

The Wigner phase-space quasi-probability distribution function
QUANTUM MECHANICS LIVES AND WORKS IN
PHASE SPACE

A complete, autonomous formulation of QM based on the standard c-number variables x and p and their functions in phase-space, which compose through a special operation.

C K Zachos

Three alternate paths to quantization:

1. Hilbert space (Heisenberg, Schrödinger, Dirac)
2. Path integrals (Dirac, Feynman)
3. Phase-space distribution function of Wigner (Wigner 1932; Groenewold 1946; Moyal 1949; Baker 1958; Fairlie 1964; ...)

$$f(x, p) = \frac{1}{2\pi} \int dy \psi^* \left(x - \frac{\hbar}{2} y \right) e^{-iyp} \psi \left(x + \frac{\hbar}{2} y \right).$$

A special representation of the density matrix (Weyl correspondence).

Useful in describing **quantum** transport/flows in phase space \leadsto quantum optics; quantum chemistry; nuclear physics; study of decoherence (eg, quantum computing).

But also signal processing (time-frequency spectrograms); Intriguing mathematical structure of relevance to Lie Algebras, M-theory,...

Properties of $f(x, p) = \frac{1}{2\pi} \int dy \psi^*(x - \frac{\hbar}{2}y) e^{-iyp} \psi(x + \frac{\hbar}{2}y) :$

⊃ Normalized, $\int dp dx f(x, p) = 1 .$

✓ Real

• Bounded: $-\frac{2}{\hbar} \leq f(x, p) \leq \frac{2}{\hbar}$ (Cauchy-Schwarz Inequality)
 \rightsquigarrow Cannot be a **spike**: **Cannot be certain!**

• p - or x -projection leads to marginal probability densities: A space-like shadow $\int dp f(x, p) = \rho(x)$; **or else** a momentum-space shadow $\int dx f(x, p) = \sigma(p)$, resp.; both positive semidefinite. But cannot be conditioned on each other. The uncertainty principle is fighting back \rightsquigarrow

↷ f can, and most often does, **go negative** (Wigner). A hallmark of **quantum interference**.

“**Negative probability**” (Bartlett; Moyal; Feynman; Bracken & Melloy).

Hiding through the uncertainty principle. Smoothing f by a filter of size larger than \hbar (eg, convolving with phase-space Gaussian) results in a positive-semidefinite function: it has been **smearred or blurred to a classical distribution** (de Bruijn, 1967). \rightsquigarrow **Negative areas are small.**

When is a real $f(x, p)$ a bona-fide Wigner function? When its Fourier transform L-R-factorizes:

$$\tilde{f}(x, y) = \int dp e^{ipy} f(x, p) = g_L^*(x - \hbar y/2) g_R(x + \hbar y/2) ,$$

$$\left(\frac{\partial^2 \ln \tilde{f}}{\partial(x - \hbar y/2) \partial(x + \hbar y/2)} = 0 \right), \quad \text{so } g_L = g_R \text{ from reality.}$$

▲ Nevertheless, it **is** a distribution: it yields **expectation values from phase-space c-number functions.**

In Weyl's association rule (1927), given an operator $\mathbf{A}(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp A(x, p) \exp(i\tau(\mathbf{p}-p) + i\sigma(\mathbf{x}-x))$, the corresponding phase-space kernel function $A(x, p)$, obtained by $\mathbf{p} \mapsto p$, $\mathbf{x} \mapsto x$, yields that operator's expectation value,

$$\langle \mathbf{A} \rangle = \int dx dp f(x, p) A(x, p).$$

Dynamical evolution of f (Moyal):

Liouville's Thm, $\partial_t f + \{f, H\} = 0$, quantum generalizes to

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar},$$

based on the \star -product (Groenewold):

$$\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)},$$

the essentially **unique** one-parameter (\hbar) associative deformation of Poisson Brackets of classical mechanics, (viz. $\hbar \rightarrow 0$). (Isomorphism:

$$\mathbf{AB} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp (A \star B) \exp(i\tau(\mathbf{p} - p) + i\sigma(\mathbf{x} - x)).$$

Systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. But time-independent pure-state Wigner functions \star -commute with H .

However, they further obey a more powerful functional \star -genvalue equation (Fairlie, 1964):

$$\begin{aligned}
 H(x, p) \star f(x, p) &= H \left(x + \frac{i\hbar}{2} \overrightarrow{\partial}_p, p - \frac{i\hbar}{2} \overrightarrow{\partial}_x \right) f(x, p) \\
 &= f(x, p) \star H(x, p) = E f(x, p) ,
 \end{aligned}$$

which amounts to a complete characterization of them:

For real functions $f(x, p)$, the Wigner form is equivalent to compliance with the \star -genvalue equation (\Re and \Im parts).

(Curtright, Fairlie, & Zachos, Phys Rev **D58** (1998) 025002)

\Rightarrow Projective orthogonality spectral properties

$$f \star H \star g = E_f f \star g = E_g f \star g.$$

For $E_g \neq E_f$, $\implies f \star g = 0$.

Precluding degeneracy, for $f = g$,

$$f \star H \star f = E_f f \star f = H \star f \star f,$$

$$\implies f \star f \propto f.$$

f s \star -project onto their space.

$$f_a \star f_b = \frac{1}{\hbar} \delta_{a,b} f_a.$$

- The normalization matters (Takabayasi, 1954): despite linearity of the equations, it prevents superposition of solutions (this is not how QM interference works here!).

$$\int dpdx f \star g = \int dpdx fg,$$

so, for different \star -genfunctions,

$$\int dpdx fg = 0.$$

\rightsquigarrow **Negative values are a feature**, not a liability. Quantum interference confined to “ \hbar -small” regions.

NB $\hookrightarrow \int H(x, p) f(x, p) dx dp = E \int f dx dp = E .$

NB $\rightsquigarrow \int f^2 dx dp = \frac{1}{\hbar} .$

In general, $\leq 1/\hbar \rightsquigarrow$ quantum: fuzzy — classical: spiky.

- For any function, $\langle |g|^2 \rangle$ need not ≥ 0 .

But $\langle g^* \star g \rangle \geq 0 \quad \hookrightarrow$ the **uncertainty principle**,

$\Delta x \Delta p \geq \hbar/2 \quad \rightsquigarrow \quad (\Delta x)^2 + (\Delta p)^2 \geq \hbar.$ Hides negative values

Curtright & Zachos, Mod Phys Lett **A16** (2001) 2381.

▼ Pf

$$H(x, p) \star f(x, p)$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left((p - i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x) \right) \int dy e^{-iy(p + i\frac{\hbar}{2} \overleftarrow{\partial}_x)} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\
 &= \frac{1}{2\pi} \int dy \left((p - i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x + \frac{\hbar}{2}y) \right) e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\
 &= \frac{1}{2\pi} \int dy e^{-iyp} \left((i \overrightarrow{\partial}_y + i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x + \frac{\hbar}{2}y) \right) \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\
 &= \frac{1}{2\pi} \int dy e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) E \psi(x + \frac{\hbar}{2}y) = \\
 &= E f(x, p);
 \end{aligned}$$

↪ Action of the effective differential operators on ψ^* turns out to be null.

$$f \star H$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int dy e^{-iyp} \left(-(\overrightarrow{\partial}_y - \frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x - \frac{\hbar}{2}y) \right) \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\
 &= E f(x, p).
 \end{aligned}$$

Conversely, the pair of \star -eigenvalue equations dictate, for $f(x, p) = \int dy e^{-iyp} \tilde{f}(x, y)$,

$$\int dy e^{-iyp} \left(-\frac{1}{2m} (\vec{\partial}_y \pm \frac{\hbar}{2} \vec{\partial}_x)^2 + V(x \pm \frac{\hbar}{2} y) - E \right) \tilde{f}(x, y) = 0.$$

\rightsquigarrow Real solutions of $H(x, p) \star f(x, p) = E f(x, p)$ ($= f(x, p) \star H(x, p)$) must be of the Wigner form, $f = \int dy e^{-iyp} \psi^*(x - \frac{\hbar}{2} y) \psi(x + \frac{\hbar}{2} y) / 2\pi$, (s.t. $\mathbf{H}\psi = E\psi$).

The wonderful fact (Groenewold): \star -multiplication of c-number phase-space functions is in **complete isomorphism** to Hilbert-space operator algebra.

SIMPLE HARMONIC OSCILLATOR

Solve **directly** for $H = (p^2 + x^2)/2$
 (with $\hbar = 1, m = 1, \omega = 1$):

$$\left(\left(x + \frac{i}{2} \partial_p \right)^2 + \left(p - \frac{i}{2} \partial_x \right)^2 - 2E \right) f(x, p) = 0.$$

Mere PDEs! Imaginary part: $(x\partial_p - p\partial_x)f = 0. \quad \rightsquigarrow f$ depends on
 only one variable, $z = 4H = 2(x^2 + p^2). \quad \rightsquigarrow$

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - E \right) f(z) = 0.$$

Set $f(z) = \exp(-z/2)L(z) \quad \implies \quad \text{Laguerre's eqn}$

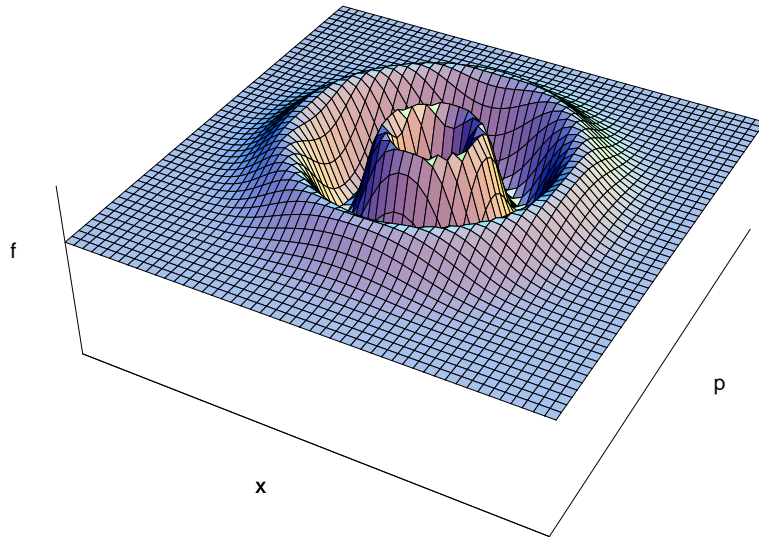
$$\left(z\partial_z^2 + (1 - z)\partial_z + E - \frac{1}{2} \right) L(z) = 0.$$

Satisfied by Laguerre polynomials, $L_n = e^z \partial^n (e^{-z} z^n) / n!$, for
 $n = E - 1/2 = 0, 1, 2, \dots \quad \rightsquigarrow \quad$ eigen-Wigner-functions are

$$f_n = \frac{(-1)^n}{\pi} e^{-2H} L_n(4H); \quad L_0 = 1, \quad L_1 = 1 - 4H,$$

$L_2 = 8H^2 - 8H + 1, \dots \quad \diamond$ not positive definite.

Oscillator Wigner Function, $n=3$



$$\sum_n f_n = \frac{1}{2\pi} .$$

Dirac's Hamiltonian factorization for algebraic solution **carries through intact in \star space:**

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} ,$$

so define

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip).$$

$$a \star a^\dagger - a^\dagger \star a = 1 .$$

★-Fock vacuum:

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 .$$

Associativity of the ★-product permits the customary ladder spectrum generation; $H \star f = f \star H$ ★-genstates:

$$f_n \propto (a^\dagger \star)^n f_0 (\star a)^n .$$

※ real, like the Gaussian ground state;

↷ left-right symmetric;

★-orthogonal for different eigenvalues;

project to themselves, since the Gaussian ground state does, $f_0 \star f_0 \propto f_0$.

TIME EVOLUTION

Isomorphism to operator algebras \rightsquigarrow associative combinatoric operations completely analogous to Hilbert space QM.

\rightsquigarrow \star -unitary evolution operator, a “ \star -exponential”, $U_\star(x, p; t) = e_\star^{itH/\hbar} \equiv$

$$1 + (it/\hbar)H(x, p) + \frac{(it/\hbar)^2}{2!}H \star H + \frac{(it/\hbar)^3}{3!}H \star H \star H + \dots,$$

$$f(x, p; t) = U_\star^{-1}(x, p; t) \star f(x, p; 0) \star U_\star(x, p; t).$$

NB Collapse to **classical** trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p ,$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \quad \implies$$

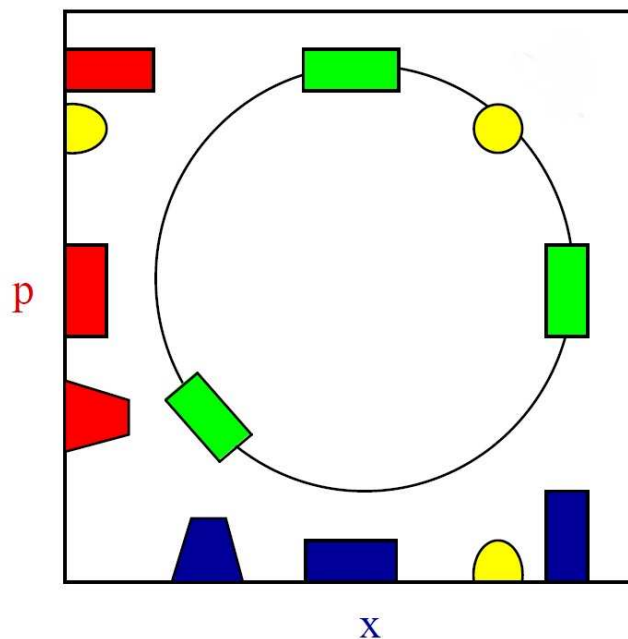
$$x(t) = x \cos t + p \sin t,$$

$$p(t) = p \cos t - x \sin t.$$

⇒ For SHO the functional form of the Wigner function is preserved along classical phase-space trajectories (Groenewold, 1946):

$$f(x, p; t) = f(x \cos t - p \sin t, p \cos t + x \sin t; 0).$$

Any Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete



quantum mechanical description.

In general, **loss of simplicity upon integration in x (or p) to yield probability densities**: the rotation induces shape variations of the oscillating probability density profile.

NB Only if (eg, coherent states) a Wigner function configuration has an additional axial $x - p$ symmetry around its **own** center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density.

THE WEYL CORRESPONDENCE BRIDGE

Weyl's correspondence map, by itself, merely provides **a change of representation between phase space and Hilbert space** \leftrightarrow Mutual language to contrast classical to quantum mechanics on common footing, and illuminate the transition.

$$\mathbf{A}(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp a(x, p) \exp(i\tau(\mathbf{p} - p) + i\sigma(\mathbf{x} - x)),$$

Inverse map (Wigner):

$$a(x, p) = \frac{1}{2\pi} \int dy e^{-iyp} \left\langle x + \frac{\hbar}{2}y \left| \mathbf{A}(\mathbf{x}, \mathbf{p}) \right| x - \frac{\hbar}{2}y \right\rangle .$$

PHASE SPACE

a

quantum \downarrow

$a \star b$

classical $\hbar=0 \downarrow$

ab

$\xrightarrow{\text{Weyl}}$

$\xrightarrow{\text{Groenewold}}$

$\xrightarrow{\text{Weyl}}$

HILBERT SPACE

\mathbf{A}

\downarrow quantum

\mathbf{AB}

\downarrow Bracken $\hbar=0$

$\mathbf{A} \odot \mathbf{B}$

\rightsquigarrow A plethora of choice-of-ordering quantum mechanics problems reduce to purely \star -product algebraic ones: varied deformations (ordering choices) can be surveyed systematically in phase space. (Curtright & Zachos, New J Phys 4 (2002) 83.1-83.16 [hep-th/0205063])

A Concise Treatise on Quantum Mechanics in Phase Space

This is a text on quantum mechanics formulated systematically in terms of position and momentum, i.e. in phase space. It is written at an introductory level, drawing on the remarkable history of the subject for inspiration and motivation. Wigner functions – density matrices in a special Weyl representation – and star products are the cornerstones of the formalism.

The resulting framework is a rich source of physical intuition. It has been used to describe transport in quantum optics, structure and dynamics in nuclear physics, chaos, and coherences in quantum computing. It is also of importance in signal processing and the mathematics of algebraic deformation. A remarkable aspect of its internal logic, pioneered by Groenewold and Moyal, has only emerged in the last quarter-century: it furnishes a third, alternative way to formulate and understand quantum mechanics, independent of the conventional Hilbert space or path integral approaches to the subject.

It is logically complete and self-consistent formalism: one need not choose sides between coordinate or momentum space variables. It works in full phase space, accommodating the uncertainty principle; and it offers unique insights into the classical limit of quantum theory. The observables in this formalism are real-valued functions in phase space instead of operators, with the same interpretation as their classical counterparts, only composed together in novel algebraic ways using star products.

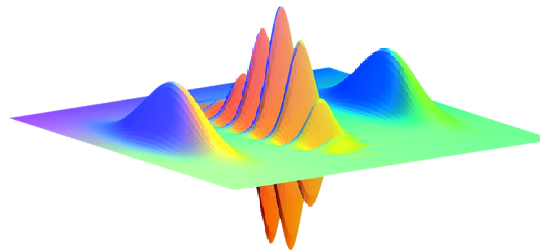
This treatise provides an introductory overview and supplementary material suitable for an advanced undergraduate or a beginning graduate course in quantum mechanics.

Curtright • Fairlie • Zachos

A Concise Treatise on Quantum Mechanics in Phase Space

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Thomas L. Curtright, David B. Fairlie and Cosmas K. Zachos



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