Poincare Sphere and a Unified Picture of Wigner's Little Groups

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Presented at the Wigner 111
Budapest, Hungary

I was here before,
but I am very happy
to be here again.

I thank the
organizers for
inviting me to this
great conference.

Toyotomi Hideyoshi (1536-98) unified Japan, appointed himself as the Emperor of China, and invaded Korea in 1591.

To humans, he looks like a monkey.

To monkeys, he looks like a human.

Japanese are Kantianists. Einstein started as a Kantianist.

This person has been writing papers since 1961. People seem to have difficulties in understanding his papers.

To opticd people, I look like a particle physicist.

To particle people, I look like an optical physicist.

Thus, I should talk about particle physics, using the language of optics.

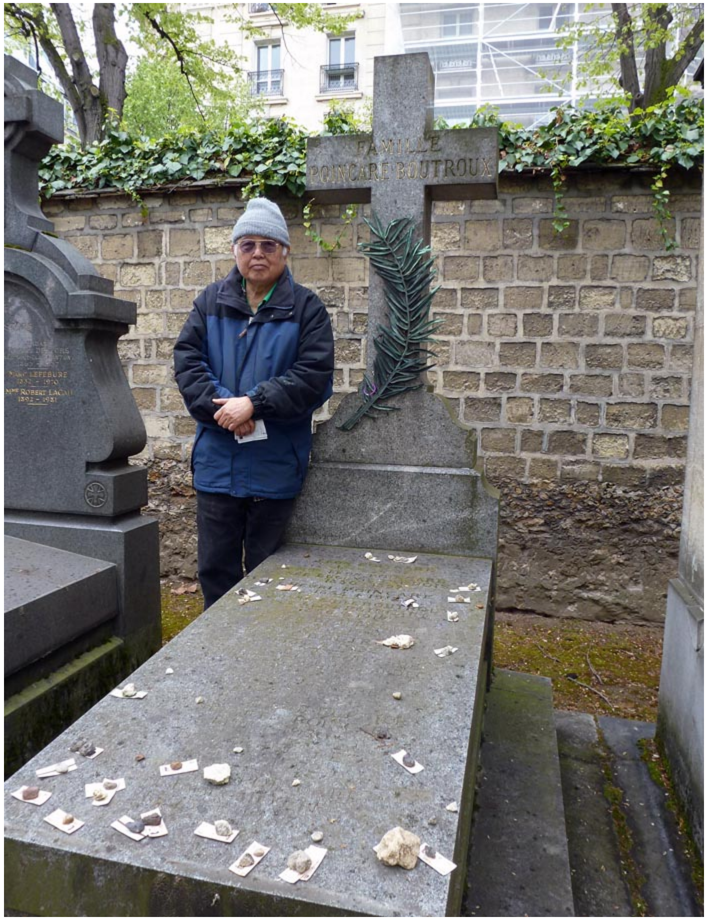
Two great physicists with many faces.

Henri Poincare (1854-1912)

Eugene Wigner (1902-2002)

How close was I to them?

Poincare’s Grave in Paris at Montparnasse Cem. (2010).



with Eugene Wigner at the University of Maryland (1986).

Poincare Sphere and Wigner's Little Groups

Polarization optics.
Well understood

The most difficult paper to read.

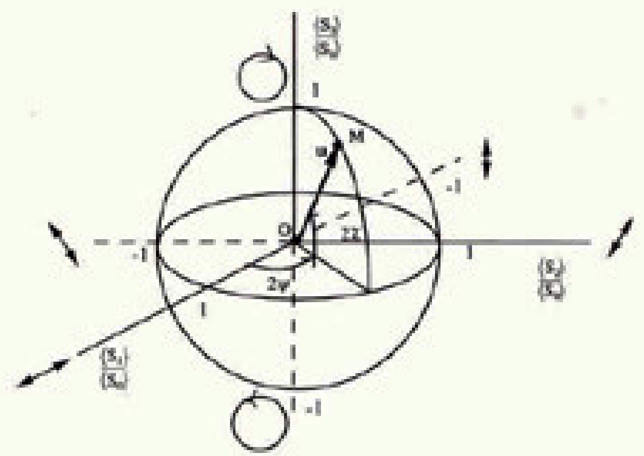


Figure 3.2.1. The Poincaré sphere Σ_1^2 is the unit sphere surrounding the origin of the Cartesian coordinate, orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$. The normalized Stokes parameters $(\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle)$ constitute the components of the Poincaré vector **a** that represents the state of polarization of an arbitrary pure state of polarization ($|\mathbf{a}| = 1$). The longitude 2ψ and latitude 2χ of point M are respectively related to the azimuth and the ellipticity angles of the polarization ellipse of the wave. Each point on Σ_1^2 corresponds to a unique state of polarization. The north pole $N = [0, 0, 1]^T$ represents right circularly polarized light. The south pole $S = [0, 0, -1]^T$ represents left circularly polarized light. Points on the equator ($2\chi = 0$) represent linearly polarized light. Elliptical polarization states lie between the poles and equator. The positive directions of the angle 2ψ and 2χ are defined according the adopted sign convention.

ANNALS OF MATHEMATICS
Vol. 40, No. 1, January, 1939

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

BY E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,[1] in which a unitary *scalar product* is defined.[2] The states are generally represented by wave functions[3] in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ, or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ, to see whether or not the state is ψ, gives the result that it is ψ. If there are two or more different experiments to decide this (e.g., essentially the same experiment,

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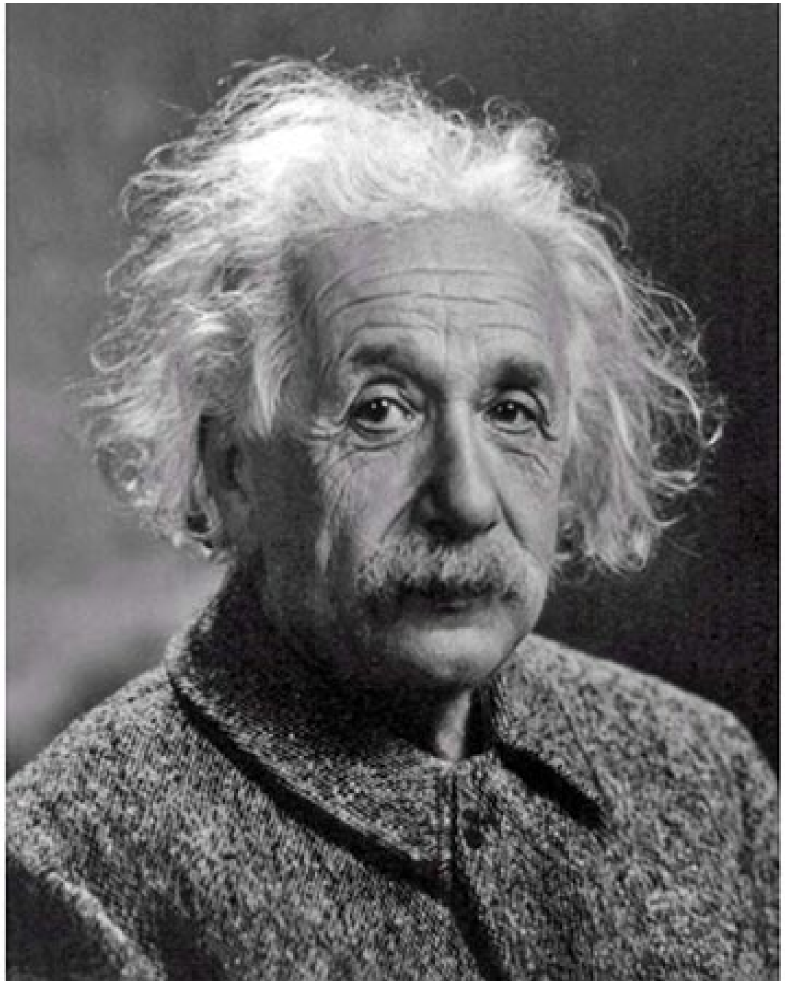
[3] The wave functions represent throughout this paper states in the sense of the "Heisenberg picture," i.e. a single wave function represents the state for all past and future. On the other hand, the operator which refers to a measurement at a certain time t contains this t as a parameter. (Cf. e.g. Dirac, l.c. ref. 2, pages 115–123). One obtains the wave function $\varphi_s(t)$ of the Schrödinger picture from the wave function φ_H of the Heisenberg picture by $\varphi_s(t) = \exp(-iHt/\hbar)\varphi_H$ The operator of the Heisenberg picture is $Q(t) = \exp(iHt/\hbar)\, Q \exp(-iHt/\hbar)$, where Q is the operator in the Schrödinger picture which does not depend on time. Cf. also E. Schrödinger, Sitz. d. Kön. Preuss. Akad. p. 418, 1930.

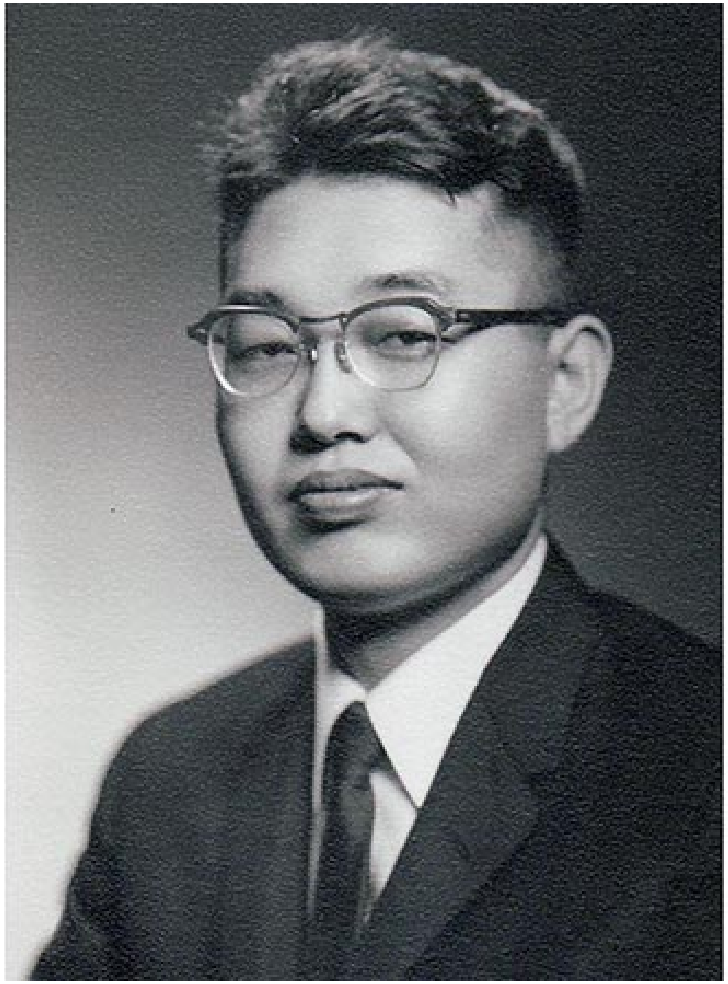
The wave functions are complex quantities and the undetermined factors in them are complex also. Recently attempts have been made toward a theory with real wave functions. Cf. E. Majorana, Nuovo Cim. *14*, 171, 1937 and P. A. M. Dirac, in print.

I never met this man, but met some who met him. O.J.Turner was a Princeton photographer.

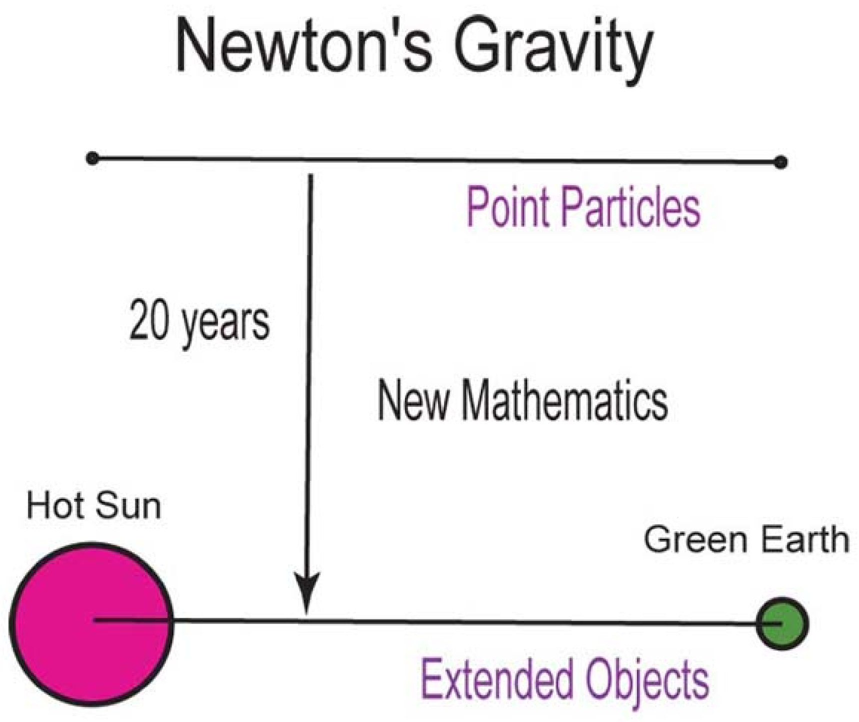
1947

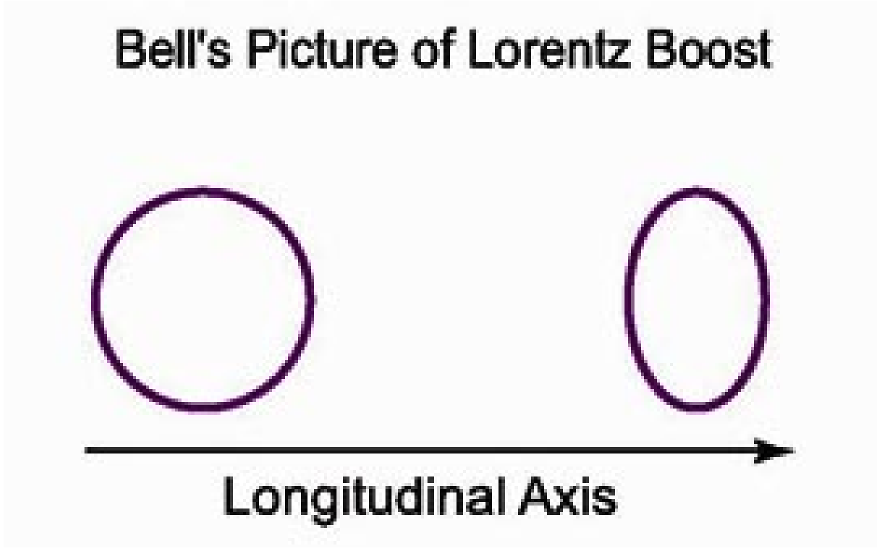


1961



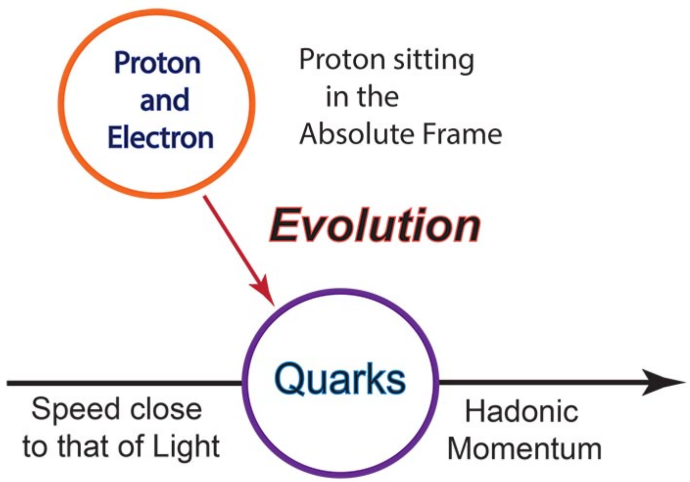
Einstein Issue: How to Lorentz-transform the hydrogen atom?



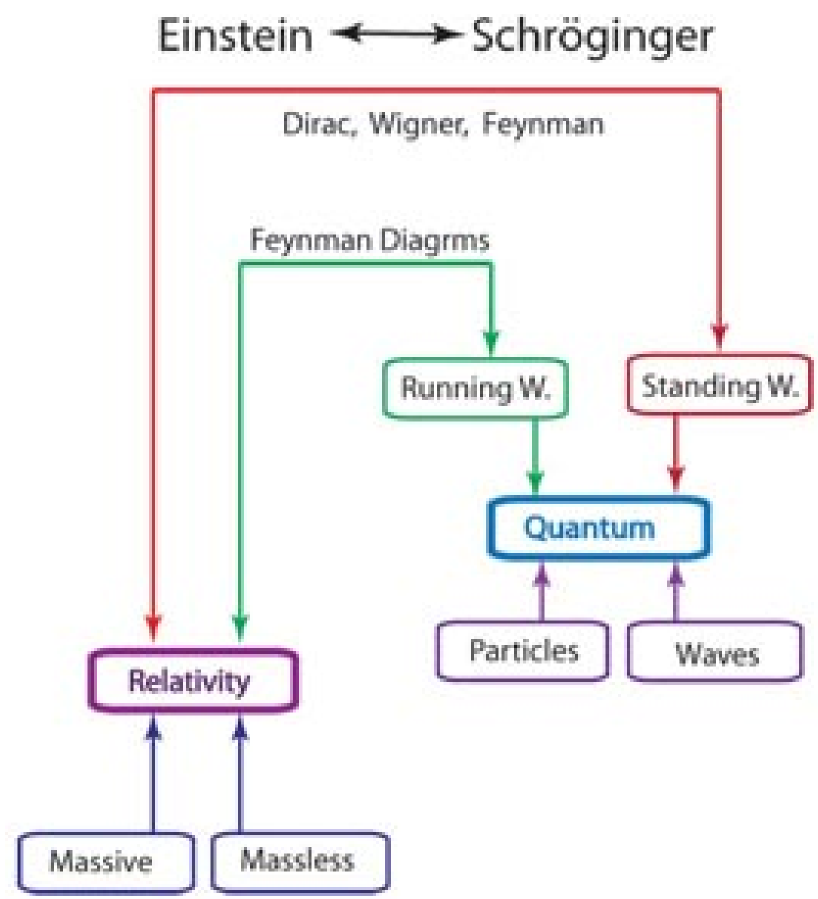


In 1927, the orbit was replaced by a standing wave. How to Lorentz-boost standing waves?

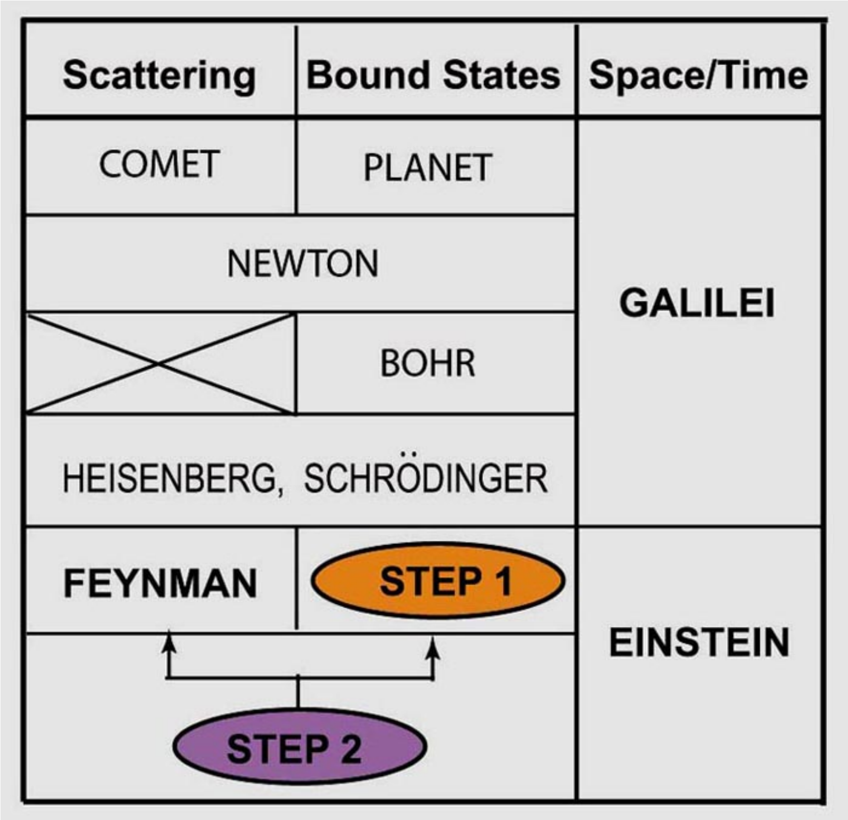
Proton replaced the hydrogen atom, because it can be accelerated. Like the H-atom, it is a bound state or standing wave.

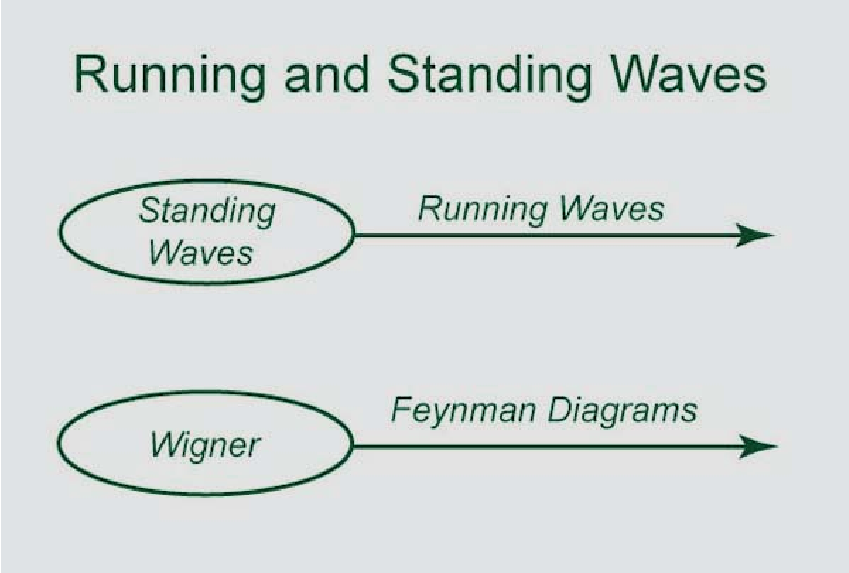


The key question is how to put Einstein and Schrodinger into one box.



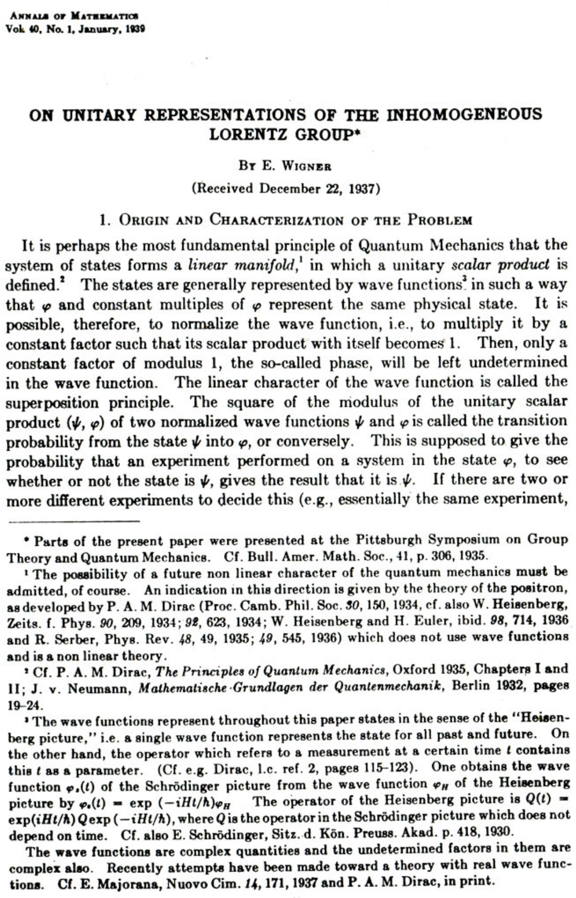
History of Physics. Where is Wigner?





Wigner’s little groups define internal space-time symmetries

Wigner’s 1939 paper



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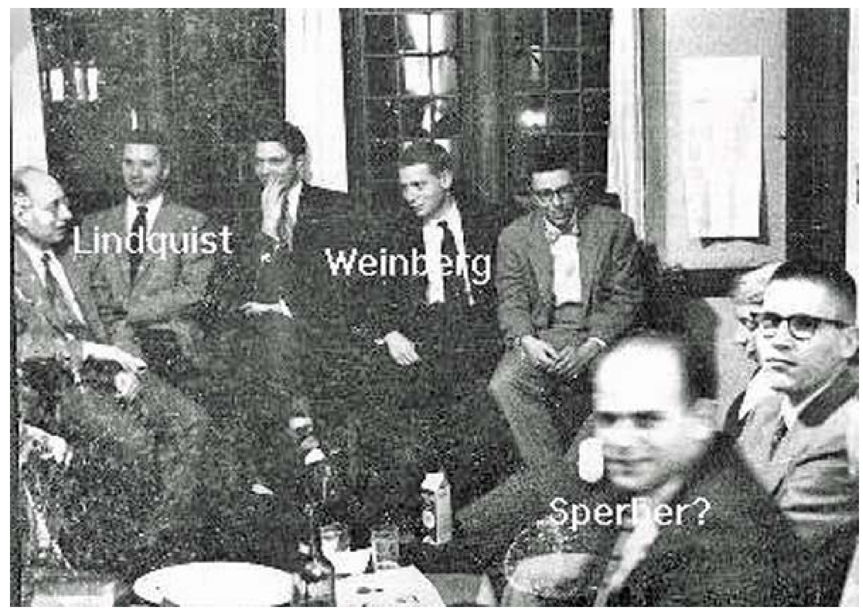
149

Steven Weinberg was interested in this paper and published many papers on this subject in 1963-65.



Sam Treiman was Weinberg's advisor. He was also my advisor.

Weinberg and Treiman

1985

The Treimans and the Kims

1987

In 1988, I gave a big conference to Wigner with 8 Nobels attending. I also invited the Hungarian ambassador to the United States.

He came with Wigner's membership certificate for the Academy of Sciences
of the Peoples Republic of Hungary.

In 2002, Hungarians gave a Wigner Centennial Conference. I am very happy to meet here two of the organizers of the 2002 conference.

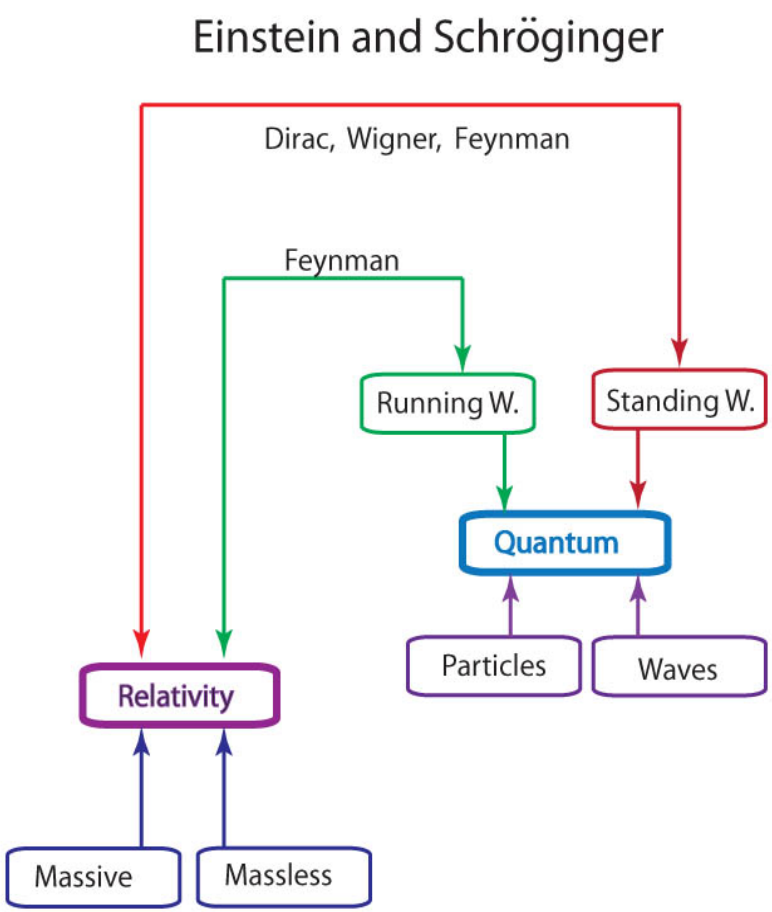
In 2002, there was also a meeting in Princeton to honor Wigner. With me in this photo are F.Seitz, C.Upton, J.Taylor.

Wigner’s family photo (1950?)

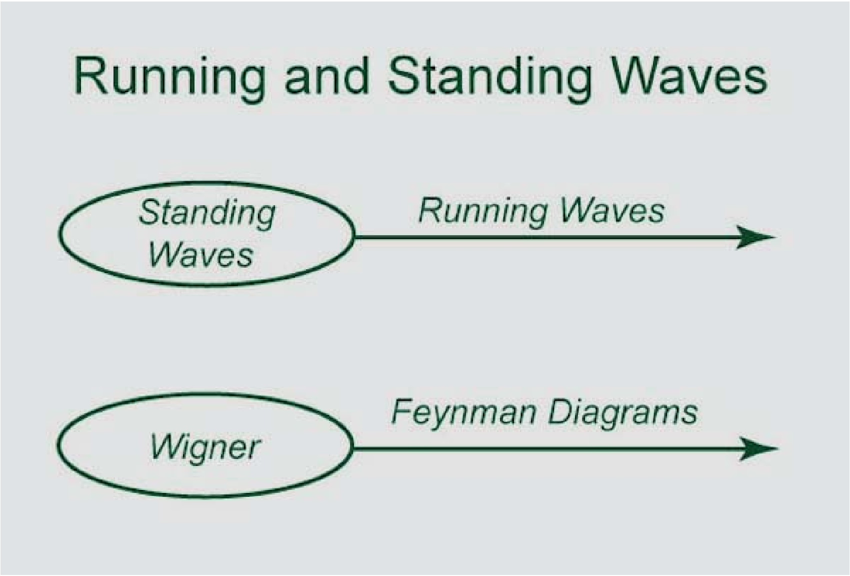
Wigner's daughter

When I said I had a photo with Wigner's daughter, young ladies at this conference wanted have photos with me. I am still available.

Wigner 1939.
Internal space-time symmetries.



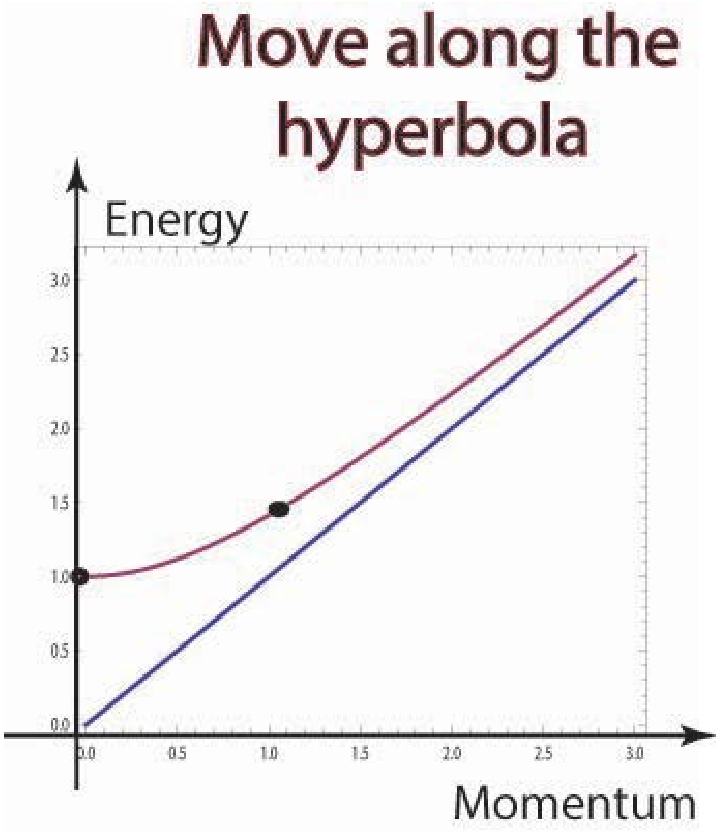
Lorentz group has six degrees of freedom, three boosts and there rotations.



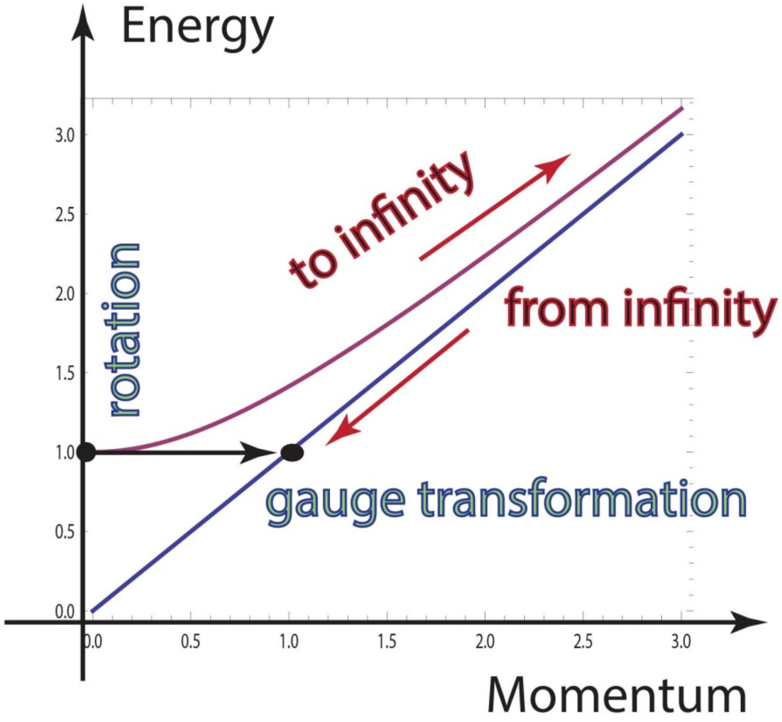
If the momentum is fixed, there are only **three degrees of freedom** left for internal symmetries.

Wigner Plan

Start from a massive particle at rest



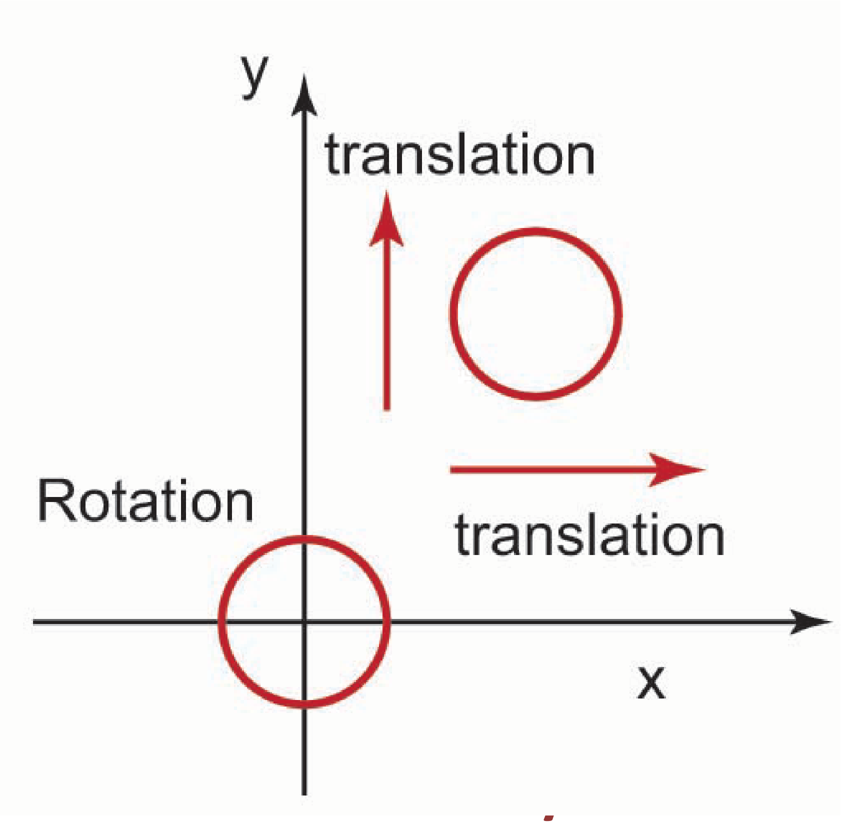
Jump the hyperbola at infinity



Wigner 1939:
Internal space-time symmetries

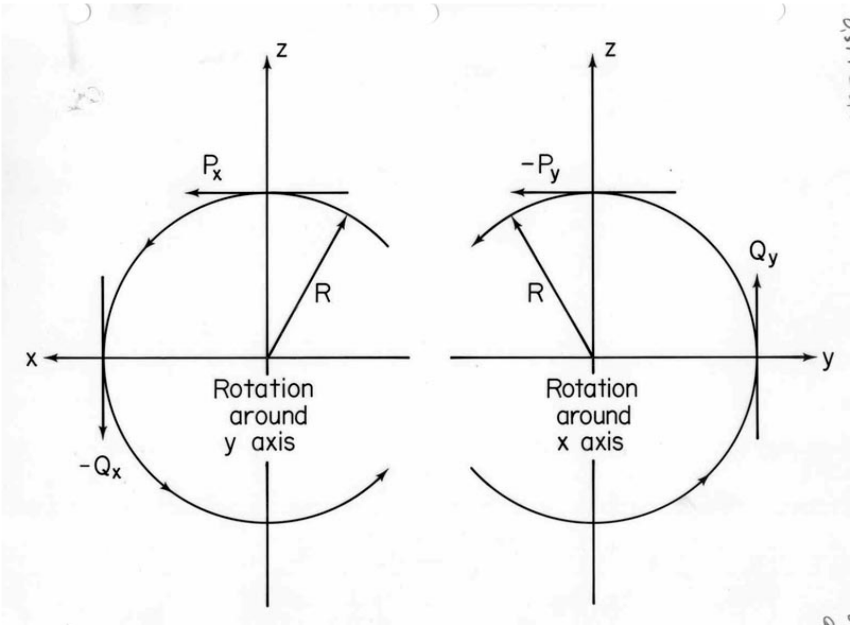
For a massive particle, it is O(3), or 3-d rotation group. This defines the particle spin in Einstein's world

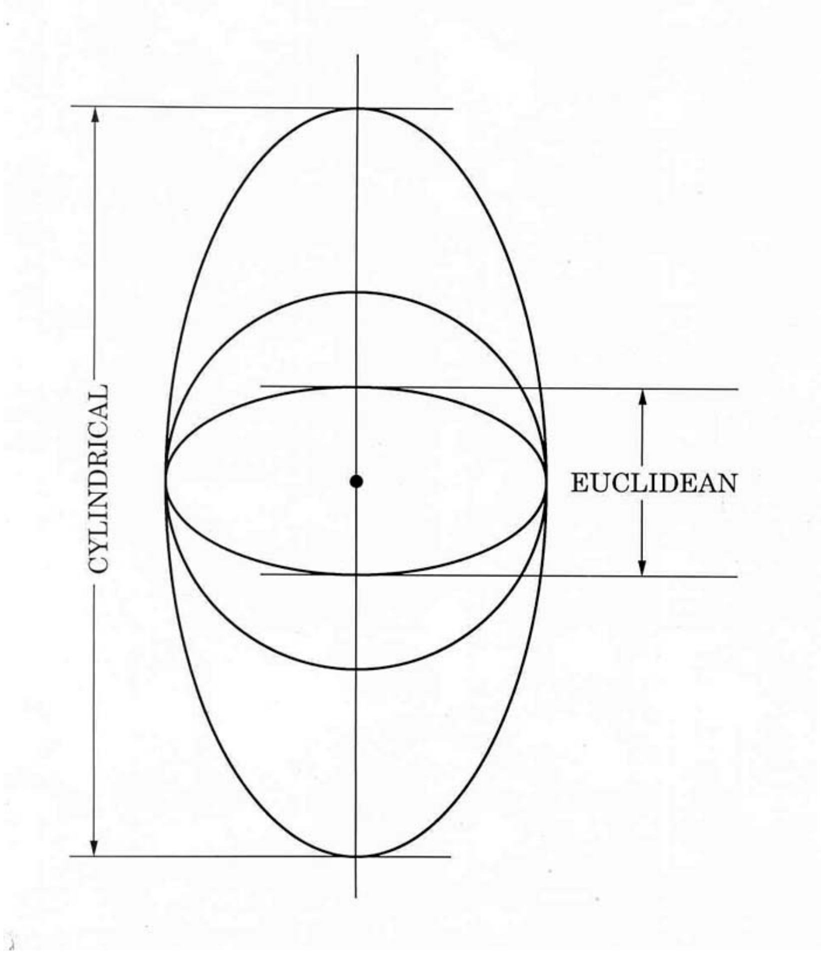
For a massless particle, the symmetry group is like E(2), 2d Euclidean group.



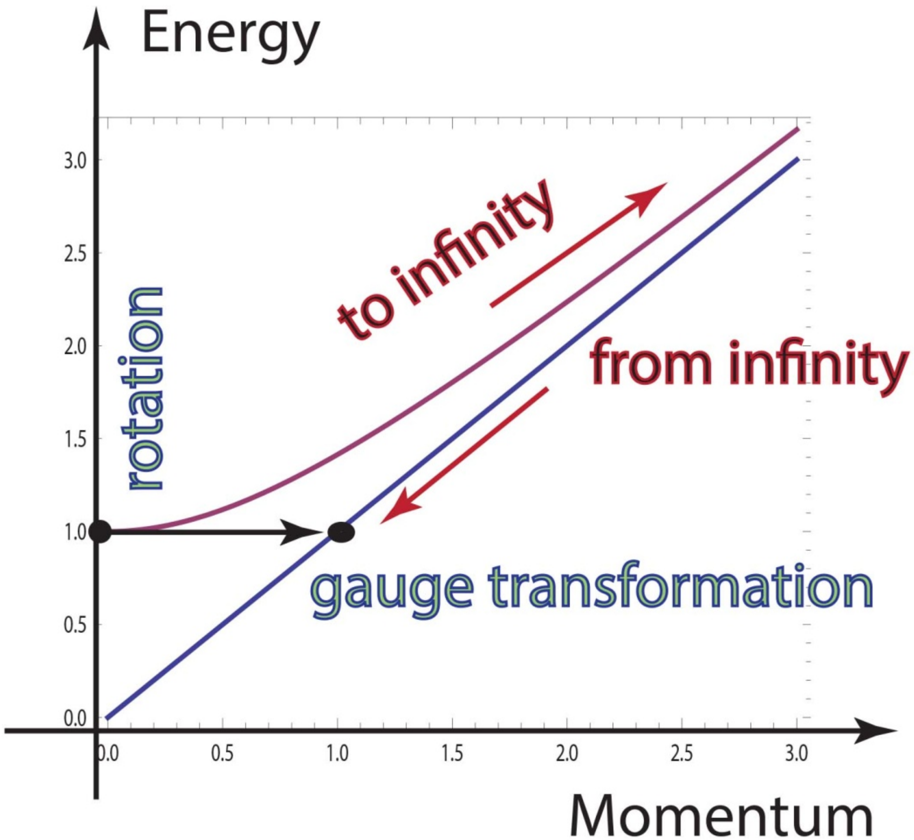
Translations: Physics?

E(2) becomes Cylindrical (1987).
Up-down translation becomes gauge transformation.





When the particle mass becomes zero



Einstein's Genealogy

	Massive Slow	between	Massless Fast
Energy Momentum	$E = \frac{p^2}{2m}$	Einstein's $E = \sqrt{m^2+p^2}$	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	Wigner's Little Group	S_3 Gauge Trans.

BUILD YOUR OWN HOUSE

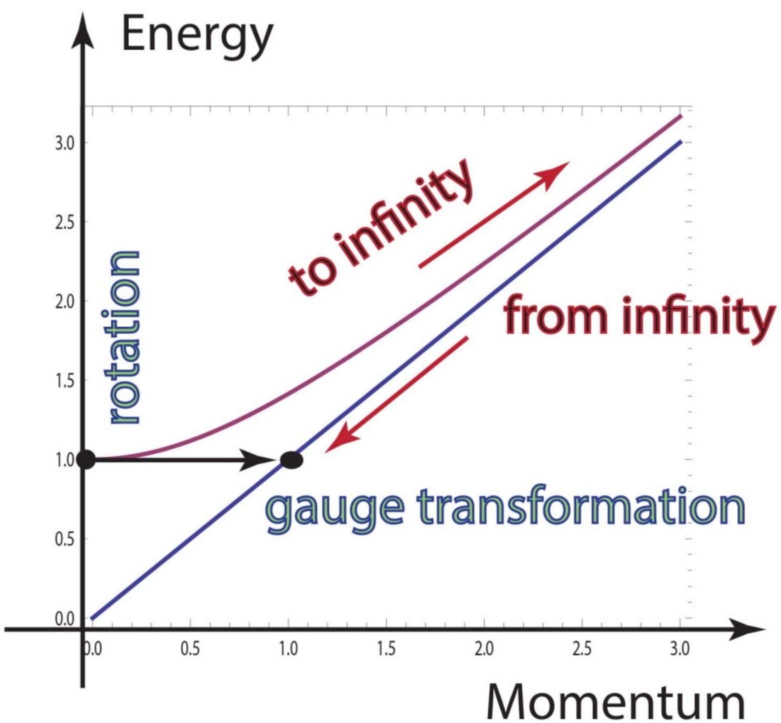
see Phys. Rev. Lett. 63, 384 (1989).

Further contents of Einstein's E = mc^2

	Massive Slow	← between →	Massless Fast
Energy Momentum	$E = \frac{p^2}{2m}$	← $E = \sqrt{m^2 + p^2}$ →	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	← Little Groups →	S_3 Gauge Trans.
Quarks Partons	Quark Model	← (Covariant Phase Space) →	Parton Model

Registered at Phys. Rev. Lett. [63] 348-351 (1989).

Massive to Massless



The Lorentz boost depends only on the relative velocity

$$\frac{\mathbf{v}}{\mathbf{c}} = \frac{\mathbf{p}}{(\mathbf{p})^2 + \mathbf{m}^2} = \frac{(\mathbf{p}/\mathbf{m})}{\sqrt{1 + (\mathbf{p}/\mathbf{m})^2}}.$$

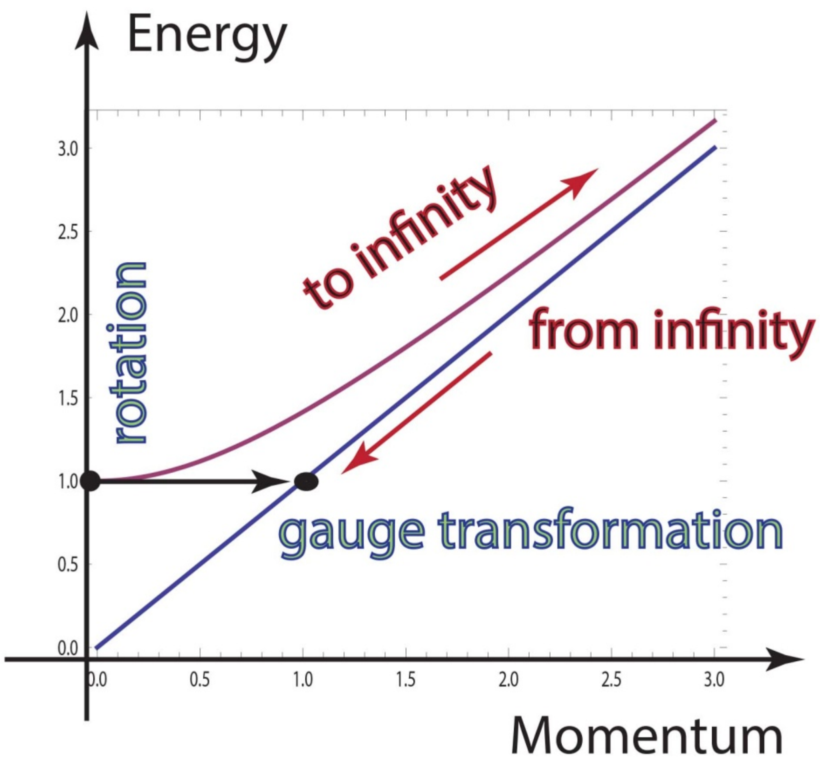
Thus, it depends only on the ratio

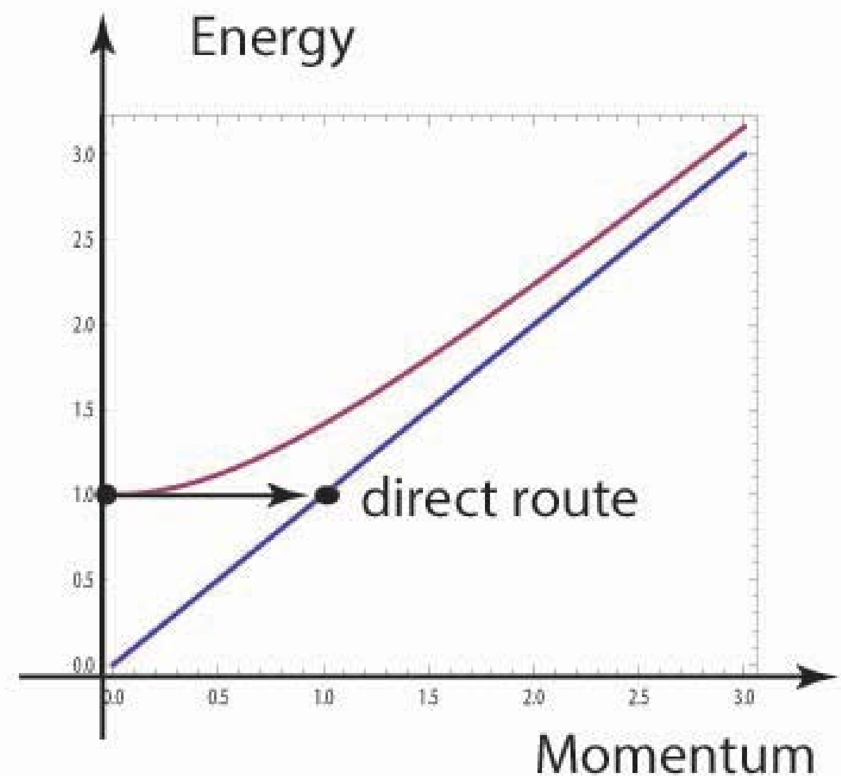
$$\frac{\mathbf{p}}{\mathbf{m}}.$$

This quantity can become large when p becomes large, or m becomes small.

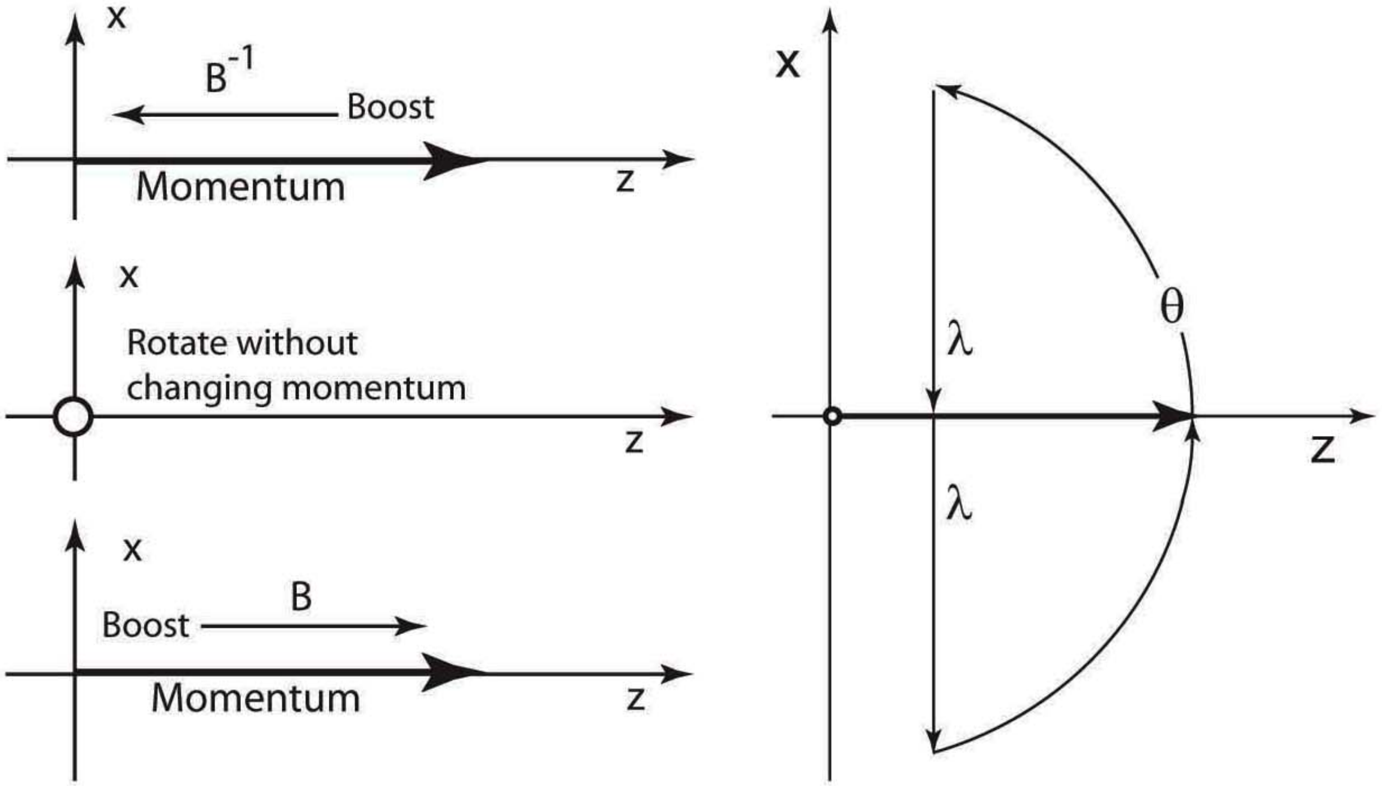
The Lorentz group does not allow us to change m. Thus, the only way to get the little group for a massless particle is to go to infinity and come back, as shown in this figure.

Is there a direct route?
We need a Poincare sphere.





Wigner's Little group whose transformations leave the **four-momentum** of a given particle **invariant**



Four-vectore in 2-by-2 representation

The Lorentz group starts with a group of four-by-four matrices performing Lorentz transformations on the Minkowskian vector space of (t, z, x, y), leaving the quantity

$$t^2 - z^2 - x^2 - y^2$$

In this two-by-two representation, we write the four-vector as a matrix

$$\mathrm{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix},$$

whose determinant is just $t^2 - z^2 - x^2 - y^2$.

The Lorentz transformation = determinant-preserving transformation.

Determinant-preserving transformations.

Let us consider

$$\boldsymbol{G} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \boldsymbol{G}^{\dagger} = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix},$$

with

$$\det(\boldsymbol{G}) = 1.$$

The $\boldsymbol{G}$ matrix starts with four complex numbers. Due to the above condition on its determinant, it has six independent parameters. The group of these $\boldsymbol{G}$ matrices is known to be locally isomorphic to the group of four-by-four matrices performing Lorentz transformations on the four-vector (t, z, x, y).

With this point in mind, we can now consider the transformation

$$\boldsymbol{X}' = \boldsymbol{G}\boldsymbol{X}\boldsymbol{G}^{\dagger}.$$

Since $\boldsymbol{G}$ is not a unitary matrix, it is not a unitary transformation. In order to tell this difference, we call this the "Naimark transformation." This expression can be written explicitly as

$$\begin{pmatrix} t' + z' & x' - iy' \\ x + iy & t' - z' \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}.$$

Be careful. G is not Hermitian, and this Lorentz transformation is not a similarity transformation.

Einstein's Four-momentum

Einstein defined the energy-momentum four-vector, and showed that it also has the same Lorentz-transformation law as the space-time four-vector. We write the energy-momentum four-vector as

$$P = \begin{pmatrix} E + p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix},$$

with

$$\det(P) = E^2 - p_x^2 - p_y^2 - p_z^2,$$

which means

$$\det(P) = m^2,$$

where m is the particle mass.

Now Einstein's transformation law can be written as

$$\boldsymbol{P}' = \boldsymbol{G}\boldsymbol{P}\boldsymbol{G}^{\dagger},$$

or explicitly

$$\begin{pmatrix} \boldsymbol{E}' + \boldsymbol{p}'_z & \boldsymbol{p}'_x - i\boldsymbol{p}'_y \\ \boldsymbol{p}'_x + i\boldsymbol{p}'_y & \boldsymbol{E}' - \boldsymbol{p}'_z \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix} \begin{pmatrix} \boldsymbol{E} + \boldsymbol{p}_z & \boldsymbol{p}_x - i\boldsymbol{p}_y \\ \boldsymbol{p}_x + i\boldsymbol{p}_y & \boldsymbol{E} - \boldsymbol{p}_z \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^* & \boldsymbol{\gamma}^* \\ \boldsymbol{\beta}^* & \boldsymbol{\delta}^* \end{pmatrix}.$$

4-by-4 from 2-by-2

It is possible to construct familiar 4-by-4 transformation matrices from the $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ parameters.

$$\begin{pmatrix} t' + z' \\ x' - iy' \\ x' + iy' \\ t' - z' \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* & \beta\alpha^* & \beta\beta^* \\ \alpha\gamma^* & \alpha\delta^* & \beta\gamma^* & \beta\delta^* \\ \gamma\alpha^* & \gamma\beta^* & \delta\alpha^* & \delta\beta^* \\ \gamma\gamma^* & \gamma\delta^* & \delta\gamma^* & \delta\delta^* \end{pmatrix} \begin{pmatrix} t + z \\ x - iy \\ x + iy \\ t - z \end{pmatrix},$$

and

$$\begin{pmatrix} t \\ z \\ x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \begin{pmatrix} t + z \\ x - iy \\ x + iy \\ t - z \end{pmatrix}.$$

Transformation	2-by-2	4-by-4
rot. around z	$Z(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{pmatrix}$
boost along z	$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}$	$\begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
rot. around y	$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \sin(\theta/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
boost along x	$S(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \sinh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh\lambda & 0 & \sinh\lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\lambda & 0 & \cosh\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Wigner's Little Groups

In 1939, Wigner considered subgroups of the Lorentz group whose tranformations leave the four-momentum of a given particle invariant. The massive particle can be brought to its rest frame, and its four-momentum becomes

$$(m, 0, 0, 0).$$

This four-momentum is invariant under three-dimensional rotations applicable only to the z, x, y coordinates. The dynamical variable associated with this rotational degree of freedom is called the spin of the particle.

In the two-by-two representation, the particle at rest has its four-momentum

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is invariant under rotation:

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If the particle is massless,

the momentum four-vector is

$$\boldsymbol{p_0(1,1,0,0)} \quad \rightarrow \quad \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and the invariant transformation

$$\begin{pmatrix} \mathbf{1} & -\gamma \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\gamma & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

This

$$\begin{pmatrix} \mathbf{1} & -\gamma \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

triangular matrix looks strange, and has a stormy history. Physically, it corresponds to a

gauge transformation.

Lorentz completion of the little group

Mathematicians call this "orbit" completion.
We are then interested in what happens when the particle moves with a non-zero momentum. If it moves along the z direction, the four-momentum takes the value

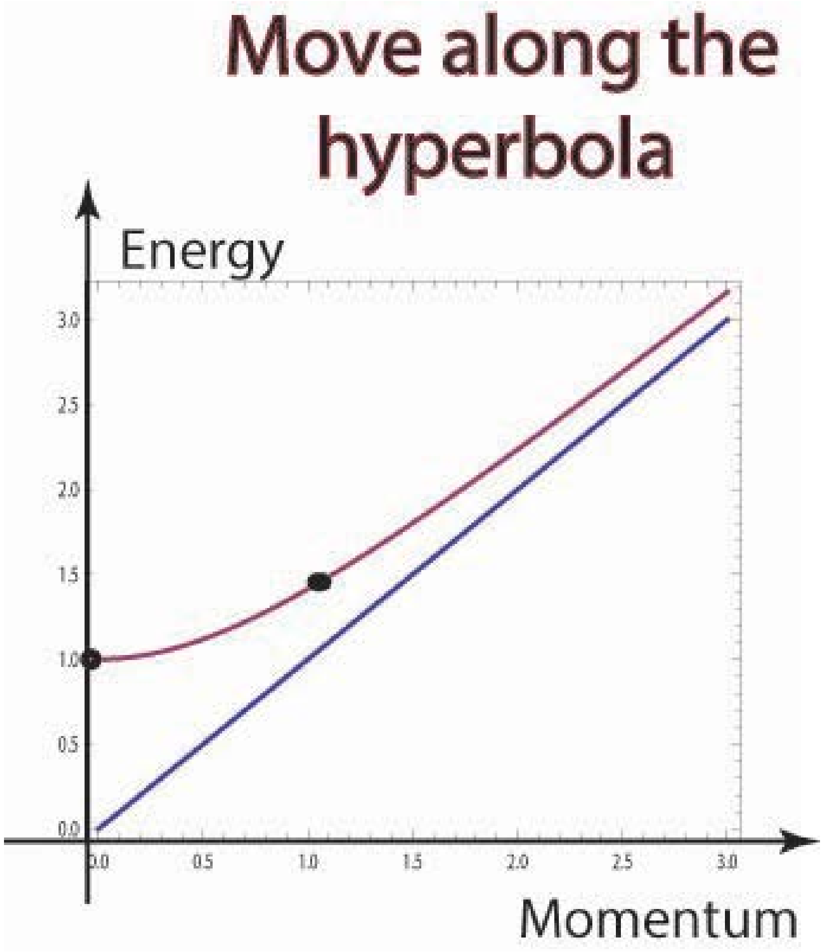
$$m(\cosh\eta, \sinh\eta, 0, 0),$$

which means

$$p_0 = m(\cosh\eta), \quad p_z = m(\sinh\eta),$$

$$e^{\eta} = \sqrt{\frac{p_0 + p_z}{p_0 - p_z}}.$$

Accordingly, the little group consists of Lorentz-boosted rotation matrices.
This aspect is not contained in Wigner's
original 1939 paper.



Go to infinite momentum and come back

In the large-η limit, the four-momentum becomes

$$P = (e^{\eta}, e^{\eta}, 0, 0).$$

In the two-by-two representation, this becomes

$$P = \begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix} \rightarrow \begin{pmatrix} e^{\eta} & 0 \\ 0 & 0 \end{pmatrix}.$$

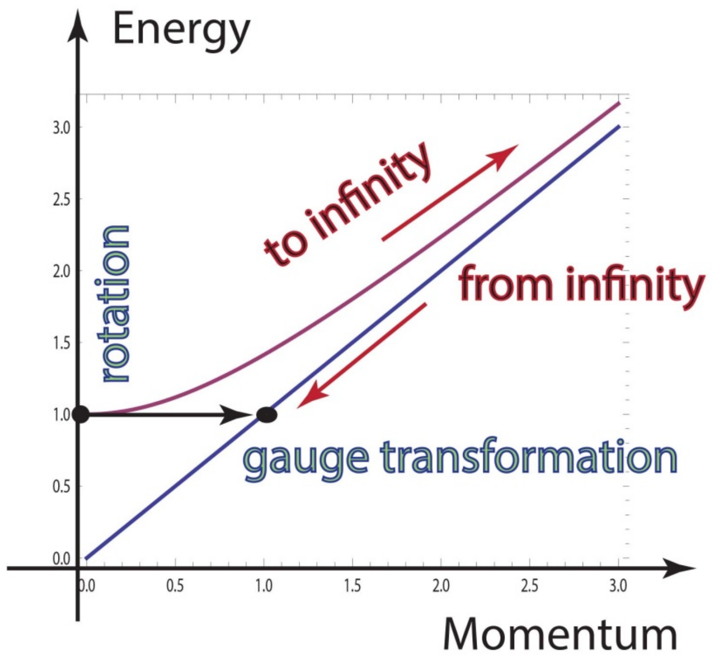
For $\eta = 0$, the momentum can come back to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for the massive particle, and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for the massless particle.



Lorentz-boosted rotation matrix

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2) & -e^{\eta}\sin(\theta/2) \\ e^{-\eta}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \cos(\theta/2) & e^{\eta}\sin(\theta/2) \\ 0 & \cos(\theta/2) \end{pmatrix}$$

However, $\theta = 0$ if the *determinant* $= 1$. Thus $e^{\eta}\sin(\theta/2) = \gamma$, and

$$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}.$$

Winger's Little Groups

Particle mass	Wigner 4-mom.	2-by2	Transform matrices
Massive	$(1,0,0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$
Massless	$(1,1,0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$
Imaginary	$(0,1,0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$

Wigner and Einstein

Massive, Slow	COVARIANCE	Massless, Fast
$E = p^2/2m$	$E = \sqrt{m^2c^4 + (cp)^2}$	$E = cp$
S_3	Wigner's Little Group	Helicity
S_1, S_2		Gauge Transformation

Wigner liked this table.

Jones vector for polarization optics

An Optical beam progpagates along the z direction with x and y components of the electric field. We start with

$$\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} = \begin{pmatrix} \exp\{i(kz-\omega t)\} \\ \exp\{i(kz-\omega t)\} \end{pmatrix}.$$

When the beam goes through a medium with different values of indexes of refraction for the x and y directions, we have to apply the matrix

$$Z(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}.$$

In the language of the space-time symmetry, this matrix performs a roation aroung the z axis.

Also along the x and y directions, the attenuation coefficients could be different. This will lead to the matrix

$$\begin{pmatrix} e^{-\eta_1} & 0 \\ 0 & e^{-\eta_2} \end{pmatrix}$$

$$= e^{-(\eta_1+\eta_2)/2} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}$$

with $\eta = \eta_2 - \eta_1$. If $\eta_1 = 0$ and $\eta_2 = \infty$, the above matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which eliminates the y component. This matrix is known as a polarizer in the textbooks, and is a special case of the attenuation.

Matrices applicable to the Jones vector

This attenuation matrix tells us that the electric fields are attenuated at two different rates. The exponential factor $e^{-(\eta_1+\eta_2)/2}$ reduces both components at the same rate and does not affect the state of polarization. The effect of polarization is solely determined by the squeeze matrix

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}.$$

In the language of space-time symmetries, this matrix performs a Lorentz boost along the z direction.

The polarization axes are not always the x and y axes. For this reason, we need the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

This matrix peroforms a rotation around the x axis in the space-time symmetry.

Among the rotation angles, the angle of 45^o plays an important role in polarization optics. Indeed, if we rotate the squeeze matrix $B(\eta)$ by 45^o, we end up with another squeeze matrix

$$S(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}.$$

In the language of space-time physics, this matrix leads to a Lorentz boost along the x axis.

Matrices for optics and Lorentz transformations

Polarization Optics	Transformation Matrix	Particle Symmetry
Phase shift δ	$\begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}$	Rotation around z.
Rotation around z	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	Rotation around y.
Squeeze along x and y	$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}$	Boost along z.
Squeeze along 45^o	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	Boost along x.
$(ab)^2 \sin^2 \chi$	Determinant	$(mass)^2$

Stokes parameters and coherency matrix

However, the Jones vector alone cannot tell us whether the two components are coherent with each other. In order to address this important degree of freedom, we use the coherency matrix

$$C = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

with

$$S_{11} = < \psi_1^* \psi_1 > = 1,$$

$$S_{22} = < \psi_2^* \psi_2 > = 1,$$

If i and j are different, the off-diagonal elements become

$$< \psi_i^* \psi_j > = \frac{1}{T} \int_0^T \psi_i^*(t+\tau) \psi_j(t) dt,$$

where T, for a sufficiently long time interval, is much larger than τ.

$$S_{12} = < \psi_1^* \psi_2 > = e^{-(\sigma + i\delta)},$$

$$S_{21} = < \psi_2^* \psi_1 > = e^{-(\sigma - i\delta)},$$

The σ parameter specifies the degree of coherency.

This coherency matrix is not always real but it is Hermitian. Thus it can be diagonalized by a unitary transformation. If this matrix is normalized so that its trace is one, it becomes a density matrix.

Stokes parameters constitute a four-vector.

The coherency matrix takes the form

$$C = \begin{pmatrix} 1 & e^{-(\sigma+i\delta)} \\ e^{-(\sigma-i\delta)} & 1 \end{pmatrix}.$$

Since $e^{-\sigma}$ is always smaller than one, we can introduce an angle χ defined as

$$\cos\chi = e^{-\sigma},$$

and call it the "decoherence angle." In terms of this angle,

$$C = \begin{pmatrix} 1 & (\cos\chi)e^{-i\delta} \\ (\cos\chi)e^{i\delta} & 1 \end{pmatrix}.$$

Starting from this simplest form of the coherence matrix, the most general form can be obtained from the Lorentz transformation.

$$C' = G\ C\ G^{\dagger} = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}.$$

We can then make the following linear combinations.

$$S_0 = \frac{S_{11} + S_{22}}{2},$$

$$S_3 = \frac{S_{11} - S_{22}}{2},$$

$$S_1 = \frac{S_{12} + S_{21}}{2},$$

$$S_2 = \frac{S_{12} - S_{21}}{2i}.$$

These quantities lead to the four-vector:

$$(S_0, S_z, S_x, S_y).$$

Poincare Sphere

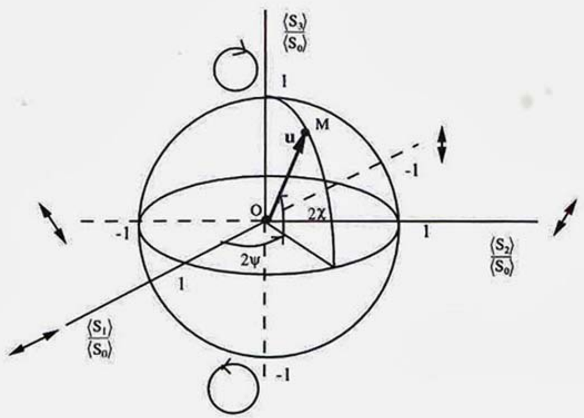


Figure 3.2.1. The Poincaré sphere Σ_1^2 is the unit sphere surrounding the origin of the Cartesian coordinate, orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$. The normalized Stokes parameters $(\langle\boldsymbol{\sigma}_1\rangle, \langle\boldsymbol{\sigma}_2\rangle, \langle\boldsymbol{\sigma}_3\rangle)$ constitute the components of the Poincaré vector $\mathbf{u}$ that represents the state of polarization of an arbitrary pure state of polarization $(|\mathbf{u}| = 1)$. The longitude 2ψ and latitude 2χ of point M are respectively related to the azimuth and the ellipticity angles of the polarization ellipse of the wave. Each point on Σ_1^2 corresponds to a unique state of polarization. The north pole $N = [0, 0, 1]^T$ represents right circularly polarized light. The south pole $S = [0, 0, -1]^T$ represents left circularly polarized light. Points on the equator $(2\chi = 0)$ represent linearly polarized light. Elliptical polarization states lie between the poles and equator. The positive directions of the angle 2ψ and 2χ are defined according the adopted sign convention.

We now have the four-vector $(\boldsymbol{S_0}, \boldsymbol{S_3}, \boldsymbol{S_1}, \boldsymbol{S_2})$, which is Lorentz-transformed like the space-time four-vector $(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y})$ or the energy-momentum four-vector. This Stokes four-vector has a three-component subspace $(\boldsymbol{S_3}, \boldsymbol{S_1}, \boldsymbol{S_2})$, which is like the three-dimensional Euclidean subspace in the four-dimensional Minkowski space. In this three-dimensional subspace, we can introduce the spherical coordinate system with

$$\boldsymbol{R = \sqrt{S_3^2 + S_1^2 + S_2^2}}$$

$$\boldsymbol{S_3 = 0}$$

$$\boldsymbol{S_1 = (\sin\chi)\cos\delta,}$$

$$\boldsymbol{S_2 = (\sin\chi)\sin\delta.}$$

Poincare sphere with two radii

The radius $\boldsymbol{R}$ is the radius of this sphere, and is

$$\boldsymbol{R} = \cos\chi.$$

with

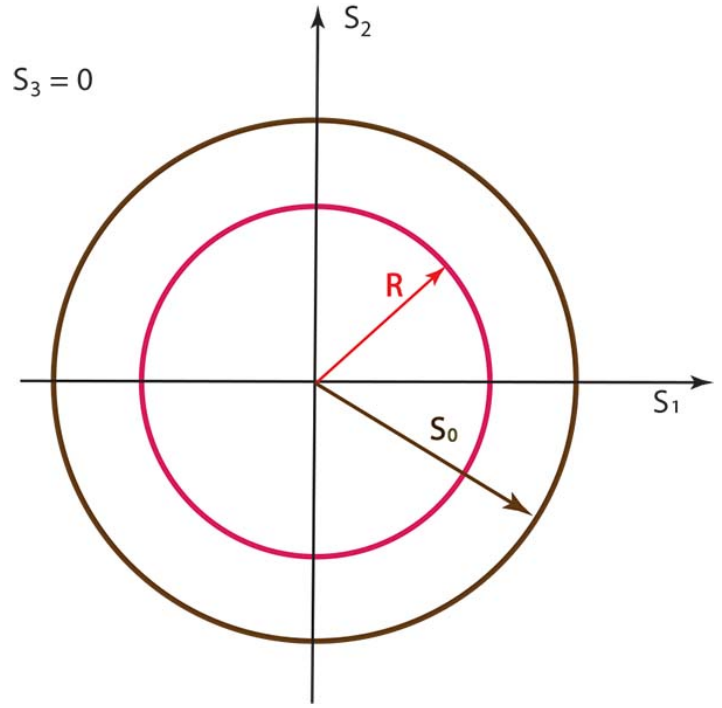
$$\boldsymbol{S_3} = 0.$$

$$\boldsymbol{S_0} = 1,$$

The radius $\boldsymbol{R}$ takes its maximum value $\boldsymbol{S_0}$ when $\chi = 0^o$. It decreases and reaches its minimum value, $\boldsymbol{S_3}$, when $\chi = 90^o$. The determinant of the coherency matrix is

$$\boldsymbol{S_0^2} - \boldsymbol{R}^2 = 1 - \cos^2\chi = \sin^2\chi.$$

This determinant is invariant under the optical (Lorentz) transformation applicable to the coherency matrix.



From massive to massless

The coherency matrix is Hermitian, and it can be diagoinalized to

$$\begin{pmatrix} 1+\cos\chi & 0 \\ 0 & 1-\cos\chi \end{pmatrix}.$$

It becomes

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

when $\chi = 0$. This form corresponds to the four-momentum for a massless particle.

It becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

when $\chi = 90^o$. This form is the same as the four-momentum for a massive particle at rest.

It is possible to go from the massive to massless case continually, but this transformation does not leave the determinant invariant since it is $(sin\chi)^2$.

Thus, the parameter χ does not belong to the Lorentz group.

Within the framwork of the Lorentz group, we take Lorentz-boost the rest-frame four-momentum:

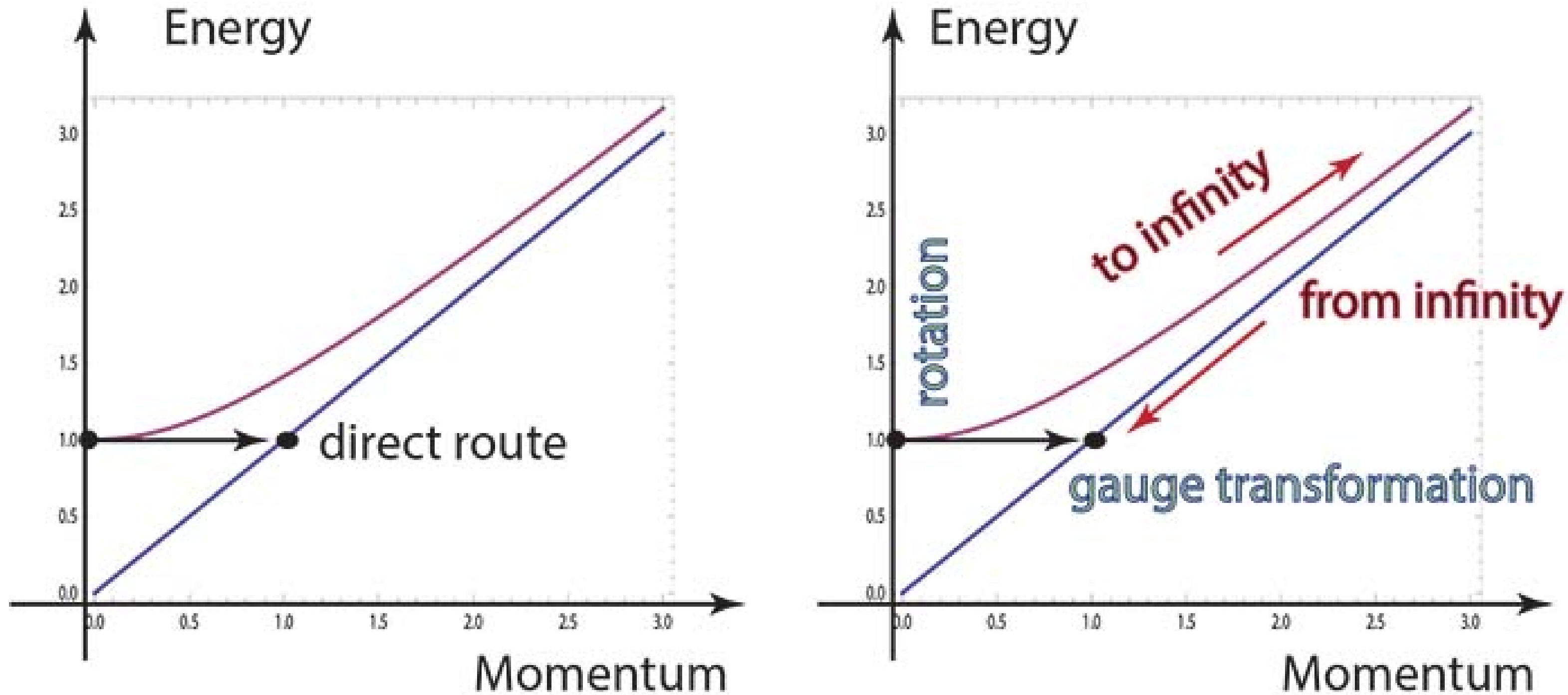
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix}.$$

For large values of η, this matrix can be written as

$$\begin{pmatrix} e^{\eta} & 0 \\ 0 & 0 \end{pmatrix}.$$

We can then come back to $\eta = 0$, and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$



The Poincaré sphere contains the symmetry of the Lorentz group. In addition, it allows the mass of the particle to take different values. We should expect more from the Poincaré sphere.

I would like to thank my long-time collegues who worked with me.

Marilyn Noz since 1970

Sibel Baskal since 1999

We all thank Wigner Jeno.

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