Investigations on the Quantum Statistics in Non-Extensive Physics and its Phenomenological Applications in High Energy Nuclear Physics

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Outline

- Theoretical Research
  - Tsallis Non-Extensive Statistics
  - Generalized Ensemble Theory
  - Particle-Hole Symmetry
  - Kaniadakis Statistics
- Applications in the High Energy Physics
  - Theoretical Basis
  - Results and Discussions
- Summary and Outlook

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  - ✓ **Tsallis Non-Extensive Statistics**
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metaphor

- $\frac{dy}{dx} = 0,\ (y(0) = 1)$ leads to $y(x) = 1.$
- $\frac{dy}{dx} = 1,\ (y(0) = 1)$ leads to $y(x) = 1 + x.$
- $\frac{dy}{dx} = y,\ (y(0) = 1)$ leads to $y(x) = \exp(x)$
- $\frac{dy}{dx} = y^q,\ (y(0) = 1)$ leads to $y(x) = [1 + (1-q)x]^{1/(1-q)},\ (q \in R)$

$$\exp(x) \longrightarrow \exp_q(x) := [1 + (1-q)x]^{1/(1-q)}, \quad (q \in R) \tag{4}$$

with its inverse function

$$\ln x \longrightarrow \ln_q x := \frac{x^{1-q} - 1}{1-q} \tag{5}$$

Tsallis Non-Extensive Statistics

In 1988, **C. Tsallis**[1] firstly proposed the non-extensive entropy

$$S_{Ts} = \ln_q W = \frac{W^{1-q} - 1}{1-q} \longleftarrow \ln W = S_{BG} \qquad (6)$$

with $p_i = \frac{1}{W}$ we obtain

$$S_q = \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} := \frac{\sum_i^W p_i^q - 1}{1-q} \longleftarrow \sum_{i=1}^{W} p_i \ln \frac{1}{p_i} = S_{Sh} \qquad (7)$$

where the normalized condition $\sum p_i = 1$ is applied.

1. C. Tsallis, J. Stat. Phys. 52, 479 (1988); Introduction to Nonextensive Statistical Mechanics.

properties

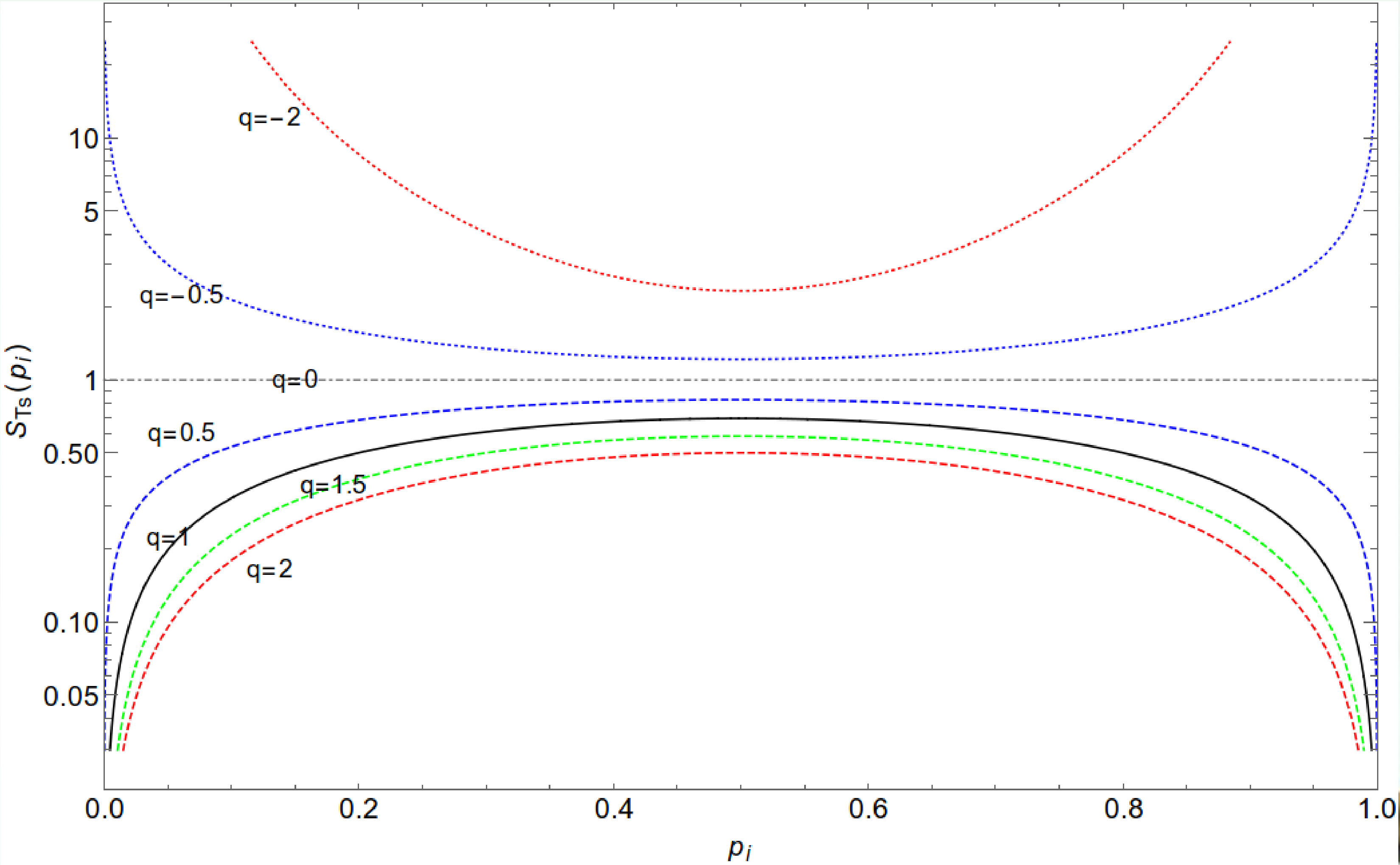
- Non-negativity
- Extremal at Equal Probabilities.

  Maximum ($q > 0$); Minimum ($q < 0$); Constant ($q = 0$): $S_{q=0} = W - 1$
- Expansibility
- Non-additivity

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A) \cdot S_q(B) \quad (8)$$

- …

properties



$W = 2$ and $S_q = (p_i^q + (1 - p_i)^q - 1)/(1 - q)$.

Thermodynamical Foundations/Applications

- Correlated Anomalous Diffusion: generalized Fokker-Planck Equation
- Central Limit Theorems[1]
- Zeroth Law[2]
- Equipartition and Virial theorems[3]
- Second Law[4]
- Quantum H-theorem[5]
- Fluctuation-Dissipation Theorem[6]

1. C. Tsallis and S. M. D. Queiros, AIP Conf. Proc. 965, 8(2007); 21(2007).
2. T. S. Biro and P. Van, Phys. Rev. E 83, 061147 (2011).
3. S. Martinez, F. Pennini and A. Plastino, Phys. Lett. A 278, 47-52 (2000).
4. S. Abe and A. K. Rajagopal, Phys. Rev. Lett. 91,12 (2003).
5. R. Silva, D. H. A. L. Anselmo and J. S. Alcaniz, EPL 89, 10004 (2010).
6. A. Chame and E. V. L. de Mello, J. Phys. A: Math. Gen. 27, 3663-3670 (1994).

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Tsallis Lagrange Multipliers Method

- Normality:

$$\sum_{i}^{W} p_i = 1$$

- As for the energy constraint,

$$\sum_i^W p_i E_i = U_1$$

$$p_i^{(1)} \propto [1-(q-1)\beta E_i]^{\frac{1}{q-1}} \quad (9)$$

Tsallis Lagrange Multipliers Method

- Normality:

$$\sum_{i}^{W} p_i = 1$$

- As for the energy constraint,

$$\sum_i^W p_i^q E_i = U_2$$

$$p_i^{(2)} \propto [1-(1-q)\beta E_i]^{\frac{1}{1-q}} \qquad (10)$$

Tsallis Lagrange Multipliers Method

- Normality:

$$\sum_{i}^{W} p_i = 1$$

- As for the energy constraint,

$$\sum_i^W \frac{p_i^q}{\sum_j^W p_j^q} E_i = U_3$$

$$p_i^{(3)} \propto \left[1 - (1-q)\frac{\beta}{\sum_j^W p_j^q}(E_i - U_3)\right]^{\frac{1}{1-q}} \quad (11)$$

Modified Lagrange Multipliers Method

It is reasonable to argue that the term of $\sum_{j=1}^{W} p_j^q$ characterizes the non-extensive properties of the whole system and should be not connected to p_i explicitly (the same as the term $\sum_{j=1}^{W} p_j = 1$ in BG case with $q \to 1$),namely,

$$\frac{\partial}{\partial p_i} \sum_{j=1}^{W} p_j^q = 0 \qquad (12)$$

Ke-Ming Shen, Ben-Wei Zhang and En-Ke Wang, Physica A 487: 215-224, (2017).

Modified Lagrange Multipliers Method

Thus we firstly add one more constraint

$$\sum_{j=1}^{W} p_j^q = C_q \qquad (13)$$

to obtain the **generalized *q*-probability function**:

$$p_i = \frac{1}{\bar{Z}_q}\left[1-(1-q)\frac{\beta^*}{\sum_j p_j^q}(E_i - U_q)\right]^{\frac{1}{1-q}} = \frac{1}{\bar{Z}_q}\exp_q[-\beta_q^*(E_i - U_q)]$$
$$= \frac{1}{Z_q}\left[1-(1-q)\frac{\beta}{\sum_j p_j^q}E_i\right]^{\frac{1}{1-q}} = \frac{1}{Z_q}\exp_q[-\beta_q E_i] \qquad (14)$$

where $\beta^* = \frac{\beta_0}{1-\gamma_0(1-q)}, \; \beta = \frac{\beta^*}{1+(1-q)\frac{\beta^*}{\sum_j p_j^q}U_q}, \; \beta_q^* = \frac{\beta^*}{\sum_j p_j^q}, \; \beta_q = \frac{\beta}{\sum_j p_j^q}.$

generalized q-thermodynamical relations

1. $\sum p_i^q = (\bar{Z}_q)^{1-q}.$
2. $S_q = \ln_q \bar{Z}_q.$
3. $Z_q = \bar{Z}_q \cdot \exp_q(-\beta_q U_q).$
4. $U_q = -\frac{\partial}{\partial \beta} \ln_q Z_q.$
5. $S_q = \ln_q Z_q - \beta \frac{\partial}{\partial \beta} \ln_q Z_q.$
6. $\beta = \frac{1}{T}.$
7. $\ldots$

generalized grand canonical ensemble theory

Consider the grand canonical ensemble theory, similarly we have

$$p_s = \frac{1}{\Xi_q} \exp_q(-\alpha_q N - \beta_q E_s) \qquad (15)$$

with its corresponding thermodynamical relationships:

$\bar{N}_q = -\frac{\partial}{\partial \alpha} \ln_q \Xi_q$, $U_q = -\frac{\partial}{\partial \beta} \ln_q \Xi_q$, $dS_q = \frac{1}{T}(dU_q - Y_q dy - \mu d\bar{N}_q)$.
Moreover, $\beta = \frac{1}{T}$, $\alpha = -\frac{\mu}{T}$.

generalized q-FDD and q-BED

Supposing there is only one kind of particle whose energy of l_{th} level is ϵ_l, and number is n_l, therefore

$$N = \sum_l n_l, \quad E = \sum_l \epsilon_l n_l$$

Then the q-average occupation number of particle in each energy level l goes like:

$$\bar{n}_l = -\frac{\partial}{\partial \alpha} \ln_q Z_q(l) \qquad (16)$$

where $Z_q(l) = \sum_{n_l} e_q^{-(\alpha_q + \beta_q \epsilon_l) n_l}$ is the l-th-partition function.

generalized q-FDD and q-BED

1. Fermion: $n_l = 0, 1$ leads to q-FDD:

$$\bar{n}_l^{FD} = -\frac{\partial}{\partial\alpha} \ln_q Z_q(l) = \frac{1}{\sum p_t^q}\left(\frac{1}{e_{2-q}^{\alpha_q+\beta_q\epsilon_l} + 1}\right)^q$$

2. Boson: $n_l = 0, 1, 2, \cdots$, so

$$\bar{n}_l^{BE} = -\frac{\partial}{\partial\alpha} \ln_q Z_q(l) = \frac{1}{\sum p_t^q}\left(\frac{1}{e_{2-q}^{\alpha_q+\beta_q\epsilon_l} - 1}\right)^q .$$

generalized *q*-FDD and *q*-BED

$$\bar{n}_l = \left(\frac{1}{e_{2-q}^{\alpha_q+\beta_q\epsilon_l} \pm 1}\right)^q \qquad (17)$$

$$\bar{n}'_l = (\bar{n}_l)^{1/q}$$

$$\bar{n}'_l = \frac{1}{e_{2-q}^{\alpha_q+\beta_q\epsilon_l} \pm 1} \qquad (18)$$

the previous *q*-FDD and *q*-BED[1]

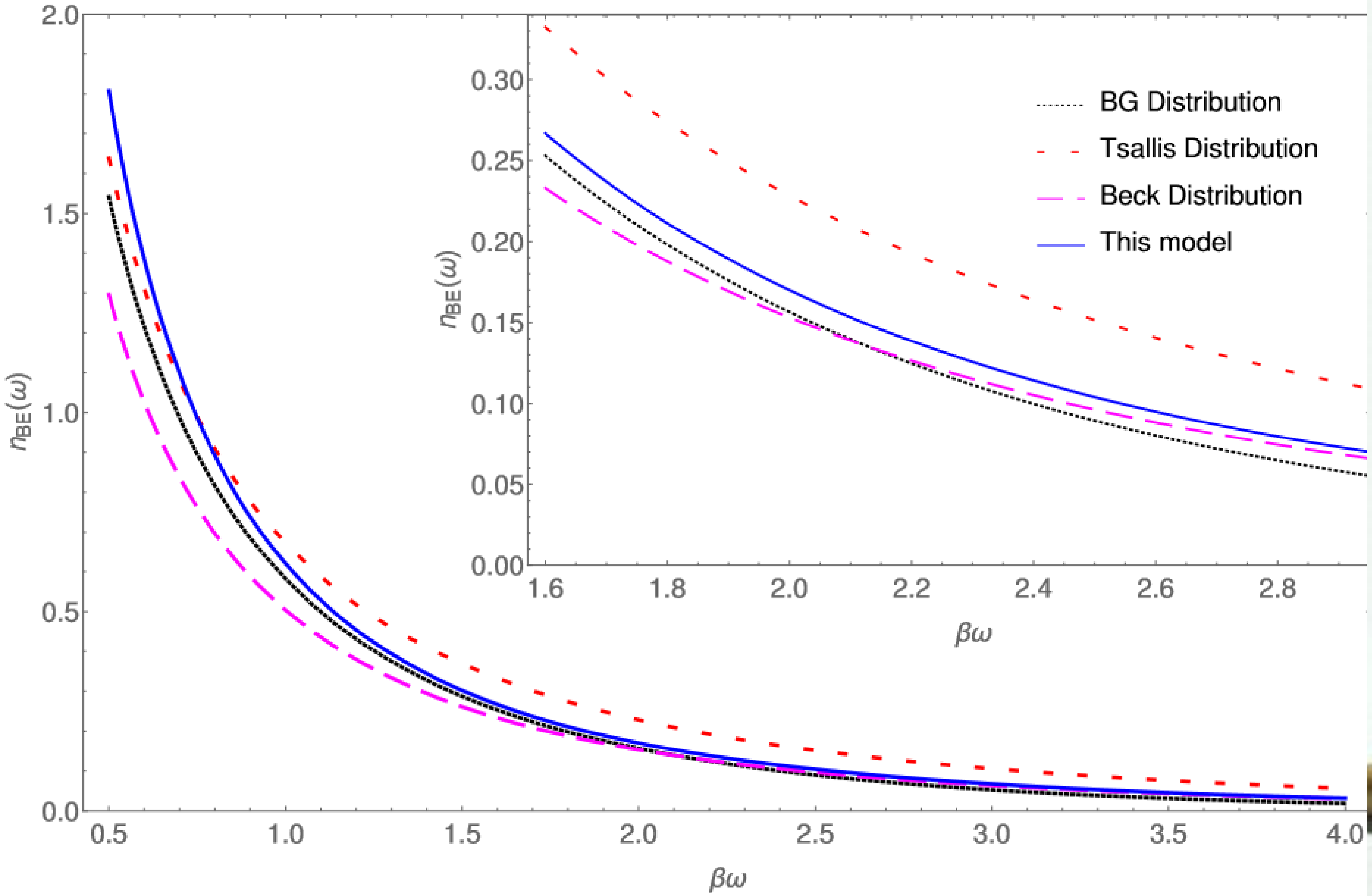
$$q \to 1$$

$$\bar{n}''_l = \frac{1}{(e_{2-q}^{\alpha_q+\beta_q\epsilon_l})^q \pm 1} \qquad (19)$$

the superstatistical *q*-FDD and *q*-BED by **C. Beck**[2]

1. F. Buyukkilic, D. Demirhan, Phys. Lett. A 181 24 (1993); *etc.*
2. C. Beck, Physica A 286 164 (2000).

generalized q-FDD and q-BED



Different BE distributions with $q = 1.2$ and $\alpha_q = 0$ for convenience.

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KMS Relation

Kubo – Martin – Schwinger:

$$
\begin{aligned}
\langle A_t B_0 \rangle &= Tr(e^{-\beta H} e^{itH} A e^{-itH} B) \\
&= Tr(e^{-\beta H} e^{itH} A e^{-itH} \mathbf{e^{\beta H} e^{-\beta H}} B) \\
&= Tr(e^{i(t+i\beta)H} A e^{-i(t+i\beta)H} e^{-\beta H} B) \\
&= Tr(e^{-\beta H} B e^{i(t+i\beta)H} A e^{-i(t+i\beta)H}) \\
&= \langle B_0 A_{t+i\beta} \rangle
\end{aligned}
$$

1. R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).
2. P. C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).

KMS Relation in quantum statistics

Consider the case with $A_t = e^{-i\omega t}a$ and $B_t = A_t^\dagger$(so $B_0 = a^\dagger$), then we have,

$$\langle aa^\dagger \rangle = \langle a^\dagger a \rangle e^{\beta\omega} \tag{20}$$

with respect to the commutator for **Bose system**, $[a, a^\dagger] = 1$ and the Hermitian operator, $n = a^\dagger a$, so

$$n(\omega) = \frac{1}{e^{\beta\omega} - 1} \tag{21}$$

which is nothing but the well-known classical Bose-Einstein distribution(BED) of the occupation number.

Particle-Hole Ratio with CPT

Missing negative energy particle = positive energy hole:

$$-n(-\omega) = 1 + n(\omega) \tag{22}$$

Thermodynamically, ***for Bosons***, the statistical weight factor with energy,

$$f(\omega) := \frac{n(\omega)}{1 + n(\omega)} \tag{23}$$

So we get the **generalized KMS relation** in quantum statistics,

$$f(\omega)f(-\omega) = 1 \tag{24}$$

Tamas S. Biro, **Ke-Ming Shen**, and Ben-Wei Zhang, Physica A 428: 410-415, (2015).

KMS Relation for q-Exponential breaks!

Certainly for BG case it satisfies this relation naturally:

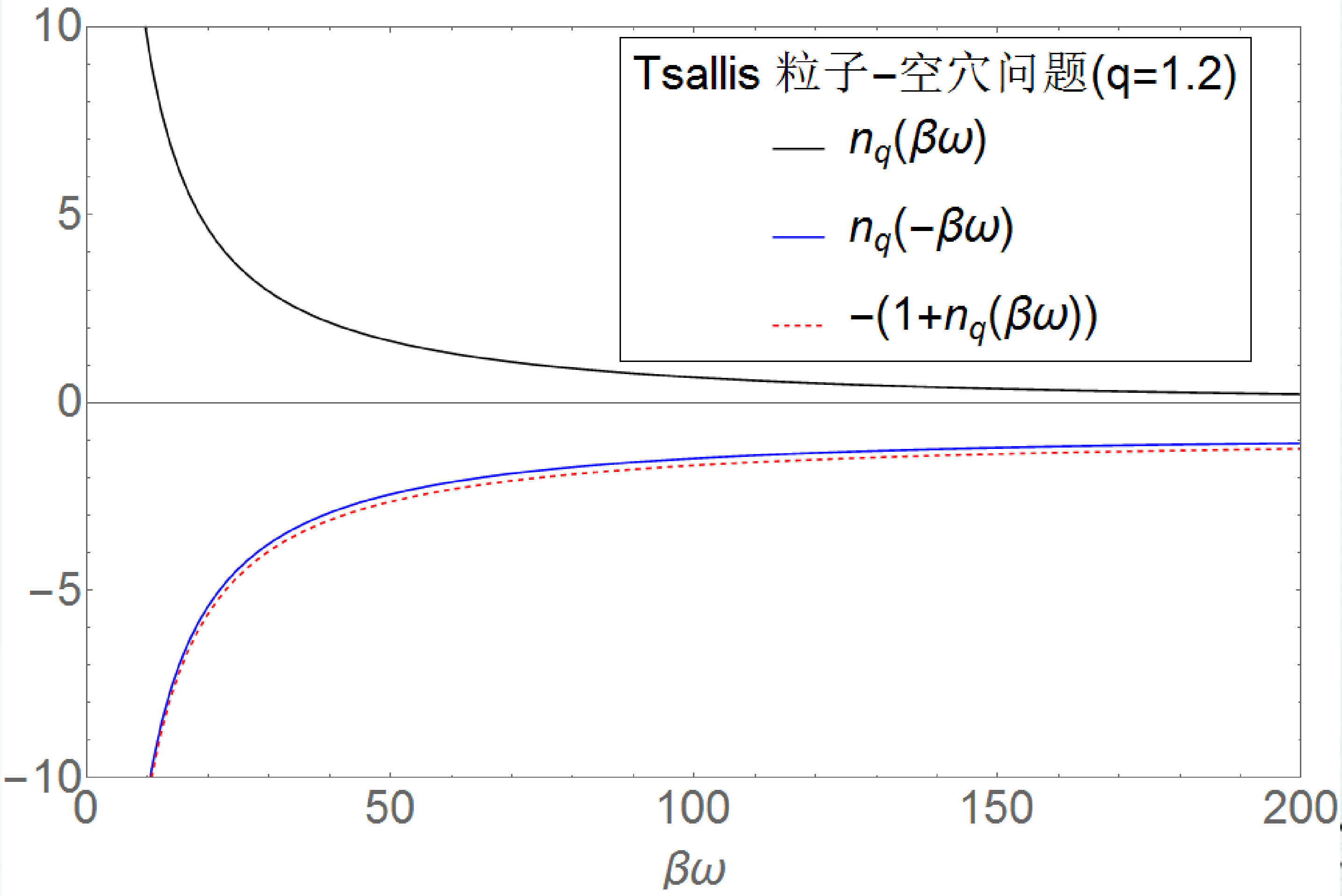
$$f(\omega)f(-\omega) = \exp(-\beta\omega)\cdot\exp(\beta\omega) = 1 \tag{25}$$

while for Tsallis case **it breaks**!

$$\begin{aligned} f_q(\omega)f_q(-\omega) &= \exp_q(-\beta\omega)\cdot\exp_q(\beta\omega) \\ &= [1-(1-q)\beta\omega]^{\frac{1}{1-q}}\cdot[1+(1-q)\beta\omega]^{\frac{1}{1-q}} \\ &\neq 1 \end{aligned} \tag{26}$$

Tamas S. Biro, **Ke-Ming Shen**, and Ben-Wei Zhang, Physica A 428: 410-415, (2015).

KMS Relation for q-Exponential breaks!



$$n_q(-\beta\omega) \neq -(1+n_q(\beta\omega)) \quad (27)$$

solutions

Assume $n_{KMS}(x) = A[n_q(x) + n_{q'}(x)] + B$, where we've known $e_q(-x) = 1/e_{q'}(x)$ with $q' = 2 - q$. To satisfy the KMS relation we get

$$n^{Lin}(\omega) = \frac{1}{2}[n_q(\omega) + n_{q'}(\omega)] \quad (28)$$

Linear Combination

with the weight factor $f^{Lin} = -\frac{n_q(\omega)+n_{q'}(\omega)}{n_q(-\omega)+n_{q'}(-\omega)}$, which satisfies the KMS relation.

Tamas S. Biro, **Ke-Ming Shen**, and Ben-Wei Zhang, Physica A 428: 410-415, (2015).

solutions

In other way, considering the fractional normalization of $q-$exponential, we can have

$$f(\omega) = e^{-\beta\omega} \rightarrow f^{Fra}(\omega) = e_q(\frac{-\beta\omega}{2})/e_q(\frac{\beta\omega}{2}) \quad (29)$$

with $n^{Fra}(\omega) = \frac{e_q(-\beta\omega/2)}{e_q(\beta\omega/2) - e_q(-\beta\omega/2)}.$

Fractional Normalization

Tamas S. Biro, **Ke-Ming Shen**, and Ben-Wei Zhang, Physica A 428: 410-415, (2015).

solutions

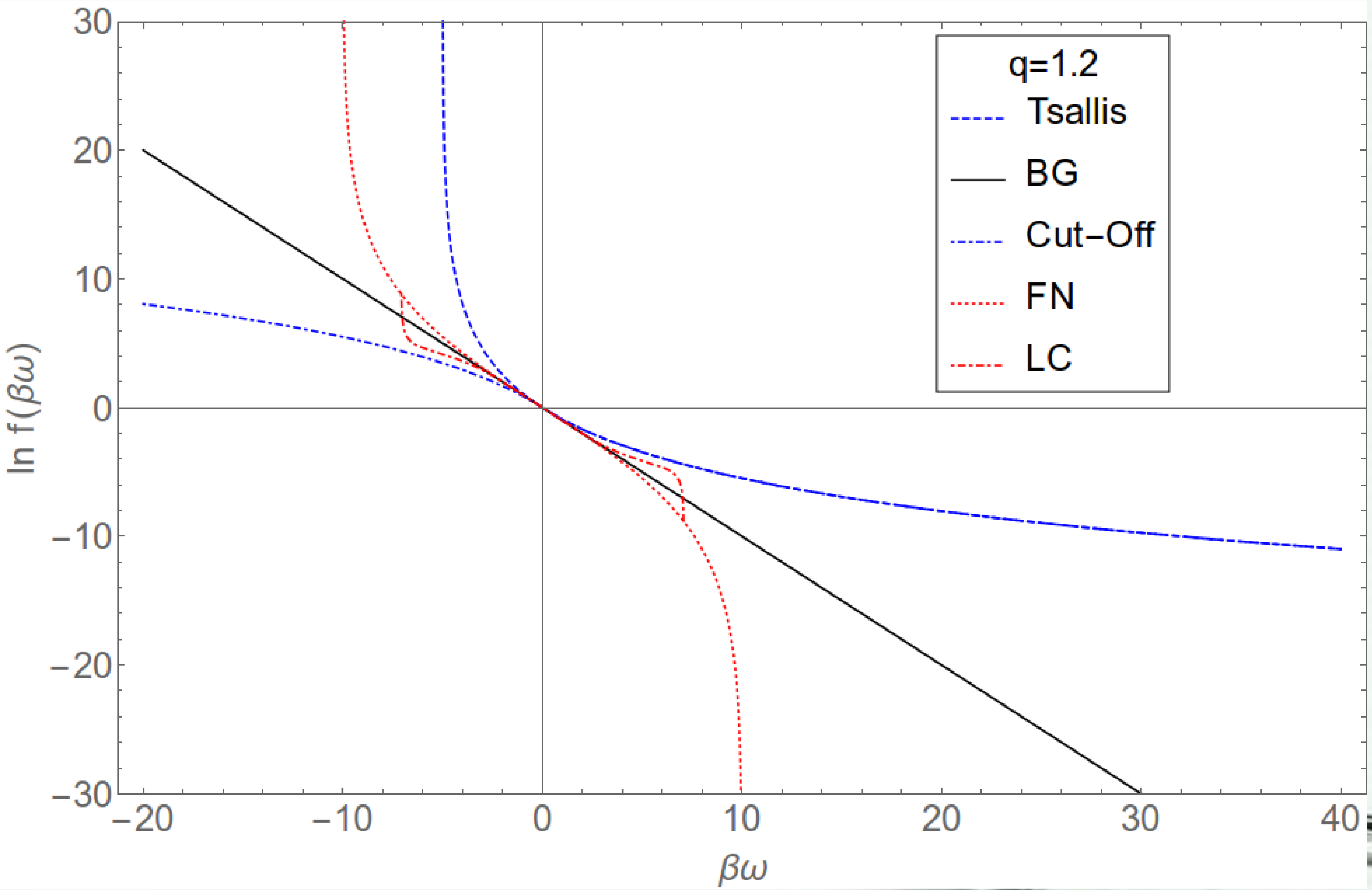
Moreover, in 2005 A. M. Teweldeberhan *et. al*.[1] improved the previous Tsallis q-exponential function and proposed a cut-off solution:

$$\widetilde{\exp}_q(x) := \begin{cases} [1+(q-1)x]^{\frac{1}{q-1}} & x > 0 \\ [1+(1-q)x]^{\frac{1}{1-q}} & x \leq 0 \end{cases} \quad (30)$$

Cut-Off

1. A. M.Teweldeberhan, *et. al.*, Phys. Lett. A 343,71-78 (2005).

Comparisons of *f*



Different weight factors with q=1.2.

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κ-exponential function

Except of what we mentioned above, we can also re-deform the exponential function into another power-law function:

- $A(x) = A_e(x) + A_o(x).$
- $\exp(x) \rightarrow \exp_\kappa(x) := A(x)^{\frac{1}{\kappa}}.$
- $A(x) \cdot A(-x) = 1.$

Following all of these conditions, the simplest one is then given as[1]

$$\exp_\kappa(x) := [\sqrt{1+(\kappa x)^2} + \kappa x]^{\frac{1}{\kappa}} \quad (31)$$

1. G. Kaniadakis, Physica A 296 405 (2001).

κ-entropy

The corresponding κ−logarithm is

$$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} \tag{32}$$

which gives out the κ−entropy:

$$S_\kappa(p_i) = \ln_\kappa W = -\sum p_i \ln_\kappa p_i \tag{33}$$

and **the κ−probability distribution function**:

$$p_i = \frac{1}{e_\kappa} \exp_\kappa(-\gamma_\kappa \beta E_i + \gamma_\kappa \alpha) \tag{34}$$

where $\gamma_\kappa \equiv \frac{1}{\sqrt{1-\kappa^2}}, e_\kappa \equiv (\frac{1+\kappa}{1-\kappa})^{1/2\kappa}$.

1. G. Kaniadakis, Physica A 296 405 (2001).

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Theoretical basis

Non-extensive statistical mechanics can be considered an appropriate basis to deal with physical phenomena where ***strong dynamical correlations***, ***long-range interactions*** and ***microscopic memory effects*** take place[1, 2].

Moreover, for such systems like QGP formed in heavy-ion collisions, the size ($\boldsymbol{N \ll N_A}$) needs be re-consideration whether Boltzmann statistics is still appropriate or not[3].

1. C. Tsallis, J. Stat. Phys. 52, 479 (1988); Introduction to Nonextensive Statistical Mechanics.
2. A. Lavagno, D. Pigato and P. Quarati, J. Phys. G: Nucl. Part. Phys. 37 (2010) 115102 (16pp); and its citations 23-26.
3. G. Biro, G. G. Barnafoldi, T. S. Biro, K. Urmossy and A. Takacs, Entropy, 19(3), 88 (2017); and refs.

Heavy-Ion Collisions

R. Hagedorn[1] proposed the QCD inspired empirical formula to describe experimental hadron production data[2]:

$$E\frac{d^3\sigma}{d^3p} = C(1+\frac{p_T}{p_0})^{-n} \to \begin{cases} \exp(-\frac{np_T}{p_0}) & p_T \to 0 \\ (\frac{p_0}{p_T})^n & p_T \to \infty \end{cases} \quad (35)$$

which coincides with

$$h_q(p_T) = C_q \exp_q(-\beta p_T) = C_q[1-(1-q)\frac{p_T}{T}]^{\frac{1}{1-q}} \quad (36)$$

for $n = 1/(q-1)$ and $p_0 = nT$.

1. R. Hagedorn, Riv. Nuovo Cim. 6N10, 1 (1983).
2. C. Y. Wong and G. Wilk, PRD 87, 114007 (2013).

Heavy-Ion Collisions

Using the κ-deformed statistics, we can also have,

$$E\frac{d^3\sigma}{d^3p} \propto \exp_\kappa(-\frac{p_T}{T}) \quad (37)$$

Thus can we study the p_T spectra with the similar steps.

Ke-Ming Shen, S. T. Hou, Gergely G. Barnafoldi, G. Biro,Tamas S. Biro, En-Ke Wang and Ben-Wei Zhang, *in preparation*.

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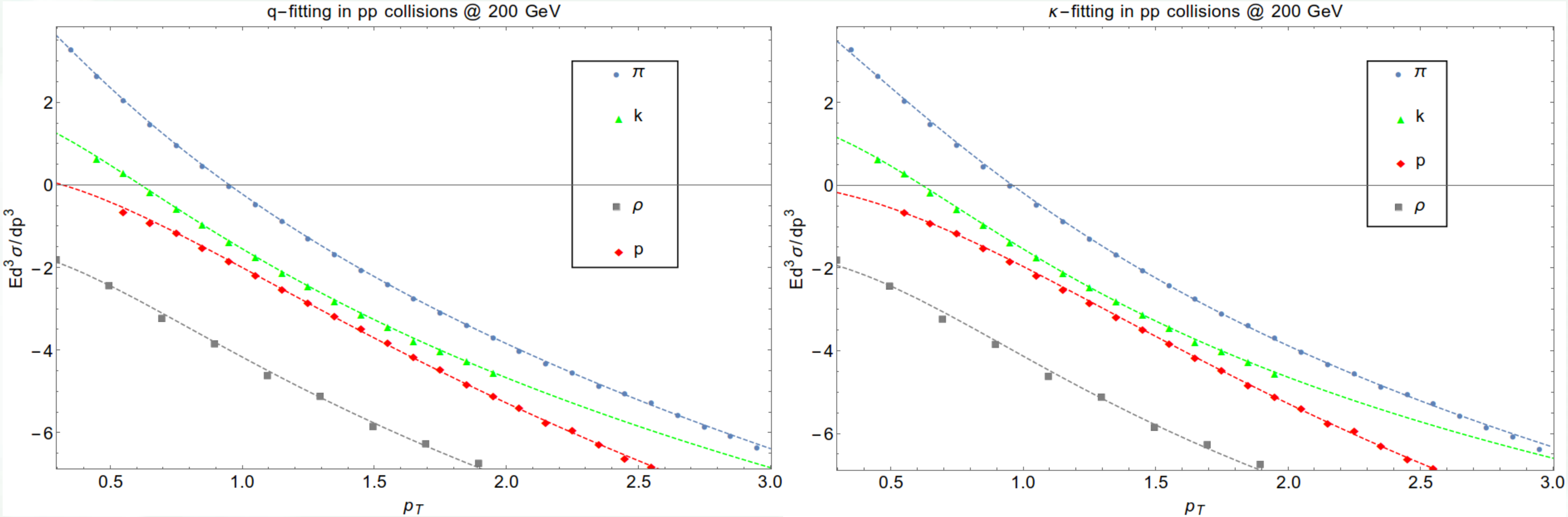
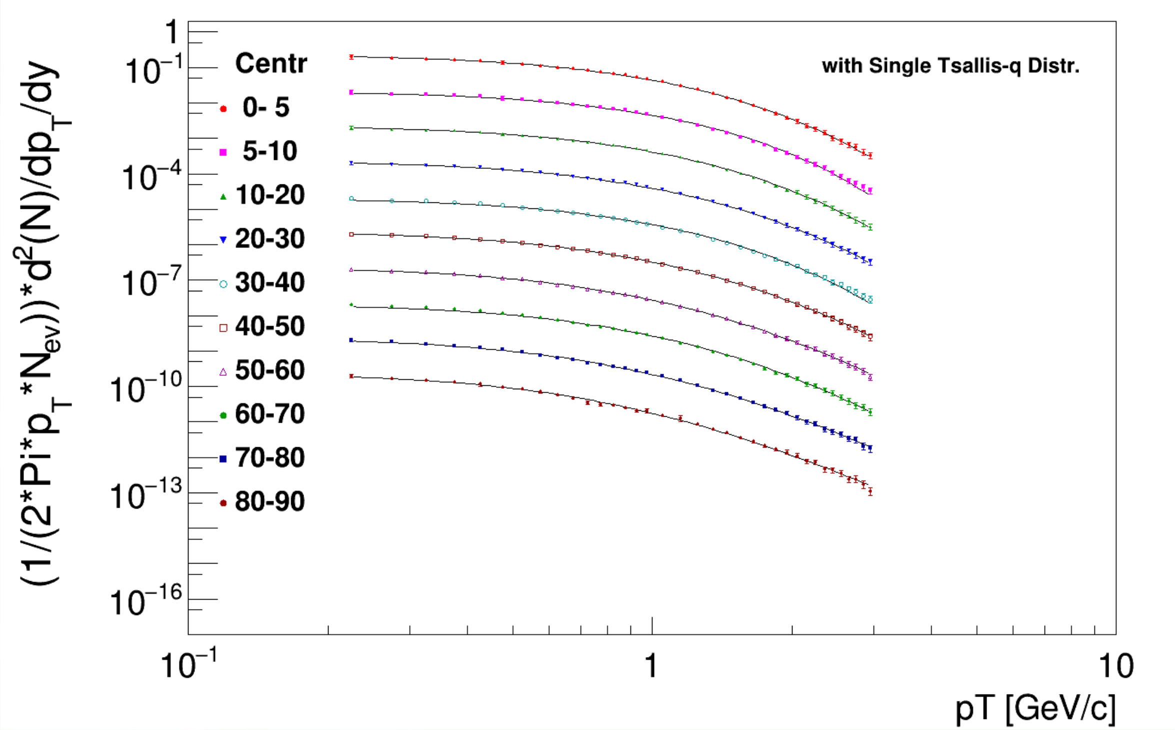
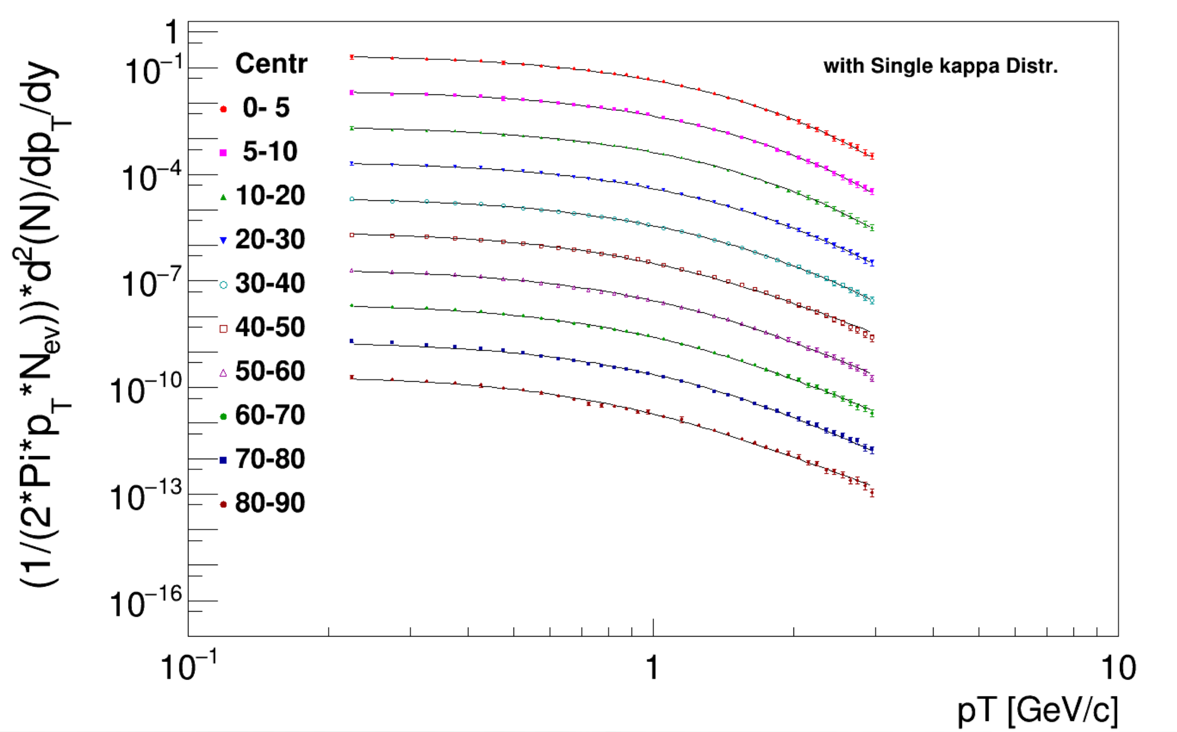
fittings in *pp* collisions



Non-extensive fittings of the p_T spectra at 200 GeV in *pp* collisions

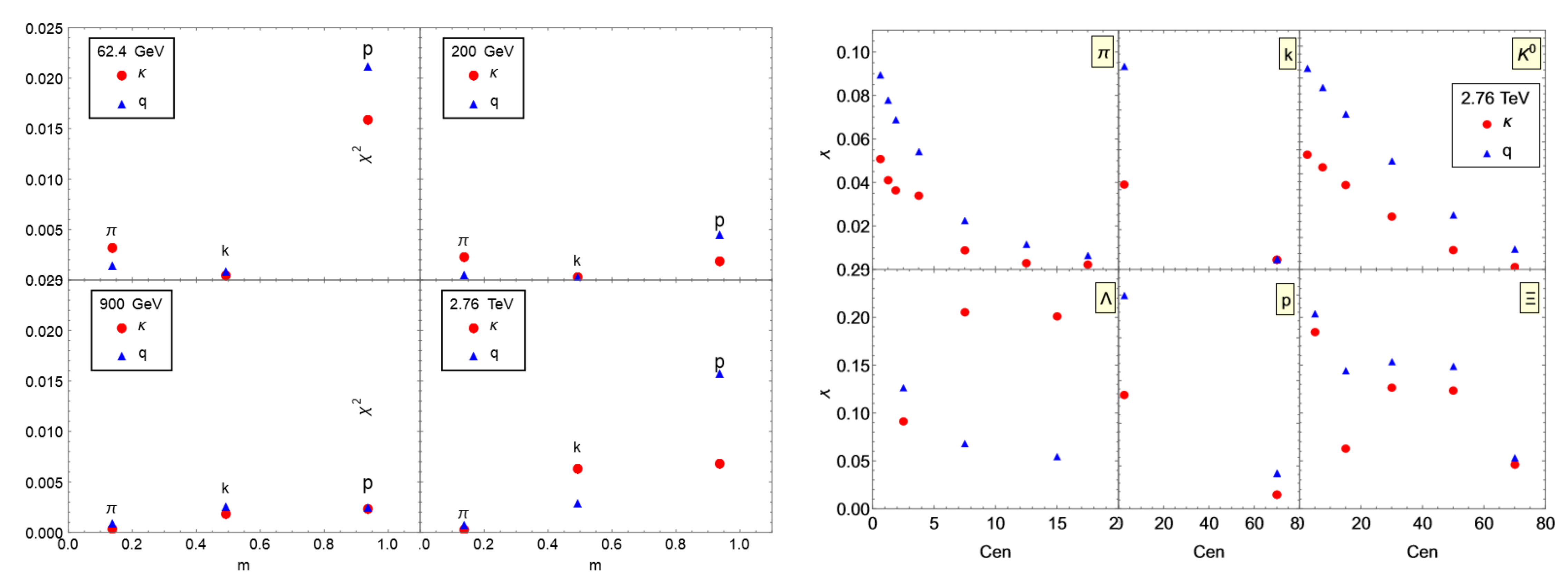
fittings in AA collisions





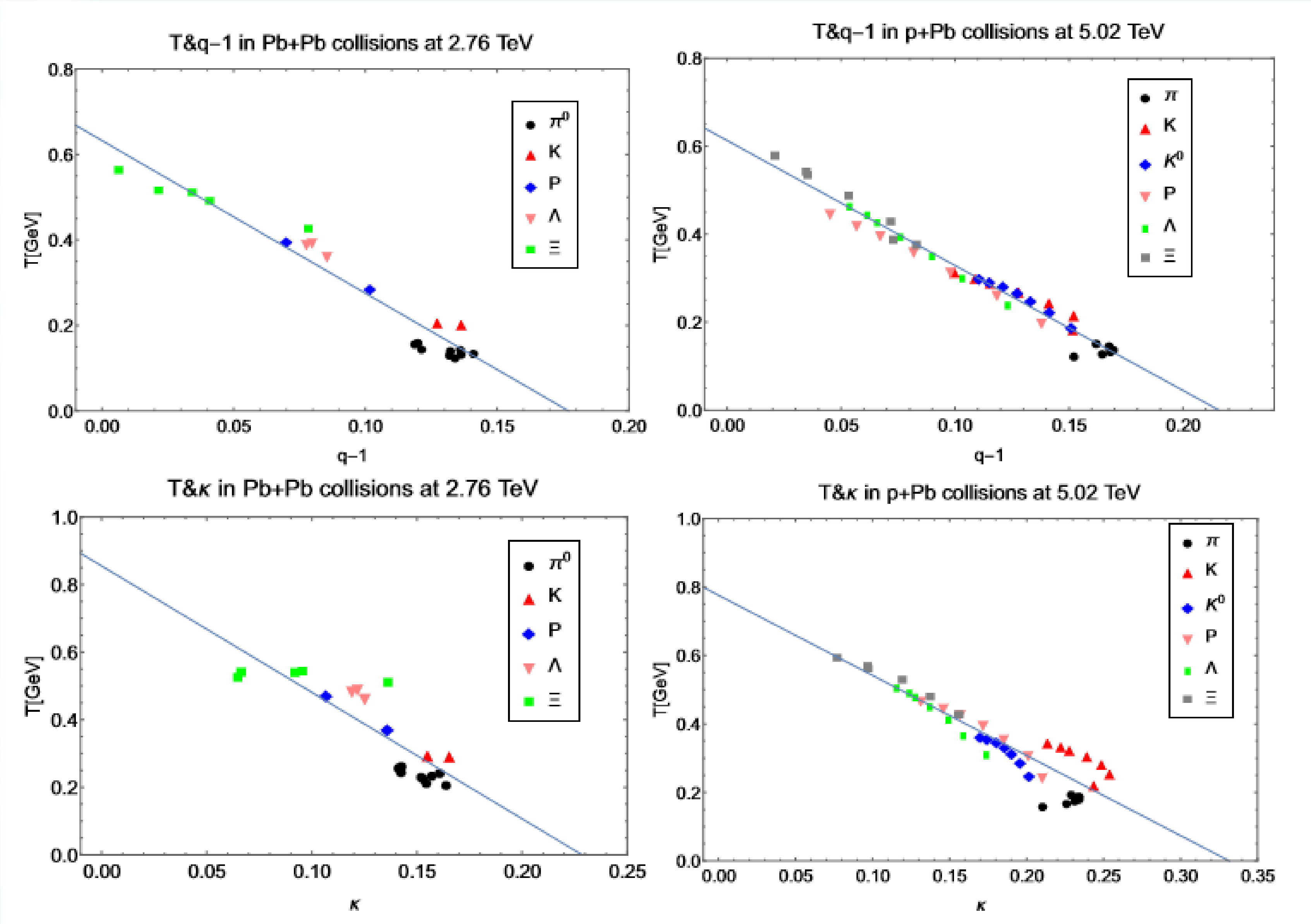
Non-extensive fittings of the p_T spectra at 2.76 TeV in *PbPb* collisions

fittings' fluctuations



Comparisons among ***q*'s and *κ*'s fitting χ^2** for different *pp* collisions and *PbPb* collisions at 2.76 TeV.

fitting parameters in *AA* collisions



Connections between the non-extensive parameter $q(\kappa)$ and fitting parameter T in different collisions.

Tsallis *q*

Firstly we think of the more generalized *q*'s relation[1]

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \quad (38)$$

- ***pp:*** Assuming now that the average occupancy of phase space by the newly produced hadrons, *f*, is constant, by the negative binomial distribution (NBD),

$$T = \frac{E}{f}(q-1) \quad (39)$$

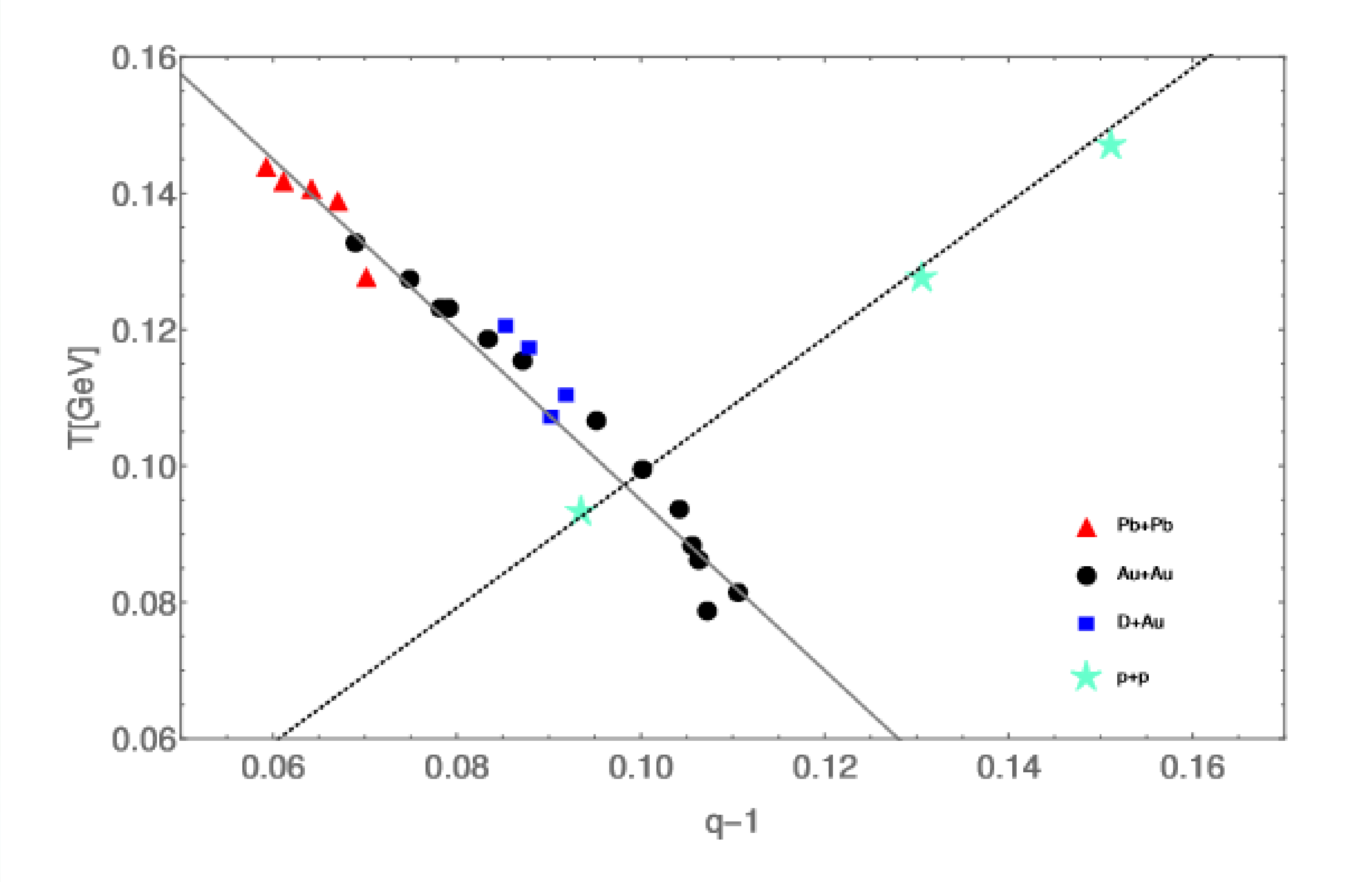
- ***AA:*** for different collisions the relative variance due to its fluctuations stays approximately constant,

$$T = E[\sigma^2 - (q-1)] \quad (40)$$

Ke-Ming Shen, Tamas S. Biro and En-Ke Wang, Physica A 492: 2353-2360 (2018).

1. T. S. Biro, G. G. Barnafoldi and P. Van, Physica A 417, 215 (2015).

compare with data results



Data are from G. Wilk's collection[1].

1. G. Wilk, Z. Wlodarczyk, AIP Conf. Proc. 1558 893 (2013); and its refs.

constant $\langle p_T^2 \rangle$

- q:

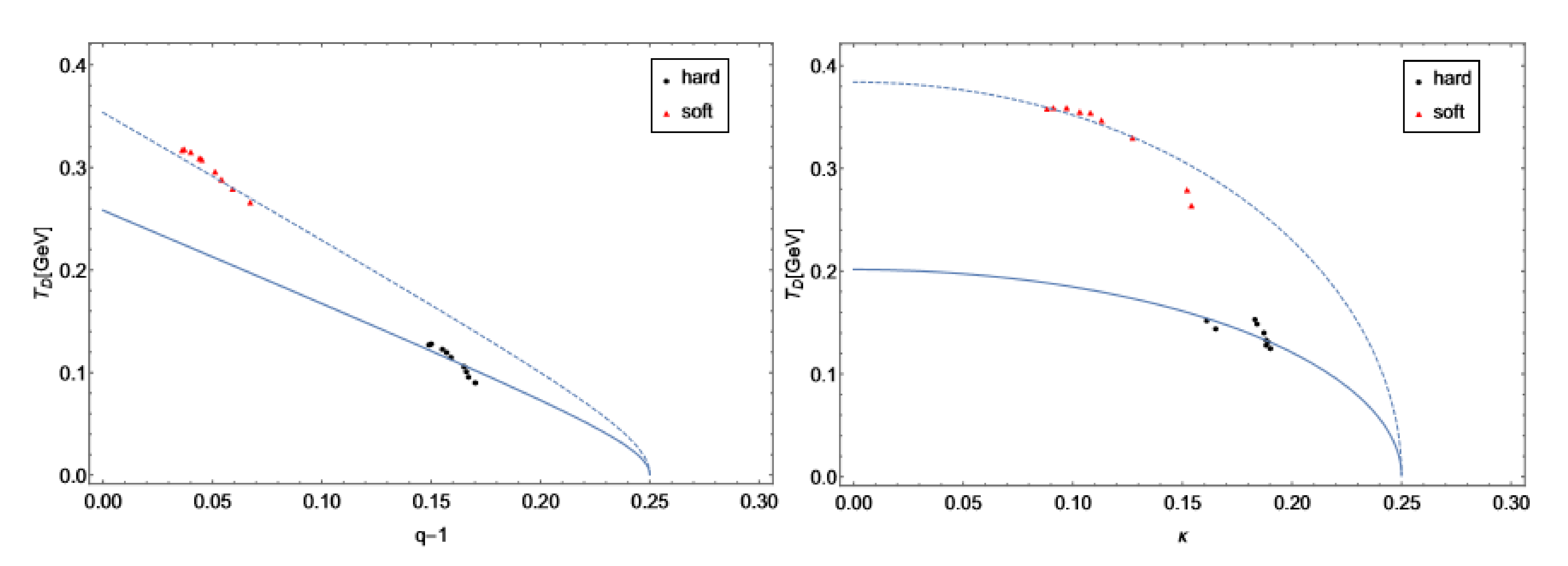
$$\langle p_T^2 \rangle \approx \frac{\int p_T^3 dp_T \exp_q(\frac{p_T}{T})}{\int p_T dp_T \exp_q(\frac{p_T}{T})} = \frac{6T^2}{(4-3q)(5-4q)} \quad (41)$$

- κ:

$$\langle p_T^2 \rangle \approx \frac{\int p_T^3 dp_T \exp_\kappa(\frac{p_T}{T})}{\int p_T dp_T \exp_\kappa(\frac{p_T}{T})} = \frac{6T^2}{1-16\kappa^2} \quad (42)$$

Ke-Ming Shen, Tamas S. Biro and En-Ke Wang, Physica A 492: 2353-2360 (2018).

compare with data results



Data are from fitting the p_T spectra in different *PbPb* collisions at 2.76 GeV using the "soft+hard" model[1].

1. G. G. Barnafoldi, K. Urmossy and G. Biro, J. Phys. Con. Series 612, 012048 (2015).

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Summary & Outlook

- Theoretical framework of Tsallis entropy and its non-extensive statistics are introduced as well as the q–probability distribution. With respect to its self-referential problem we firstly add one more constraint and generalize the corresponding ensemble theory. Modified q-BED and q-FDD are then well set-up.
- Tsallis' particle-hole symmetry breaks for its q-exponential functions. Two deformed distributions are proposed and well satisfy it. Furthermore, κ–exponential in Kaniadakis statistics is investigated as a better solution.
- p_T spectra are well fitted with non-extensive distributions for not pp but also AA collisions. The fitting parameters are not independent at all, and their connections are nicely built up in our models.
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- …

Summary & Outlook

- Theoretical framework of Tsallis entropy and its non-extensive statistics are introduced as well as the q–probability distribution. With respect to its self-referential problem we firstly add one more constraint and generalize the corresponding ensemble theory. Modified q-BED and q-FDD are then well set-up.
- Tsallis' particle-hole symmetry breaks for its q-exponential functions. Two deformed distributions are proposed and well satisfy it. Furthermore, κ–exponential in Kaniadakis statistics is investigated as a better solution.
- p_T spectra are well fitted with non-extensive distributions for not *pp* but also *AA* collisions. The fitting parameters are not independent at all, and their connections are nicely built up in our models.
- …

韶华逝兮光阴不可偷
日月易兮白云亦苍狗
尤记初入博门似昨昔
却看今朝姑妄成将侯
三载寒窗兮贵有恒忧
四时风光兮无语焉留
左遇良师兮更比益友
右得同袍者情同足手
又是一年秋月冬雨雪
且著拙文春花夏杨柳
仓促间定有笔墨不筹
望君指点之余再顿首

THANK YOU !

hypothesis

- **Boltzmann entropy** (k_B = 1 for simplicity):

$$S_{BG} = \ln W \qquad (1)$$

- **Molecular chaos hypothesis**: The velocities of colliding particles are uncorrelated, and independent of position:

$$N(\vec{v}_1, \vec{v}_2) \propto N^2 f(\vec{v}_1) f(\vec{v}_2) dt d\vec{v}_1 d\vec{v}_2 \qquad (2)$$

- **Ergodic hypothesis**: Over long periods of time, the time spent by a system in some region of the phase space of microstates with the same energy is proportional to the volume of this region, i.e., that all accessible microstates are equi-probable over a long period of time.

$$p_i = \frac{1}{W} \qquad (3)$$

background

- **Limitations of hypothesis**

  Ideal gas, thermal equilibrium[1], …

- **Complexity of systems**

  Long-range interactions[2], memory effects[3], (multi)fractal boundary conditions[4], …

1. E. P. Borges, et. al., Phys. Rev. Lett. 89, 254103, (2002); and so on.
2. R. K. Pathria, Statistical Mechanics(2nd Edt.)[M], Elsevier Pte Ltd., 2003.
3. M. A. Fuentes and M. O. Caceres, Phys. Lett. A 372, 1236 (2008).
4. M. L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).

Generalized Statistics

Here I just list some of them[1]:

Shannon-1948[2],

$$H_1(p_i) = \frac{\sum_i^W (-p_i \ln p_i)}{\sum_i^W p_i} = -\sum_i^W p_i \ln p_i$$

Renyi-1961[3]

$$H_2(p_i) = \frac{1}{1-r} \ln \frac{\sum_i^W p_i^r}{\sum_i^W p_i} = \frac{1}{1-r} \ln \sum_i^W p_i^r$$

and so on.

Measure	$h(x)$	$\varphi_1(x)$	$\varphi_2(x)$	v_i
1	x	$-x \log x$	x	v
2	$(1-r)^{-1} \log x$	x^r	x	v
3	x	$-x^r \log x$	x^r	v
4	$(s-r)^{-1} \log x$	x^r	x^s	v
5	$(1/s) \arctan x$	$x^r \sin(s \log x)$	$x^r \cos(s \log x)$	v
6	$(m-r)^{-1} \log x$	x^{r-m+1}	x	v
7	$(m(m-r))^{-1} \log x$	$x^{r/m}$	x	v
8	$(1-t)^{-1} \log x$	x^{t+s-1}	x^s	v
9	$(1-s)^{-1}(x-1)$	x^s	x	v
10	$(t-1)^{-1}(x^t-1)$	$x^{1/t}$	x	v
11	$(1-s)^{-1}(e^x-1)$	$(s-1)x \log x$	x	v
12	$(1-s)^{-1}(x^{\frac{s-1}{r-1}}-1)$	x^r	x	v
13	x	$-x^r \log x$	x	v
14	$(s-r)^{-1}x$	$x^r - x^s$	x	v
15	$(\sin s)^{-1}x$	$-x^r \sin(s \log x)$	x	v
16	$\left(1+\frac{1}{\lambda}\right) \log(1+\lambda) - \frac{x}{\lambda}$	$(1+\lambda x) \log(1+\lambda x)$	x	v
17	x	$-x \log \left(\frac{\sin(sx)}{2\sin(s/2)}\right)$	x	v
18	x	$-\frac{\sin(xs)}{2\sin(s/2)} \log \left(\frac{\sin(sx)}{2\sin(s/2)}\right)$	x	v
19	x	$-x \log x$	x	w_i
20	x	$-\log x$	1	v_i
21	$(1-r)^{-1} \log x$	x^{r-1}	1	v_i
22	$(1-s)^{-1}(e^x-1)$	$(s-1) \log x$	1	v_i
23	$(1-s)^{-1}(x^{\frac{r-1}{s-1}}-1)$	x^{r-1}	1	v_i

$$H_{h,v}^{\varphi_1,\varphi_2}(p_i) = h\left(\frac{\sum_i^W v_i \varphi_1(p_i)}{\sum_i^W v_i \varphi_2(p_i)}\right)$$

1. M. D. Esteban and D. Morales, A Summary on Entropy Statistics, (1991).
2. C. E. Shannon, Bell System Tech. J. 27, 379 and 623 (1948).
3. A. Renyi, Proc. 4th Berkeley Symp. Math. Statist. and Prob., 1, 547-561 (1961).