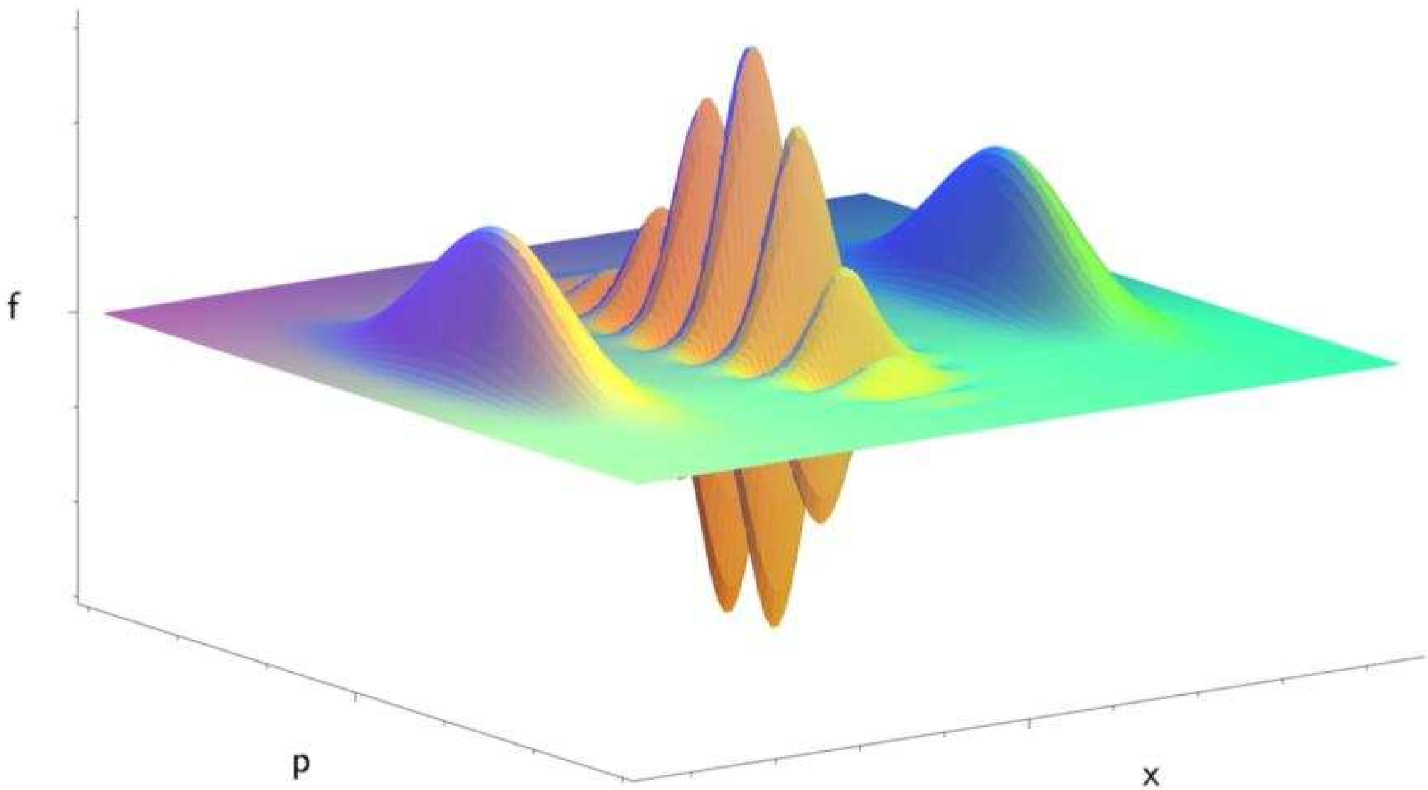

f
p
x

The Wigner phase-space quasi-probability distribution function

QUANTUM MECHANICS LIVES AND WORKS IN

PHASE SPACE

A complete, autonomous formulation of QM based on the standard c-number variables x and p and their functions in phase-space, which compose through a special operation.

C K Zachos

Three alternate paths to quantization:

1. Hilbert space (Heisenberg, Schrödinger, Dirac)
2. Path integrals (Dirac, Feynman)
3. Phase-space distribution function of Wigner (Wigner 1932; Groenewold 1946; Moyal 1949; Baker 1958; Fairlie 1964; ...)

$$f(x,p) = \frac{1}{2\pi}\int dy\ \psi^*\Big(x - \frac{\hbar}{2}y\Big)\ e^{-iyp}\psi\Big(x + \frac{\hbar}{2}y\Big).$$

A special representation of the density matrix (Weyl correspondence).

Useful in describing **quantum** transport/flows in phase space $\rightsquigarrow$ quantum optics; quantum chemistry; nuclear physics; study of decoherence (eg, quantum computing).

But also signal processing (time-frequency spectrograms); Intriguing mathematical structure of relevance to Lie Algebras, M-theory,...

Properties of $f(x,p) = \frac{1}{2\pi}\int dy\ \psi^*(x - \frac{\hbar}{2}y)\ e^{-iyp}\psi(x + \frac{\hbar}{2}y)$:

⋔ Normalized, $\int dpdx f(x,p) = 1$.

✓ Real

- Bounded: $-\frac{2}{h} \leq f(x,p) \leq \frac{2}{h}$ (Cauchy-Schwarz Inequality)
  $\rightsquigarrow$ Cannot be a spike: Cannot be certain!

- p- or x-projection leads to marginal probability densities: A space-like shadow $\int dp\ f(x,p) = \rho(x)$; **or else** a momentum-space shadow $\int dx\ f(x,p) = \sigma(p)$, resp.; both positive semidefinite. But cannot be conditioned on each other. The uncertainty principle is fighting back $\rightsquigarrow$

↬ f can, and most often does, **go negative** (Wigner). A hallmark of **quantum interference**.
"Negative probability" (Bartlett; Moyal; Feynman; Bracken & Melloy).

Hiding through the uncertainty principle. Smoothing f by a filter of size larger than $\hbar$ (eg, convolving with phase-space Gaussian) results in a positive-semidefinite function: it has been smeared or blurred to a classical distribution (de Bruijn, 1967). $\rightsquigarrow$ Negative areas are **small**.

When is a real $f(x,p)$ a bona-fide Wigner function? When its Fourier transform L-R-factorizes:

$$\tilde{f}(x,y) = \int dp \; e^{ipy} f(x,p) = g_L^*(x-\hbar y/2) \; g_R(x+\hbar y/2) \; ,$$

$\left(\frac{\partial^2 \ln \tilde{f}}{\partial(x-\hbar y/2)\partial(x+\hbar y/2)} = 0 \right)$, so $g_L = g_R$ from reality.

▲ Nevertheless, it **is** a distribution: it yields expectation values from phase-space c-number functions.

In Weyl's association rule (1927), given an operator $\mathbf{A}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\, A(x,p) \exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x))$, the corresponding phase-space kernel function $A(x,p)$, obtained by $\mathbf{p} \mapsto p$, $\mathbf{x} \mapsto x$, yields that operator's expectation value,

$$\langle \mathbf{A} \rangle = \int dx dp\ f(x,p)\ A(x,p).$$

Dynamical evolution of f (Moyal):

Liouville's Thm, $\partial_t f + \{f, H\} = 0$, quantum generalizes to

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar} \ ,$$

based on the $\star$-product (Groenewold): $\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}$,

the essentially **unique** one-parameter ($\hbar$) associative deformation of Poisson Brackets of classical mechanics, (viz. $\hbar \to 0$). (Isomorphism: $\mathbf{AB} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp (A \star B) \exp(i\tau(\mathbf{p}-p) + i\sigma(\mathbf{x}-x))$.)

Systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. But time-independent pure-state Wigner functions $\star$-commute with H.

However, they further obey a more powerful functional $\star$-genvalue equation (Fairlie, 1964):

$$H(x,p) \star f(x,p) = H\left(x + \frac{i\hbar}{2}\vec{\partial}_p, p - \frac{i\hbar}{2}\vec{\partial}_x\right) f(x,p)$$

$$= f(x,p) \star H(x,p) = E \; f(x,p) \; ,$$

which amounts to a complete characterization of them:

For real functions $f(x,p)$, the Wigner form is equivalent to compliance with the $\star$-genvalue equation ($\Re$ and $\Im$ parts).

(Curtright, Fairlie, & Zachos, Phys Rev **D58** (1998) 025002)

$\Rrightarrow$ Projective orthogonality spectral properties

$$f \star H \star g = E_f \; f \star g = E_g \; f \star g.$$

For $E_g \neq E_f$, $\Longrightarrow$ $f \star g = 0$.

Precluding degeneracy, for $f = g$,

$$f \star H \star f = E_f \; f \star f = H \star f \star f,$$

$$\Longrightarrow \qquad f \star f \propto f.$$

fs $\star$-project onto their space.

$$\boxed{f_a \star f_b = \frac{1}{h} \; \delta_{a,b} \; f_a.}$$

• The normalization matters (Takabayasi, 1954): despite linearity of the equations, it prevents superposition of solutions (this is not how QM interference works here!).

$$\int dp dx \; f \star g = \int dp dx \; f g,$$

so, for different $\star$-genfunctions,

$$\int dp dx \; f g = 0.$$

$\leadsto$ Negative values are a feature, not a liability. Quantum interference confined to "$\hbar$-small" regions.

NB $\hookrightarrow$ $$\int H(x,p) f(x,p) \; dxdp = E \int f \; dxdp = E \; .$$

NB $\rightsquigarrow$ $$\int f^2 \;\; dxdp = \frac{1}{h} \; .$$

In general, $\leq 1/h$ $\rightsquigarrow$ quantum: fuzzy — classical: spiky.

• For any function, $\langle |g|^2 \rangle$ need not ≥ 0.

But $\langle g^* \star g \rangle \geq 0$ $\hookrightarrow$ the **uncertainty principle**,

$\Delta x \Delta p \geq \hbar/2$ $\rightsquigarrow$ $(\Delta x)^2 + (\Delta p)^2 \geq \hbar$. Hides negative values

Curtright & Zachos, Mod Phys Lett **A16** (2001) 2381.

▼ **Pf**

$$H(x,p) \star f(x,p)$$

$$= \frac{1}{2\pi}\left((p - i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m + V(x)\right)\int dy\ e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)$$
$$= \frac{1}{2\pi}\int dy\ \left((p - i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m + V(x+\frac{\hbar}{2}y)\right)e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)$$
$$= \frac{1}{2\pi}\int dy\ e^{-iyp}\left((i\overrightarrow{\partial}_y + i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m + V(x+\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)$$
$$= \frac{1}{2\pi}\int dy\ e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ E\ \psi(x+\frac{\hbar}{2}y) =$$

$$= E\ f(x,p);$$

↷Action of the effective differential operators on ψ^* turns out to be null.

$$f \star H$$

$$= \frac{1}{2\pi}\int dy\ e^{-iyp}\left(-(\overrightarrow{\partial}_y - \frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m + V(x-\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)$$

$$= E\ f(x,p).$$

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p) = \int dy\ e^{-iyp}\tilde{f}(x,y)$,

$$\int dy\ e^{-iyp}\left(-\frac{1}{2m}(\overrightarrow{\partial}_y \pm \frac{\hbar}{2}\overrightarrow{\partial}_x)^2 + V(x \pm \frac{\hbar}{2}y) - E\right)\tilde{f}(x,y) = 0.$$

$\rightsquigarrow$ Real solutions of $H(x,p) \star f(x,p) = E\ f(x,p)$ $(= f(x,p) \star H(x,p))$ must be of the Wigner form, $f = \int dy\ e^{-iyp}\psi^*(x - \frac{\hbar}{2}y)\psi(x + \frac{\hbar}{2}y)/2\pi$, (s.t. $\mathbf{H}\psi = E\psi$).

The wonderful fact (Groenewold): $\star$-multiplication of c-number phase-space functions is in complete isomorphism to Hilbert-space operator algebra.

SIMPLE HARMONIC OSCILLATOR

Solve **directly** for $H = (p^2 + x^2)/2$
(with $\hbar = 1$, $m = 1$, $\omega = 1$):

$$\left((x + \frac{i}{2}\partial_p)^2 + (p - \frac{i}{2}\partial_x)^2 - 2E \right) f(x,p) = 0.$$

Mere PDEs! Imaginary part: $(x\partial_p - p\partial_x) f = 0$. $\rightsquigarrow$ f depends on only one variable, $z = 4H = 2(x^2 + p^2)$. $\rightsquigarrow$

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - E \right) f(z) = 0.$$

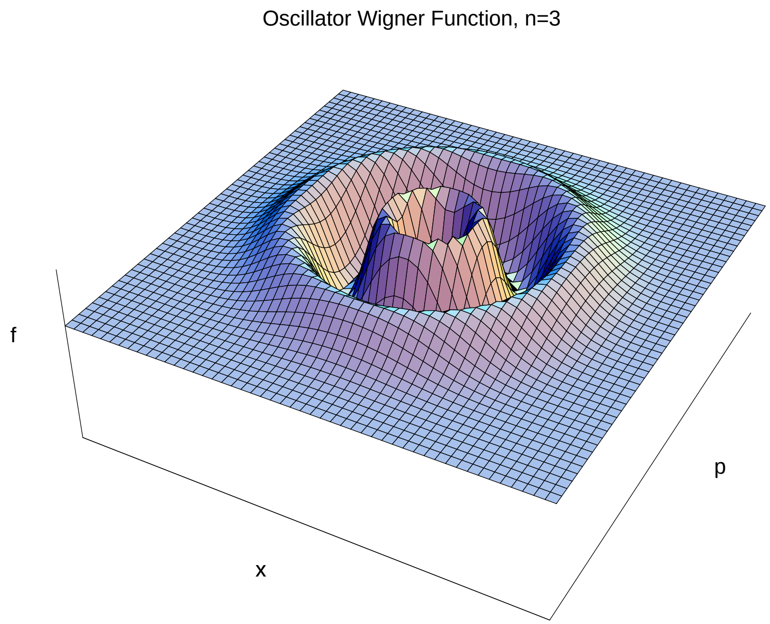
Set $f(z) = \exp(-z/2) L(z)$ $\Longrightarrow$ Laguerre's eqn

$$\left(z\partial_z^2 + (1 - z)\partial_z + E - \frac{1}{2} \right) L(z) = 0.$$

Satisfied by Laguerre polynomials, $L_n = e^z \partial^n (e^{-z} z^n)/n!$, for $n = E - 1/2 = 0, 1, 2, \ldots$ $\rightsquigarrow$ eigen-Wigner-functions are

$$f_n = \frac{(-1)^n}{\pi}\, e^{-2H}\; L_n(4H); \qquad L_0 = 1, \quad L_1 = 1 - 4H,$$

$L_2 = 8H^2 - 8H + 1, \ldots$ $\Diamond$ not positive definite.



$$\sum_n f_n = \frac{1}{2\pi} \quad .$$

Dirac's Hamiltonian factorization for algebraic solution **carries through intact in $\star$ space**:

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \ ,$$

so define

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad\qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip).$$

$$a \star a^\dagger - a^\dagger \star a = 1 \ .$$

$\star$-Fock vacuum:

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \ .$$

Associativity of the $\star$-product permits the customary ladder spectrum generation; $H \star f = f \star H$ $\quad$ $\star$-genstates:

$$f_n \propto (a^\dagger \star)^n \ f_0 \ (\star a)^n \ .$$

$\divideontimes$ real, like the Gaussian ground state;

$\leadsto$ left-right symmetric;

$\star$-orthogonal for different eigenvalues;

project to themselves, since the Gaussian ground state does, $f_0 \star f_0 \propto f_0$.

TIME EVOLUTION

Isomorphism to operator algebras $\rightsquigarrow$ associative combinatoric operations completely analogous to Hilbert space QM.

$\rightsquigarrow$ $\star$-unitary evolution operator, a "$\star$-exponential", $U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv$

$$1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!} H \star H + \frac{(it/\hbar)^3}{3!} H \star H \star H + ...,$$

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t).$$

NB Collapse to **classical** trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p \ ,$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \qquad \Longrightarrow$$

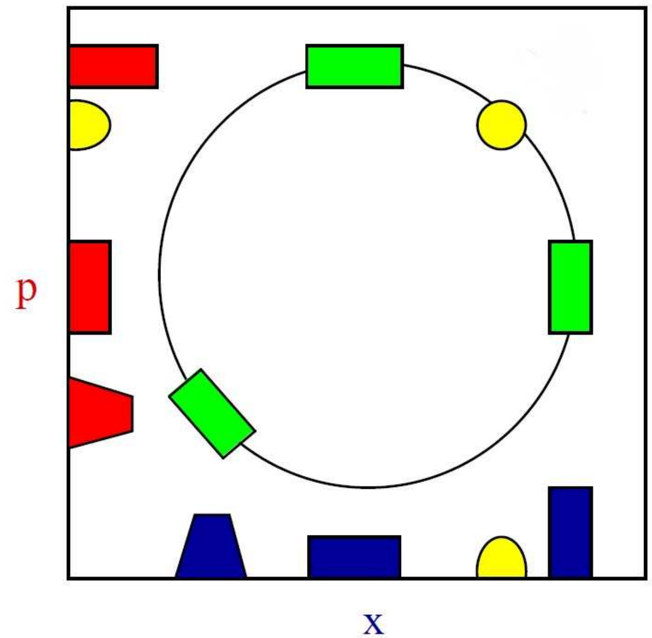
$$x(t) = x \cos t + p \sin t,$$

$$p(t) = p \cos t - x \sin t.$$

$\Longrightarrow$ For SHO the functional form of the Wigner function is preserved along classical phase-space trajectories (Groenewold, 1946):

$$f(x,p;t) = f(x\cos t - p\sin t, p\cos t + x\sin t; 0).$$

Any Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete



quantum mechanical description. In general, loss of simplicity upon integration in x (or p) to yield probability densities: the rotation induces shape variations of the oscillating probability density profile. NB Only if (eg, coherent states) a Wigner function configuration has an additional axial $x-p$ symmetry around its own center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density.

THE WEYL CORRESPONDENCE BRIDGE

Weyl's correspondence map, by itself, merely provides **a change of representation between phase space and Hilbert space** $\hookrightarrow$ Mutual language to contrast classical to quantum mechanics on common footing, and illuminate the transition.

$$\mathbf{A}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\; a(x,p)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x)),$$

Inverse map (Wigner):

$$a(x,p) = \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\mathbf{A}(\mathbf{x},\mathbf{p})\left|x-\frac{\hbar}{2}y\right\rangle \;.$$

$$\begin{array}{ccc}
\text{PHASE SPACE} & & \text{HILBERT SPACE} \\
a & \xrightarrow{Weyl} & \mathbf{A} \\
quantum\downarrow & & \downarrow quantum \\
a\star b & \xrightarrow{Groenewold} & \mathbf{AB} \\
classical\;\hbar{=}0\downarrow & & \downarrow Bracken\;\hbar{=}0 \\
ab & \xrightarrow{Weyl} & \mathbf{A}\odot\mathbf{B}
\end{array}$$

$\rightsquigarrow$ A plethora of choice-of-ordering quantum mechanics problems reduce to purely $\star$-product algebraic ones: varied deformations (ordering choices) can be surveyed systematically in phase space. (Curtright & Zachos, New J Phys **4** (2002) 83.1-83.16 [hep-th/0205063])