Branched Hamiltonians and Supersymmetry

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Wigner 111 seminar, 12 November 2013

Some examples of branched Hamiltonians are explored, as recently advocated by Shapere and Wilczek. These are actually cases of switchback potentials, albeit in momentum space, as previously analyzed for quasi-Hamiltonian dynamical systems in a classical context. A basic model, with a pair of Hamiltonian branches related by supersymmetry, is considered as an interesting illustration, and as stimulation.

> "It is quite possible ... we may discover that in nature the relation of past and future is so intimate ... that no simple representation of a present may exist." — R P Feynman

Based on work with Cosmas Zachos, Argonne National Laboratory

Introduction to the problem

In quantum mechanics

$$H = p^2 + V(x) \tag{1}$$

is neither more nor less difficult than

$$H = x^2 + V(p) \tag{2}$$

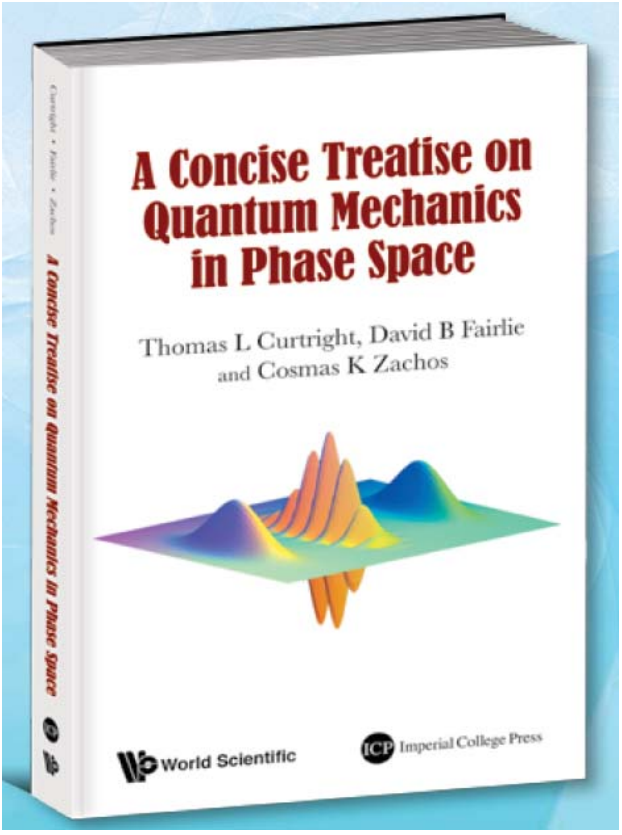
by reason of x, p duality, i.e. the Fourier transform:

$$\left.\begin{array}{c} \psi(x) \\ x \\ -i\hbar\partial/\partial x \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \phi(p) \\ +i\hbar\partial/\partial p \\ p \end{array}\right.$$

This equivalence of (1) and (2) is manifest in the QMPS formalism, as initiated by Wigner (1932),

$$\begin{aligned} f(x,p) &= \frac{1}{\pi\hbar}\int dy\ \langle x+y|\ \boldsymbol{\rho}\ |x-y\rangle\ e^{-2ipy/\hbar} \\ &= \frac{1}{\pi\hbar}\int dk\ \langle p+k|\ \boldsymbol{\rho}\ |p-k\rangle\ e^{2ixk/\hbar} \end{aligned}$$

where x and p are on an equal footing, and where even more general $H(x,p)$ can be considered. See CZ to follow, and other talks at this conference. Or even better, in addition to the excellent books cited at the conclusion of Professor Schleich's talk yesterday morning, please see our new book on the subject ...


A Concise Treatise on Quantum Mechanics in Phase Space
Thomas L Curtright, David B Fairlie and Cosmas K Zachos
World Scientific
Imperial College Press

Even in *classical* Hamiltonian mechanics, (1) and (2) are equivalent under a classical canonical transformation on phase space:

$$(x, p) \Longleftrightarrow (p, -x)$$

But upon transitioning to Lagrangian mechanics, the equivalence between the two theories becomes obscure.

Here is the issue:

A *Legendre transformation* from (x, p, H) to (x, v, L) is complicated for *non-convex* $V(p)$. The resulting L is multi-valued, in general, with several branches.

Or, if you are like Feynman in his youth, and keen to proceed from the start with a given single-valued $L(x, v)$, then you too will face similar complications if you are dealing with

$$L = x^2 - V(v) \tag{3}$$

instead of the usual

$$L = v^2 - V(x) \tag{4}$$

If you construct the Hamiltonian for (3) by Legendre transformation, in general you will encounter multi-valued-ness. (Shapere and Wilczek, 2012)

If $V(v)$ is non-convex, then the LT $v \Longleftrightarrow p$ gives a multi-valued Hamiltonian, i.e. several branches for $H(x, p)$.

Again, the same issue:

Starting from single-valued $H(x, p)$ or starting from single-valued $L(x, v)$ — either way — if the p or v dependence is non-convex then multi-valued, branched functions will arise upon Legendre transforming between Hamiltonian and Lagrangian formulations.

Let's consider some examples.

Miami fedora kinetic energy

In dimensionless variables, define

$$\begin{aligned} L(x,v) &= -\exp\left(-v^2/2\right) + 1 - V(x) \\ &= \frac{1}{2}v^2 - V(x) + O\left(v^4\right) \end{aligned} \tag{5}$$

e.g. $V(x) = x^2$ gives classical Euler-Lagrange equations

$$\begin{aligned} \frac{dv}{dt} &= \frac{-2x}{(1-v^2)} \exp\left(v^2/2\right) \\ &= -2x + O\left(v^2\right) \end{aligned}$$

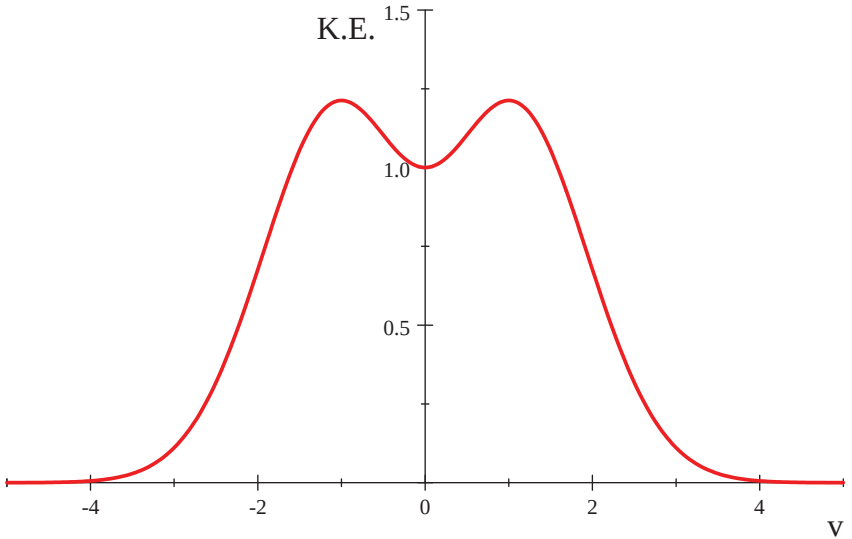
The RHS is a restorative force so long as $v^2 < 1$ For larger speeds the solutions explode.

The energy in terms of x and v is

$$E = V(x) - 1 + \left(1 + v^2\right) e^{-v^2/2}$$

This is conserved, given the E-L equations, and so it gives the trajectories as constant E curves on the (x, v) plane.

I will refer to the v-dependent terms in E as the "Miami fedora" kinetic energy for obvious reasons.



Miami fedora kinetic energy $\left(1 + v^2\right) e^{-v^2/2} = \frac{1}{2}v^2 + O\left(v^4\right)$

e.g. $V = x^2$ gives $x\,(v)$:

$$x = \pm\sqrt{E - K.E.} = \pm\sqrt{E - (1+v^2)\,e^{-v^2/2}}$$

Or take $E = x^2 + (1+v^2)\,e^{-v^2/2}$, and solve for $v\,(x)$. The solution is:

$$v = \pm\sqrt{-1 - 2\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{e}}\,(x^2 - E)\right)}$$

For real v it is necessary here for the Lambert function, with negative argument, to return negative values. That is to say, either the principal branch, $\mathrm{LambertW}\,(0, z)$ or the lower branch $\mathrm{LambertW}\,(-1, z)$ with $-1/e \le z \le 0$.

So, $v\,(x)$ is multi-valued due to the different $\sqrt{\cdots}$ and $\mathrm{LambertW}\,(\cdots)$ branches.

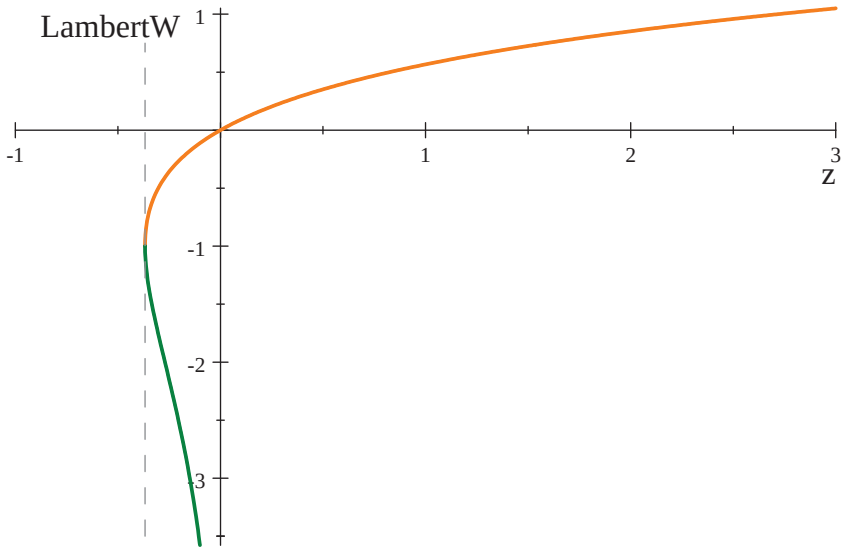
Lambert function refresher course: The solution of

$$ye^y = z$$

is

$$\begin{aligned} y(z) &= \text{LambertW}(k,z) \mid k \in \mathbb{Z} \quad \text{if} \quad z \neq 0 \\ y(z) &= 0 \qquad\qquad\qquad\qquad\quad \text{if} \quad z = 0 \end{aligned}$$

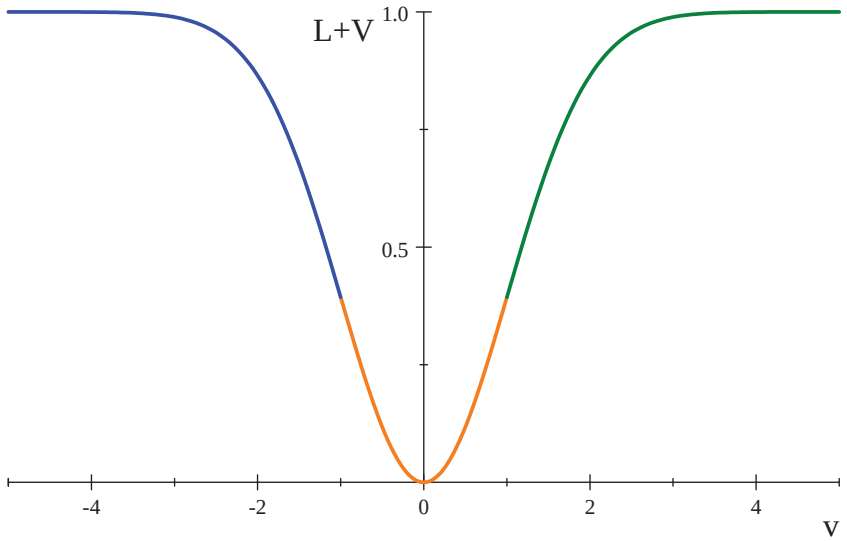
where $k = 0$ and $k = -1$ give the two real branches as shown below.



$\text{LambertW}(0,z)$ and $\text{LambertW}(-1,z)$ in orange and green, resp.

Legendre transforming to obtain the Hamiltonian for the model

The velocity dependent term is a union of three convex functions, defined on $v \in [-\infty, -1]\,,\ [-1, 1]\,,$ and $[1, \infty]$.



As a consequence, the Hamiltonian will be multi-valued.

$$H = vp - L$$

$$p = \frac{\partial L}{\partial v} = v \exp\left(-v^2/2\right)$$

So as a function of v

$$H\left(x, v\right) = \left(1 + v^2\right) \exp\left(-v^2/2\right) + V\left(x\right) - 1$$

which we recognize as the previous E, of course.

But we want $H\left(x, p\right)$.

So we need $v\left(p\right)$.

The velocity as function of p again involves a Lambert function, as well as both branches of a $\sqrt{\cdots}$.

$$v = \pm\sqrt{-\operatorname{LambertW}\left(-p^2\right)}$$

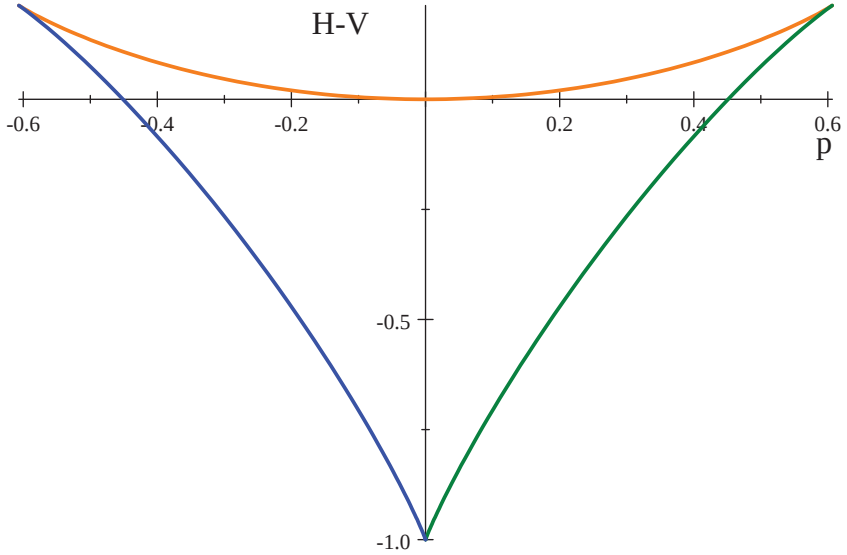
Again, for real v, negative values are required for LambertW, so either the principal branch, $\operatorname{LambertW}\left(0, z\right)$ or the lower branch $\operatorname{LambertW}\left(-1, z\right)$ will do, with $-1/e \leq -p^2 \leq 0$. That is to say, the momentum lies in the finite interval

$$-\frac{1}{\sqrt{e}} \leq p \leq \frac{1}{\sqrt{e}}$$

The result for $H\left(x, p\right)$ is multi-valued on this momentum interval, because of the square root and the Lambert function branches.

$$H = \pm p\left(\sqrt{-\operatorname{LambertW}\left(-p^2\right)} + \frac{1}{\sqrt{-\operatorname{LambertW}\left(-p^2\right)}}\right) + V\left(x\right) - 1$$

Note $\exp\left(-v^2/2\right) = p/v$.



The real branches of $H - V$ versus $p \in [-1/\sqrt{e}, 1/\sqrt{e}] \approx [-0.607, 0.607]$

$H(x, p)$ may be thought of as the union of three convex functions of p.

Classically, a particle switches H during the course of its trajectory. Different branches of H govern the motion at different times.

In a previous context (2010) — a study of the evolution of chaotic dynamical systems — Zachos and I found the problem was mathematically equivalent to particle motion in which the Hamiltonian (the potential $V(x)$, actually) switched when the particle encountered turning points, with continuous evolution of $x(t)$, $v(t)$, and $p(t)$ at the switching points. (Please see the literature cited below in the reference list.)

The present example illustrates the *same switchback effects*, only here it is the momentum dependent part of H that switches.

For quantum mechanics, the Hamiltonian to be used in the Schrödinger equation also switches. The only additional issue is what are the boundary conditions on the wave function where the switches occur. Shapere and Wilczek impose conditions so that the probability current is conserved.

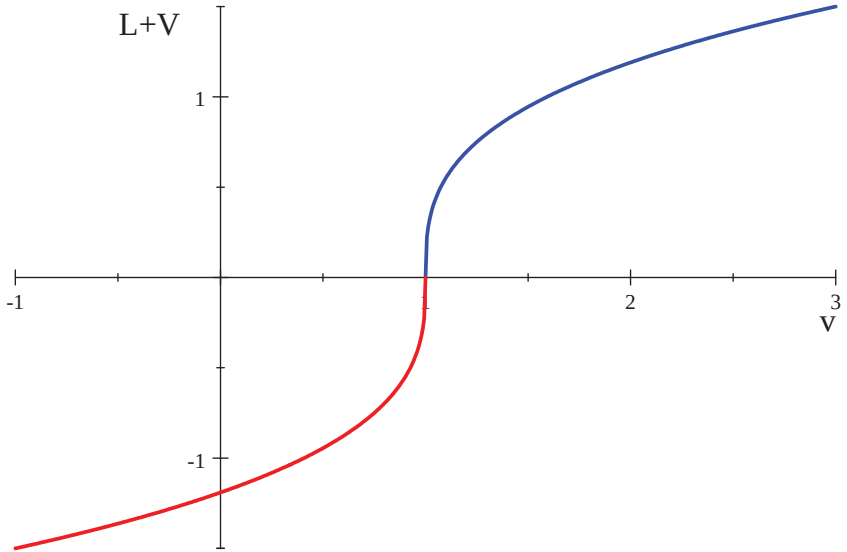
To shed light on such Hamiltonian switching quantum systems, I will *switch* gears from the Florida hat model to a more tractable example ...

Supersymmetric QM

Consider the single-valued real function, for $-\infty \leq v \leq +\infty$,

$$L = C\ (v-1)^{1/3} - V(x) \tag{6}$$

where $C = 3/\sqrt[3]{16} \approx 1.19$. Take real roots on the real v axis.



$$L + V = C\,(v-1)^{\frac{1}{3}}$$

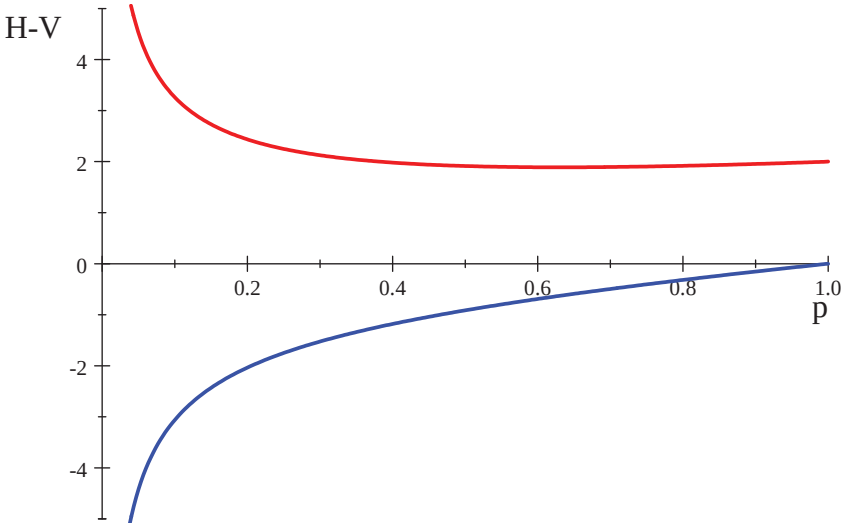
Proceeding with the Legendre transform

$$p = \frac{\partial L}{\partial v} \quad \Rightarrow \quad v_{\pm}(p) \equiv 1 \mp \frac{1}{4}\left(\frac{1}{\sqrt{p}}\right)^3$$

and the Hamiltonian is ... double valued.()

$$H_{\pm} = p \pm \frac{1}{2\sqrt{p}} + V(x) \tag{7}$$

From the shape of $(L+V)(v)$ curve, note that $p \geq 0$.



$H_{\pm} - V(x) = p \pm \frac{1}{2\sqrt{p}}$ in red/blue. There is a cusp at $p = \infty$.

Following the suggestions of Shapere and Wilczek, we define the associated *quantum* theory with $p \geq 0$ as a restriction, with various boundary conditions imposed on the wave functions, $\psi(p)$, at $p = 0$, such that there is no probability flow to negative p.

When the potential $V(x)$ is harmonic, this is a *supersymmetric* quantum mechanical system when viewed in momentum space.

$$V(x) = x^2 \underset{\text{QM in } p \text{ space}}{\longmapsto} -\frac{d^2}{dp^2}$$

Quantum features The momentum space supersymmetric pair of QM Hamiltonian operators for this case is therefore expressible in the standard form:

$$H_{\pm} = -\frac{d^2}{dp^2} + w_0^2(p) \pm w_0'(p) = \left(\frac{d}{dp} \pm w_0(p)\right)\left(-\frac{d}{dp} \pm w_0(p)\right) \tag{8}$$

$$w_0(p) = \sqrt{p}$$

This has the feature that the true — square-integrable — ground state of the system is non-vanishing for only one of the branches, namely, H_-.

As an algebraic system, for $p \geq 0$, the two Hamiltonians are related in a familiar fashion by

$$H_- = a^\dagger a$$
$$H_+ = a a^\dagger$$

$$H_+ = H_- + \left[a, a^\dagger\right]$$

$$a = \frac{d}{dp} + \sqrt{p}\ , \quad a^\dagger = -\frac{d}{dp} + \sqrt{p}\ , \quad \left[a, a^\dagger\right] = \frac{1}{\sqrt{p}}\ .$$

Obviously, either energy spectrum is non-negative.

The zero-energy ground state of H_- is given by

$$a\psi_0(p) = 0 \ , \quad \psi_0(p) = N_0 \exp\left(-\frac{2}{3}p^{3/2}\right) \tag{9}$$

$N_0 = 6^{1/6}/\sqrt{\Gamma\left(\frac{2}{3}\right)} \approx 1.16$ ensures that $\int_0^\infty |\psi_0(p)|^2\, dp = 1$.

The ground state obeys the boundary condition $\psi_0'(0) = 0$.

On the other hand, the zero-energy state for H_+, namely, $\phi(x) = \exp\left(+\frac{2}{3}p^{3/2}\right)$, is not admissible, because it has infinite norm.

The higher energy states are degenerate, with $H_\pm \psi^{(\pm)} = E\psi^{(\pm)}$ eigenstates for $E > 0$ mutually related by

$$\psi_E^{(+)} = \frac{1}{\sqrt{E}}\, a\psi_E^{(-)} \ , \quad \psi_E^{(-)} = \frac{1}{\sqrt{E}}\, a^\dagger \psi_E^{(+)} \ ,$$

so as to have equal norms.

In particular the first excited state for H_- is degenerate with the lowest energy state for H_+, with $E_1 = 1.89379$, as determined by numerical analysis.

The degenerate $H_\pm$ eigenfunctions obey different boundary conditions at $p = 0$. If one is Dirichlet, the other is Neumann. For example, the first H_- excited state and its degenerate H_+ partner eigenstate satisfy $\psi_{E_1}^{(-)}\Big|_{p=0} = 0 = d\psi_{E_1}^{(+)}/dp\Big|_{p=0}$, while for the next excited states, $d\psi_{E_2}^{(-)}/dp\Big|_{p=0} = 0 = \psi_{E_2}^{(+)}\Big|_{p=0}$, etc.

All this conforms with *well-known* expectations for general supersymmetric QM, but the common *single-component* L underlying both branches of the Hamiltonian, in this case, is a new observation, so far as I know.

The flipping of the boundary conditions actually has a practical benefit due to the $1/\sqrt{p}$ singularity in both $H_{\pm}$: It is more straightforward to perform an accurate numerical computation of the energy eigenvalue using the boundary condition $\psi_E(0) = 0 \neq \psi'_E(0)$ than it is using the condition $\psi_E(0) \neq 0 = \psi'_E(0)$. The degeneracy of the eigenfunctions permits one to always choose the $\psi_E(0) = 0$ condition, along with the corresponding H_+ or H_-.

These higher energy states may be thought of as a single nontrivial state defined on a unified covering space — a double covering of the half-line $\mathbb{R}_+$ by $\mathbb{R}$ — obtained by unfolding the two Hamiltonian branches to obtain a single H globally defined on $\mathbb{R}$. However, as is clear from the preceding discussion, the true ground state of the system is $\psi_0(p) \cup 0$ on the unfolded space. The latter, somewhat unusual feature is possible because the two Hamiltonians on the half-lines join together in a cusp at $p = \infty$, where ψ_0 and all its derivatives vanish. So too vanish all the higher $\psi_E^{(\pm)}$ and all their derivatives at $p = \infty$.

For this reason, it would be excusable not to have thought of the degenerate eigenstates on the half-line as two branches of a single function. However, the unified picture provided by joining them together on a covering real line, with Neumann and Dirichlet boundary conditions at opposite ends, is a more compelling point of view, in our opinion. Perhaps more importantly, this omniscient view of the system becomes natural when the common Lagrangian underpinning both $H_\pm$ is considered.

Classical features It is also instructive to survey essential features of the classical trajectories for the model. But time does not permit me to do this here. (Please see my paper with Cosmas Zachos, when it appears on the arXiv.)

Discussion

As emphasized by Shapere and Wilczek, "many worlds" systems with branched Hamiltonians are by no means rare, in theory. Here, I have displayed some simple unified Lagrangian prototype systems which, by virtue of non-convexity in their velocity dependence, branch into double-valued (but still self-adjoint) Hamiltonians.

I have surveyed the spectral and boundary conditions involved for a supersymmetric model, in a uniform framework, by utilizing the eigenstate-linking "supercharge" ladder operators. These particular branched Hamiltonians — although governing "two worlds" — are nevertheless paired into a uniform isospectral system, in the very same Hilbert space.

Yet they are inexorably separated, in some analogy to fermionic and bosonic sectors, as the respective dynamical intervals only connect at $p = \infty$. In this respect, this particular supersymmetric system differs from more typical constructions given by Shapere and Wilczek, which exhibit similar operator branching structures but connect for finite p.

However, in addition, I have outlined another model whose branches lie on a compact, closed momentum manifold with coalescing cusps at finite p. The quantum features of this other model are in line with the cases considered by Shapere and Wilczek.

While only double-valued, H for the Miami fedora model is clearly the union of *three* convex functions, defined on three overlapping momentum intervals: H_-, H_0, and H_+ for $p \in [-1/\sqrt{e}, 0]$, $[-1/\sqrt{e}, 1/\sqrt{e}]$, and $[0, 1/\sqrt{e}]$, as displayed in a previous Figure in blue, orange, and green, respectively.

This unified 3-fold structure brings to mind some previous theories exhibiting triality, as discussed by Shankar (1981). However, to my knowledge the gaussian model shows no compelling signs of supersymmetry. Still, it would be quite interesting to find a simple, three-Hamiltonian, single-particle quantum system, based on a single unifying Lagrangian, that could be partitioned into pairs of supersymmetric Hamiltonians, with state-linking operators of the type analyzed above.

Acknowledgements:

I tip my hat to the organizers of this meeting for the excellent work they have done, and I thank them for the opportunity to visit Budapest to present this material.

This research was supported in part by NSF Award PHY-1214521; and in part, the submitted manuscript has been created by UChicago Argonne, LLC, Operator of Argonne National Laboratory. Argonne, a U.S. Department of Energy Office of Science laboratory, is operated under Contract No. DE-AC02-06CH11357. The U.S. Government retains for itself, and others acting on its behalf, a paid-up nonexclusive, irrevocable worldwide license in said article to reproduce, prepare derivative works, distribute copies to the public, and perform publicly and display publicly, by or on behalf of the Government. TLC was also supported in part by a University of Miami Cooper Fellowship.

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