.

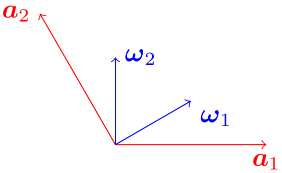


Figure: Simple roots and fundamental weights for $N = 3, p = 1$.

Root system notation

Introduce the following weighted sum of fundamental weights

$$\boldsymbol{\rho}_p = \beta g(\boldsymbol{\omega}_{1,p} + \cdots + \boldsymbol{\omega}_{N-p,p}) + \left(\beta g - \frac{2\pi}{\alpha}\right)(\boldsymbol{\omega}_{N-p+1,p} + \cdots + \boldsymbol{\omega}_{N-1,p}).$$

Then the configuration space $\Sigma_{g,p}$ is the simplex consisting of points of the form

$$\boldsymbol{x} = \boldsymbol{\rho}_p + \operatorname{sgn}(M) \sum_{j=1}^{N-1} m_j \boldsymbol{\omega}_{j,p}, \quad \text{with} \quad m_j > 0, \quad \sum_{j=1}^{N-1} m_j < |M|.$$

Let us also introduce the functions

$$V_{\boldsymbol{\nu}}(\boldsymbol{x}) = \prod_{\substack{\boldsymbol{a} \in A_{N-1} \\ \langle \boldsymbol{a}, \boldsymbol{\nu} \rangle = 1}} \frac{\sin \frac{\alpha}{2}(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + \beta g)}{\sin \frac{\alpha}{2} \langle \boldsymbol{a}, \boldsymbol{x} \rangle}.$$

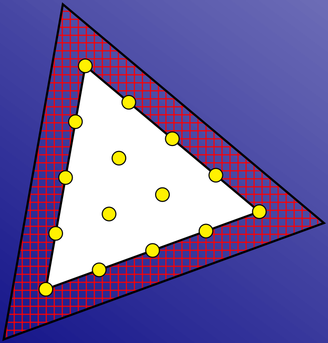
Then the Hamiltonians can be written (in the center-of-mass frame E) as

$$\mathcal{H}_r(\boldsymbol{x}, \boldsymbol{p}) = \sum_{\boldsymbol{\nu} \in S_N(\boldsymbol{\omega}_r)} \cos(\beta \langle \boldsymbol{\nu}, \boldsymbol{p} \rangle) \sqrt{V_{\boldsymbol{\nu}}(\boldsymbol{x}) V_{\boldsymbol{\nu}}(-\boldsymbol{x})}, \quad r = 1, \ldots, N-1.$$

The products $V_{\boldsymbol{\nu}}(\boldsymbol{x}) V_{\boldsymbol{\nu}}(-\boldsymbol{x})$ are positive in $\Sigma_{g,p}$ and vanish at certain boundary points.

Compactified Trigonometric RS Model

Quantum Case



The Hilbert space of lattice functions

Consider the uniform lattice $\Lambda_{p,M}$ consisting of points

$$\boldsymbol{x} = \boldsymbol{\rho}_p + \operatorname{sgn}(M) \sum_{j=1}^{N-1} m_j \boldsymbol{\omega}_{j,p}, \quad \text{with} \quad m_j \in \mathbb{N}_0, \quad \sum_{j=1}^{N-1} m_j \leq |M|.$$

This lattice fits the classical configuration space $\Sigma_{g,p}$ iff the following ***quantisation condition*** is satisfied

$$M = \frac{2\pi}{\alpha} p - Ng \in \mathbb{Z} \setminus \{0\}.$$

Let $L^2(\Lambda_{p,M})$ denote the finite-dimensional vector space of lattice functions

$$\phi\colon \Lambda_{p,M} \to \mathbb{C},$$

equipped with the inner product

$$(\phi, \psi)_{p,M} = \sum_{\boldsymbol{x} \in \Lambda_{p,M}} \phi(\boldsymbol{x}) \overline{\psi(\boldsymbol{x})}.$$

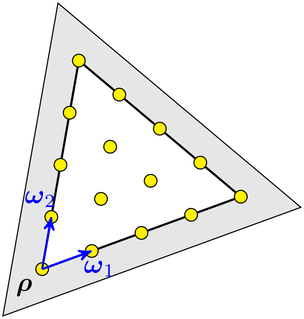


Figure: The 3-particle lattice $\Lambda_{1,4}$.

Its dimension equals the cardinality of $\Lambda_{p,M}$, which is $\binom{N-1+|M|}{|M|}$.

The quantum Hamiltonians

The following difference operators commute [Ruijsenaars '87]:

$$\hat{\mathcal{H}}_r = \sum_{\boldsymbol{\nu} \in S_N(\boldsymbol{\omega}_r)} V_{\boldsymbol{\nu}}^{1/2}(\boldsymbol{x}) \hat{T}_{\boldsymbol{\nu}} V_{\boldsymbol{\nu}}^{1/2}(-\boldsymbol{x}), \quad r = 1, \ldots, N-1,$$

where $\hat{T}_{\boldsymbol{\nu}} = \exp(\langle \boldsymbol{\nu}, \partial/\partial \boldsymbol{x} \rangle)$ is the translation operator acting on ϕ as

$$\big(\hat{T}_{\boldsymbol{\nu}} \phi\big)(\boldsymbol{x}) = \phi(\boldsymbol{x} + \boldsymbol{\nu}).$$

Let us introduce the operators

$$\hat{\mathcal{H}}_{r,M} \equiv \sum_{\boldsymbol{\nu} \in S_N(\boldsymbol{\omega}_r)} V_{\boldsymbol{\nu}}^{1/2}(\boldsymbol{x}) \hat{T}_{\mathrm{sgn}(M)\boldsymbol{\nu}} V_{\boldsymbol{\nu}}^{1/2}(-\boldsymbol{x}).$$

Proposition. 1. $\hat{\mathcal{H}}_{N-r,M}$ is the formal adjoint of $\hat{\mathcal{H}}_{r,M}$. **2.** The operators

$$\hat{H}_{r,M} = \frac{1}{2}(\hat{\mathcal{H}}_{r,M} + \hat{\mathcal{H}}_{N-r,M}), \quad r = 1, \ldots, N-1$$

are well-defined and self-adjoint on the Hilbert space $L^2(\Lambda_{p,M})$.

A factorised joint eigenfunction

Consider the lattice function $\Delta_p\colon \Lambda_{p,M} \to \mathbb{R}$ given by

$$\Delta_p(\boldsymbol{x}) = \prod_{\boldsymbol{a}\in A^+_{N-1,p}} \frac{\sin\frac{\alpha}{2}\langle \boldsymbol{a},\boldsymbol{x}\rangle}{\sin\frac{\alpha}{2}\langle \boldsymbol{a},\boldsymbol{\rho}_p\rangle} \frac{(\langle \boldsymbol{a},\boldsymbol{\rho}_p\rangle + \operatorname{sgn}(M)g : \sin_\alpha)_{\langle \boldsymbol{a},\boldsymbol{x}-\boldsymbol{\rho}_p\rangle}}{(\langle \boldsymbol{a},\boldsymbol{\rho}_p\rangle + 1 - \operatorname{sgn}(M)g : \sin_\alpha)_{\langle \boldsymbol{a},\boldsymbol{x}-\boldsymbol{\rho}_p\rangle}},$$

where $(z : \sin_\alpha)_m$ stands for the trigonometric Pochhammer symbol

$$(z : \sin_\alpha)_m = \begin{cases} 1, & \text{if } m = 0,\\ \sin\frac{\alpha}{2}(z)\ldots\sin\frac{\alpha}{2}(z+m-1), & \text{if } m = 1,2,\ldots\\ \dfrac{1}{\sin\frac{\alpha}{2}(z-1)\ldots\sin\frac{\alpha}{2}(z+m)}, & \text{if } m = -1,-2,\ldots \end{cases}$$

Recurrence relations. For any $\boldsymbol{x}\in\Lambda_{p,M}$ and $\boldsymbol{\nu}\in S_N(\boldsymbol{\omega}_r)$, $r = 1,\ldots,N-1$ satisfying $\boldsymbol{x} + \operatorname{sgn}(M)\boldsymbol{\nu}\in\Lambda_{p,M}$, we have

$$\frac{\Delta_p(\boldsymbol{x}+\operatorname{sgn}(M)\boldsymbol{\nu})}{\Delta_p(\boldsymbol{x})} = \frac{V_{\boldsymbol{\nu}}(\boldsymbol{x})}{V_{\boldsymbol{\nu}}(-\boldsymbol{x}-\operatorname{sgn}(M)\boldsymbol{\nu})}.$$

Corollary. $\Delta_p(\boldsymbol{x})^{1/2}$ is a joint eigenfunction of the quantum Hamiltonians $\hat{H}_{r,M}$.

Joint eigenfunctions

We define the lattice functions $\Psi_{\boldsymbol{y},p}\colon \Lambda_{p,M} \to \mathbb{C}$ by letting

$$\Psi_{\boldsymbol{y},p}(\boldsymbol{x}) = \frac{1}{\mathcal{N}_0^{1/2}} \Delta_p(\boldsymbol{x})^{1/2} \Delta_p(\boldsymbol{y})^{1/2} P_{\sigma_p(\boldsymbol{y})}(\check{\boldsymbol{x}}),$$

where $\check{\boldsymbol{x}} = \mathrm{sgn}(M)(\boldsymbol{x} - \frac{2\pi}{\alpha}\sum_{j=1}^{N-1}\boldsymbol{\omega}_{j,p})$ and $P_{\boldsymbol{\lambda}}$ denote the **self-dual A_{N-1} Macdonald polynomials** with parameters $t = e^{\mathrm{i}\alpha \mathrm{sgn}(M) g}, q = e^{\mathrm{i}\alpha}$.
The self-dual property of $P_{\boldsymbol{\lambda}}$ entails that for any $\boldsymbol{x}, \boldsymbol{y} \in \Lambda_{p,M}$ we have

$$\Psi_{\boldsymbol{y},p}(\boldsymbol{x}) = \Psi_{\boldsymbol{x},p}(\boldsymbol{y}),$$

which in turn can be used to show that $\Psi_{\boldsymbol{y},p}$ are **joint eigenfunctions** of the quantum Hamiltonians:

$$\hat{H}_{r,M}\Psi_{\boldsymbol{y},p} = E_r(\boldsymbol{y})\Psi_{\boldsymbol{y},p}, \quad r = 1, \ldots, N-1.$$

Finally, the orthogonality of the Macdonald polynomials implies that $\Psi_{\boldsymbol{y},p}$ form an **orthonormal eigenbasis** in $L^2(\Lambda_{p,M})$.

Summary and plans for future work

In conclusion, we considered the new compact forms of trigonometric RS models with type (i) coupling parameters and

- **defined the appropriate quantum Hamiltonians** as difference operators acting on a finite-dimensional Hilbert space of lattice functions,
- explicitly **solved the corresponding eigenvalue problem** in terms of A_{N-1} Macdonald polynomials.

We intend to generalise these results to

- the case of type (ii) coupling parameters (**in progress**),
- compactified models attached to root systems other than A_{N-1},
- finite-dimensional representations of SL(2,$\mathbb{Z}$),
- new quantum elliptic models?