Hidden symmetries of scattering amplitudes in gauge theories

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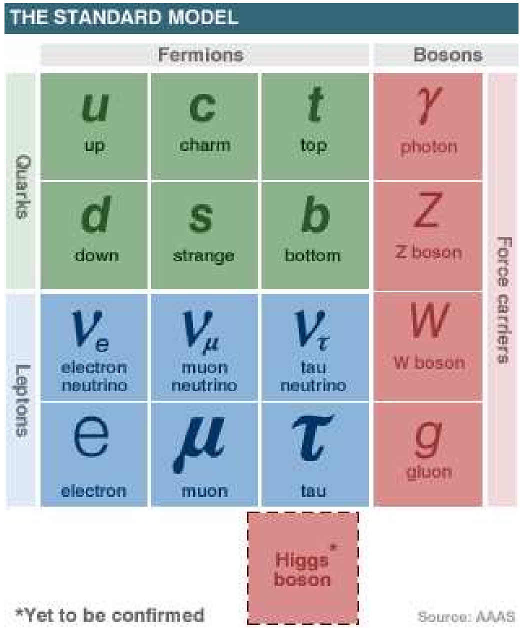
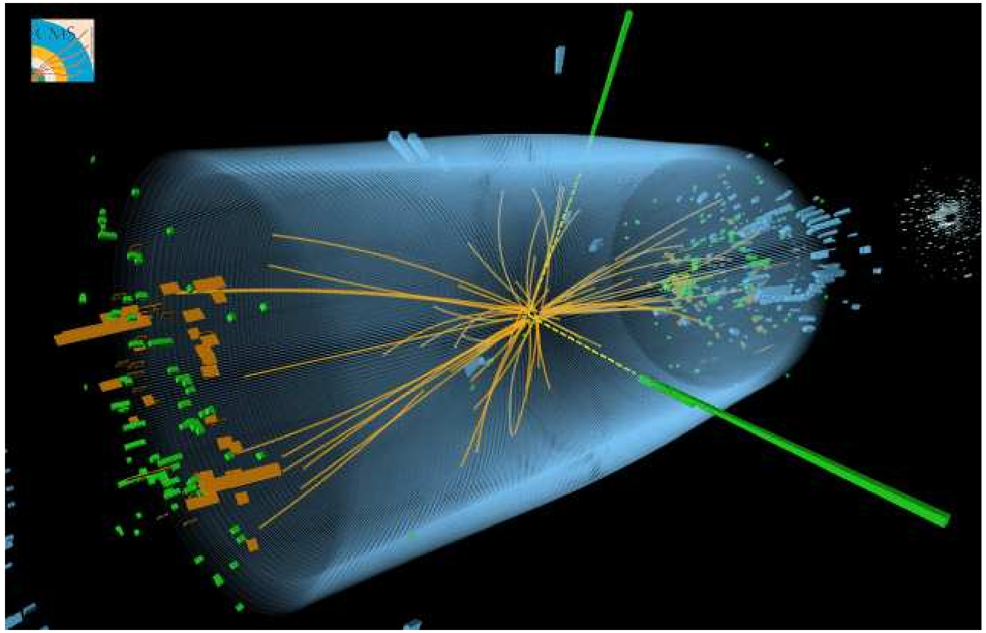
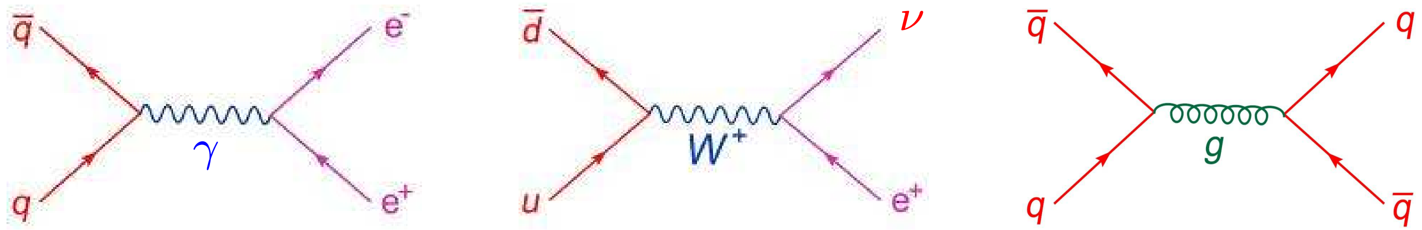
The Standard Model





- ✔ All matter is composed of spin$-1/2$ fermions
- ✔ All forces (except gravity) is carried by spin-1 vector bosons:



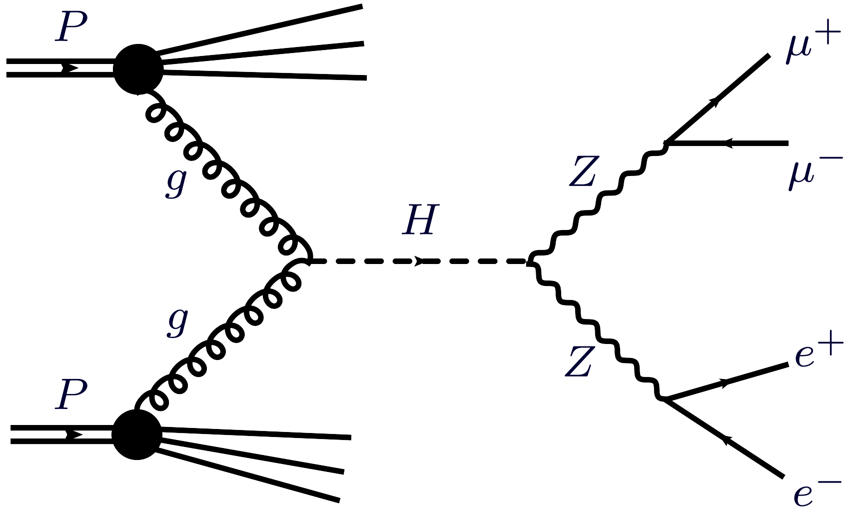
electromagnetism　　weak　　strong (QCD)

- ✔ Gauge theory with the symmetry group $SU(3) \times SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$
- ✔ Until July 4th, 2012 the only missing piece was the Higgs boson

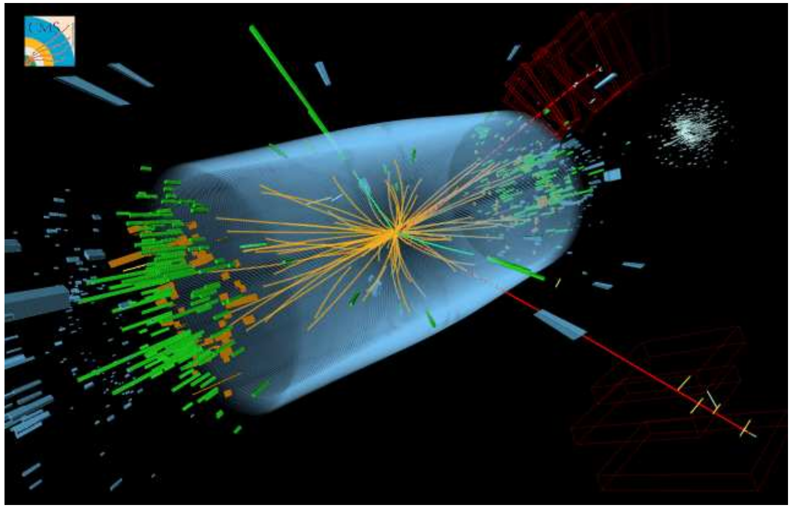
Higgs at LHC

One of the decay modes of the Higgs boson:

$$\text{gluon} + \text{gluon} \to \text{Higgs} \to e^+ + e^- + \mu^+ + \mu^-$$



Theory

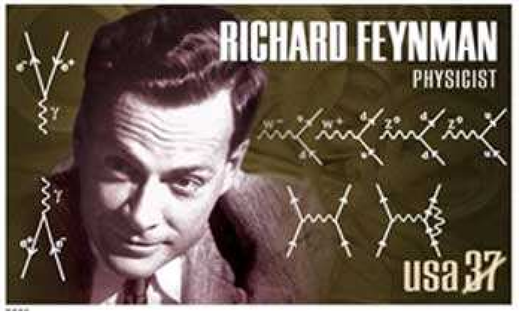


Experiment

- ✔ Specific feature of proton colliders – lots of produced quarks and gluons in the final state leading to large background
- ✔ Identification of Higgs boson requires detailed understanding of scattering amplitudes for many scattered high-energy particles – especially quarks and gluons of QCD
- ✔ Theory should provide solid basis for a successful physics program at the LHC

Theory toolkit

Feynman diagrams:



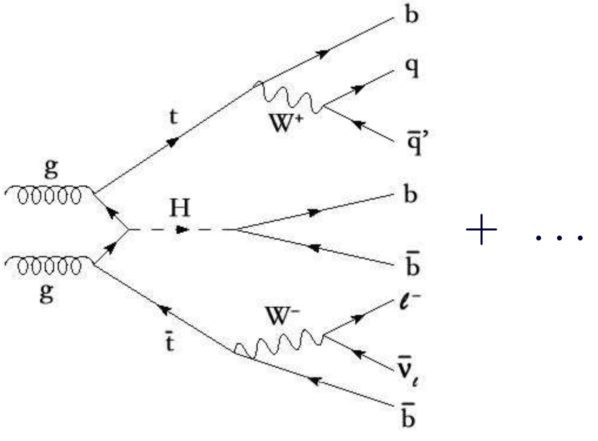
- ✔ intuitive graphical representation of the scattering amplitudes
- ✔ bookkeeping device for simplifying lengthy calculations in *perturbation theory* (in coupling constants)

For LHC physics we need scattering amplitudes with many particles involved!

We know how to do this in principle:

(1) draw all Feynman diagrams

(2) compute them!



Reality is more complicated however...

Most often the computation of multi-leg/loop Feynman diagrams is too hard :

☞ *Feynman diagrams are not optimized for the processes with many particles* involved

☞ *Important to find more efficient methods making use of hidden symmetries*

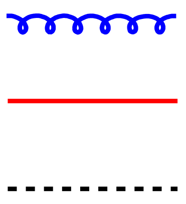
☞ *Try to consider first the simplest gauge theory*

The simplest gauge theory

Maximally supersymmetric Yang-Mills theory

- ✔ Most (super)symmetric theory possible (without gravity)
- ✔ Uniquely specified by local internal symmetry group - e.g. number of colors N_c for $SU(N_c)$
- ✔ Exactly scale-invariant field theory for any coupling (Green functions are powers of distances)
- ✔ Weak/strong coupling duality (AdS/CFT correspondence, gauge/string duality)

Particle content:

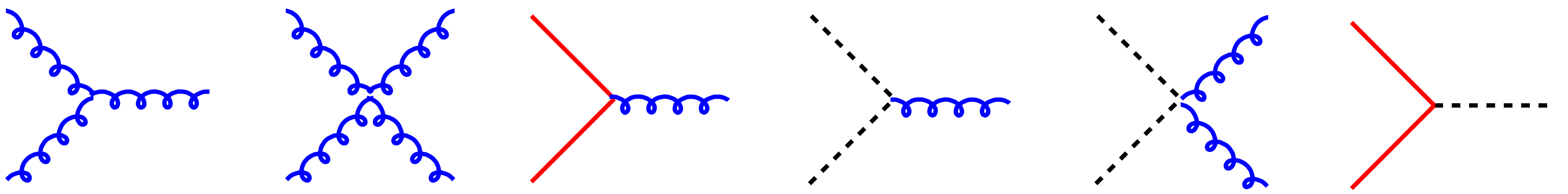


massless spin-1 gluon (= the same as in QCD)

4 massless spin-1/2 gluinos (= cousin of the quarks)

6 massless spin-0 scalars

Interaction between particles:



All proportional to same dimensionless coupling g_{YM} and related to each other by supersymmetry

Why Maximally supesymmetric Yang-Mills theory is interesting?

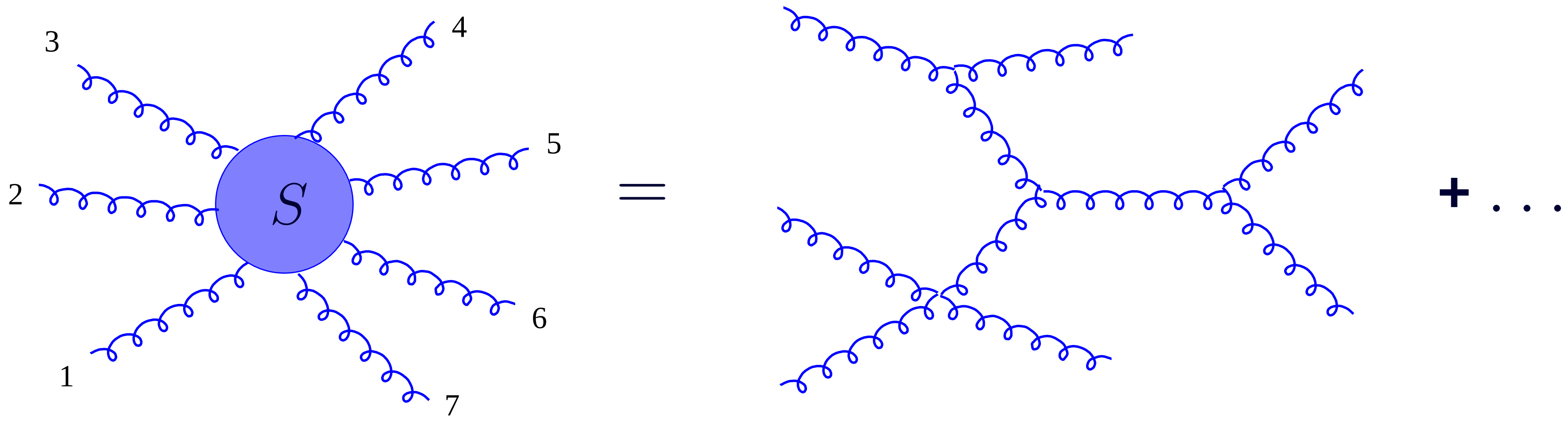
- ✔ Four-dimensional gauge theory with extended spectrum of physical states/symmetries
- ✔ An excellent testing ground for QCD in the perturbative regime relevant for collider physics
- ✔ Is equivalent to QCD at *tree level* and serves as one (most complicated) piece of QCD *all-loop* computation
- ✔ Why $\mathcal{N} = 4$ SYM theory is fascinating?
  - ✗ *At weak coupling*,
    - the number of contributing Feynman diagrams is *MUCH* bigger compared to QCD
    - ... but the final answer is *MUCH* simpler
  - ✗ *At strong coupling*, the conjectured gauge/string duality (AdS/CFT correspondence)

    Strongly coupled planar $\mathcal{N} = 4$ *SYM* $\quad\Longleftrightarrow\quad$ *Weakly coupled 'dual' string theory on* $\mathrm{AdS}_5 \times \mathrm{S}^5$
- ✔ Final goal (dream):

Maximally supesymmetric Yang-Mills theory is an example of the
four-dimensional gauge theory that can be/ should be/ would be solved
exactly for arbitrary value of the coupling constant!!!

Conventional approach

Simplest example: Gluon scattering amplitudes



Number of external gluons	4	5	6	7	8	9	10
Number of 'tree' diagrams	4	25	220	2485	34300	559405	10525900

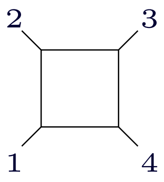
- ✔ Number of diagrams grows factorially for large number of external gluons/number of loops
- ✔ If one spent 1 second for each diagram, computation of 10 gluon amplitude would take 121 days!
- ✔ ... but the final expression for tree amplitudes looks remarkably simple

$$A_n^{\text{tree}}(1^+2^+3^-\dots n^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\,, \qquad \left[\text{spinor notations: } \langle ij\rangle = \lambda^\alpha(p_i)\lambda_\alpha(p_j)\right]$$

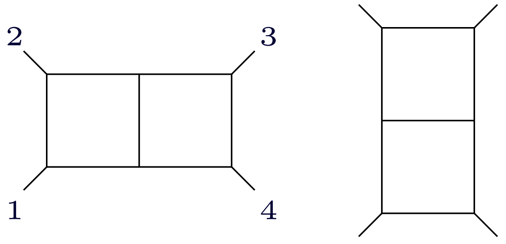
Four-gluon planar amplitude in $\mathcal{N}=4$ SYM at weak coupling

‘Mirracle’ at weak coupling: number of Feynman diagrams increases with loop level but their sum can be expressed in terms of a few ‘special’ scalar box-like integrals

- ✔ One loop:

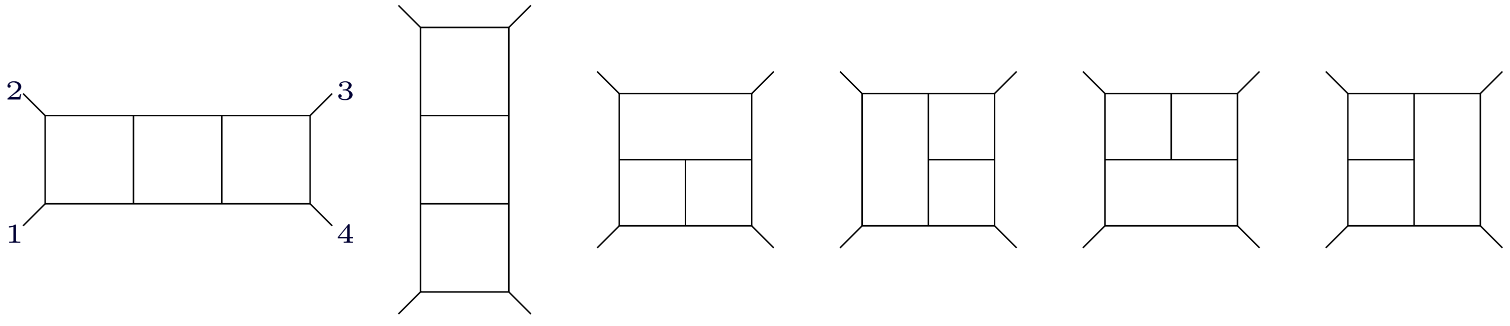


- ✔ Two loops:



all-loop iteration structure conjectured

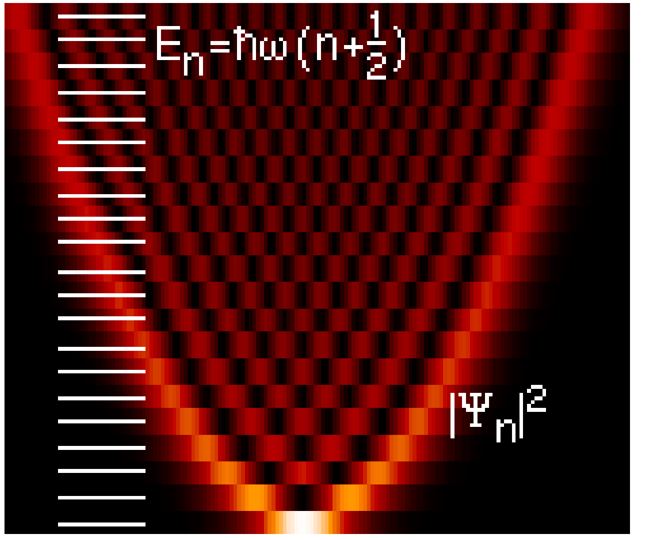
- ✔ Three loops:



Little hope to get an exact all-loop analytical solution... unless there is some hidden symmetry

Hint for hidden symmetry: Harmonic oscillator

- One of the few quantum mechanical systems for which a simple exact solution is known



- The quantum mechanical analogue of the classical harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

- Surprising duality between coordinates and momenta

$$p \to (m\omega)x\,, \qquad\qquad x \to -(m\omega)^{-1}p$$

- The wave function looks alike in the coordinate and momentum representations

$$\Psi_0(x) \sim \exp\left(-\frac{x^2}{2}\frac{m\omega}{\hbar}\right) \qquad \Leftrightarrow \qquad \tilde{\Psi}_0(p) \sim \exp\left(-\frac{p^2}{2}\frac{(m\omega)^{-1}}{\hbar}\right)$$

- The dual symmetry is sufficient to find the eigenstates

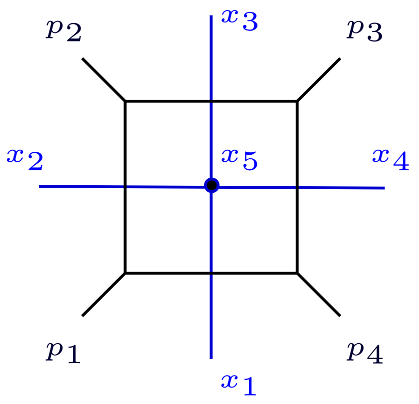
$$\Psi_\ell(x) = \tilde{\Psi}_\ell(p)\Big|_{p=xm\omega}$$

Back to scattering amplitudes: dual conformal symmetry

Examine one-loop 'scalar box' diagram

✔ Change variables to go to a *dual 'coordinate space'* picture :

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^D k\, (p_1+p_2)^2 (p_2+p_3)^2}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2} = \int \frac{d^D x_5\, x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Unexpected conformal invariance under inversion (for $D = 4$)

$$x_i^\mu \to x_i^\mu / x_i^2\,, \qquad d^4 x_5 \to d^4 x_5 / (x_5^2)^4\,, \qquad x_{ij}^2 \to x_{ij}^2 / (x_i^2 x_j^2)$$

✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!

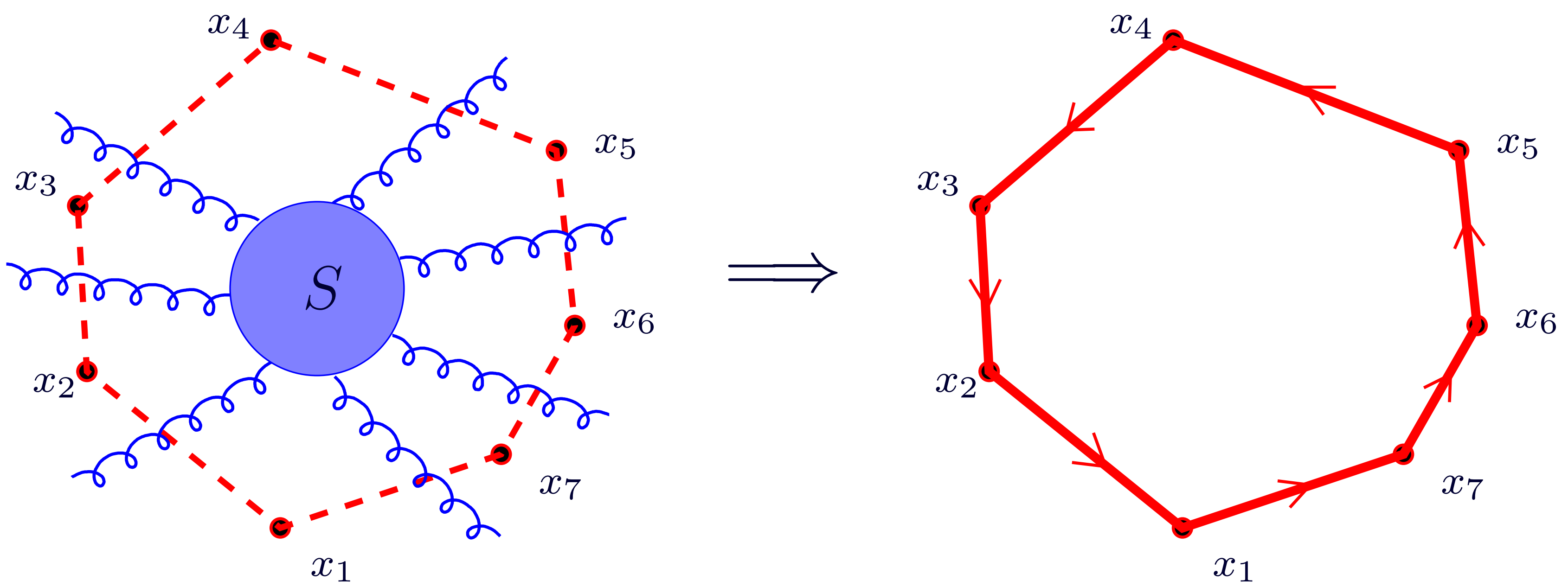
✔ The symmetry *is not related* to conformal $SO(2,4)$ symmetry of $\mathcal{N} = 4$ SYM

- ✗ Conventional conformal transformations act *locally on the coordinates* (preserve angles)
- ✗ After Fourier transform, conformal transformations act *nonlocally on the momenta*
- ✗ Dual conformal transformations act *locally on the momenta*

✔ ***The dual conformal symmetry is powerful enough to determine four- and five-gluon planar scattering amplitudes to all loops!***

Dual description of scattering amplitudes

Let us introduce dual variables :



Closed contour in Minkowski space-time ($n-$gon)

$$C_n = [x_1, x_2] \cup \ldots \cup [x_n, x_1]$$

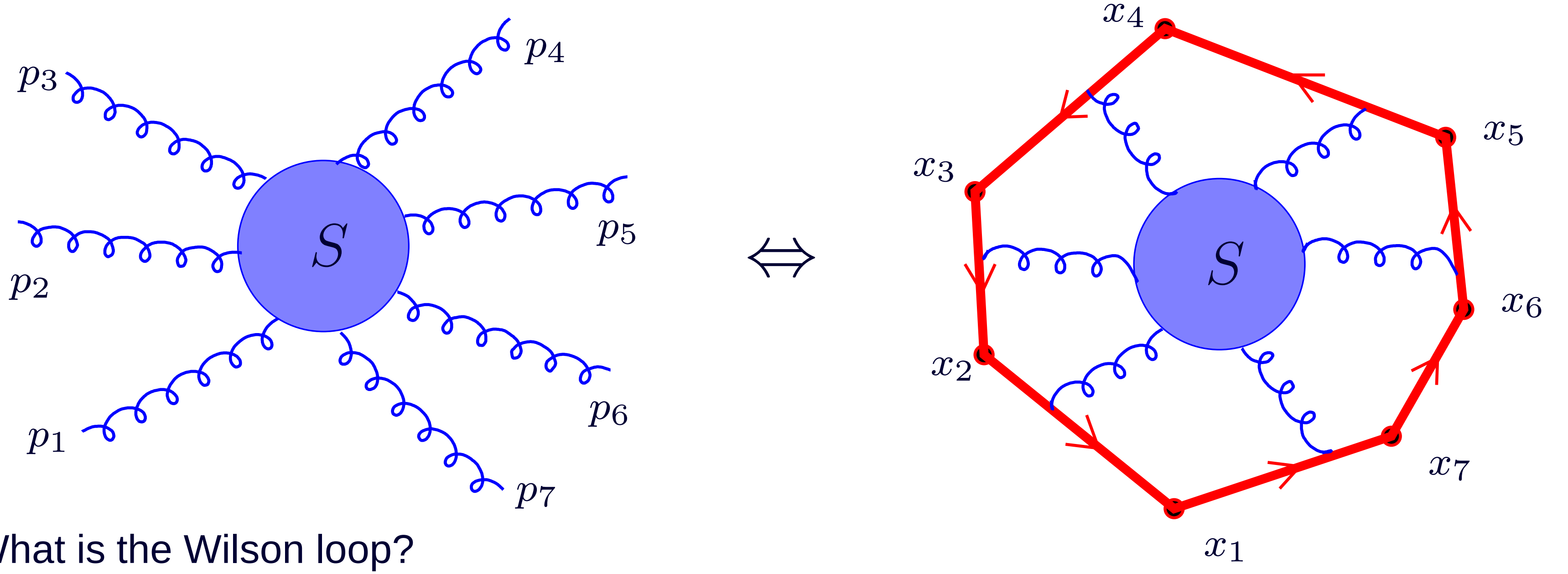
Built from light-like segments defined by gluon momenta

$$p_i = x_i - x_{i+1} \,, \qquad p_i^2 = 0$$

Who “lives” on polygon light-like contour in the dual description?

Scattering amplitudes/Wilson loops duality

Gluon (MHV) scattering amplitude $=$ Wilson loop in Minkowski space-time



✔ What is the Wilson loop?

$$W_n = \langle 0 | \exp\left(ig \oint_{C_n} dx^\mu A_\mu(x) \right) |0\rangle$$

- ✗ Circulation of (nonabelian) gauge field along polygon like contour
- ✗ Interaction of a test particle moving along the closed contour C_n with its own radiation
- ✗ Gauge-invariant scalar function of dual distances = Mandelstam invariants

$$(x_1 - x_2)^2 = 0\,, \qquad (x_1 - x_3)^2 = (p_1 + p_2)^2\,, \qquad (x_1 - x_4)^2 = (p_1 + p_2 + p_3)^2\,, \quad \ldots$$

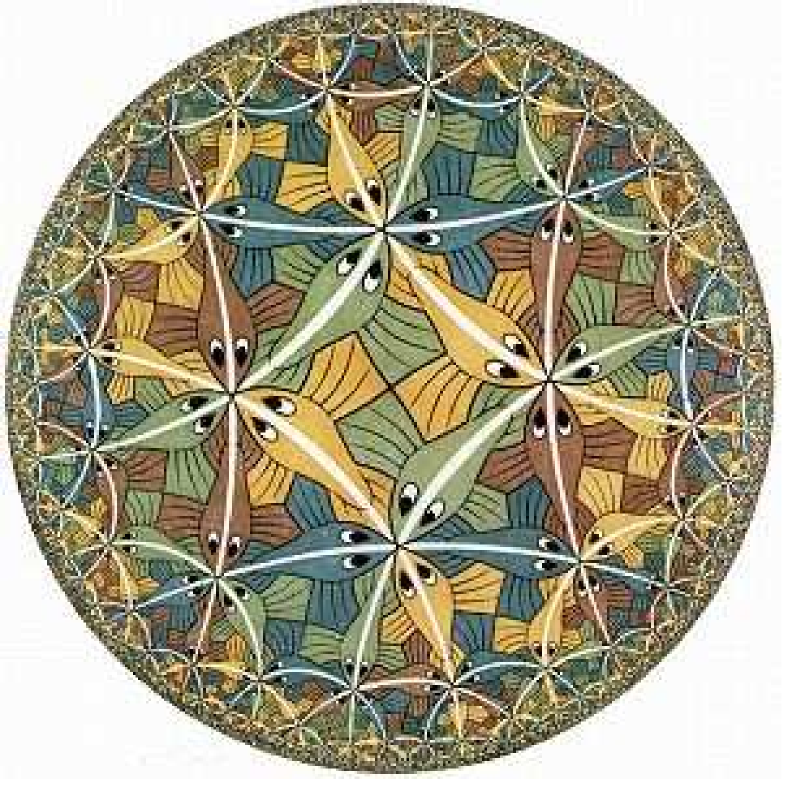
Gauge/string duality

Surprising correspondence of strongly interacting quantum field theories with gravitational theories:

Strongly coupled maximally supersymmetric gauge theory	$\Leftrightarrow$	Weakly coupled 'dual' string theory on anti-de Sitter space

Dual space is 5 dimensional hyperbolic space with constant negative curvature

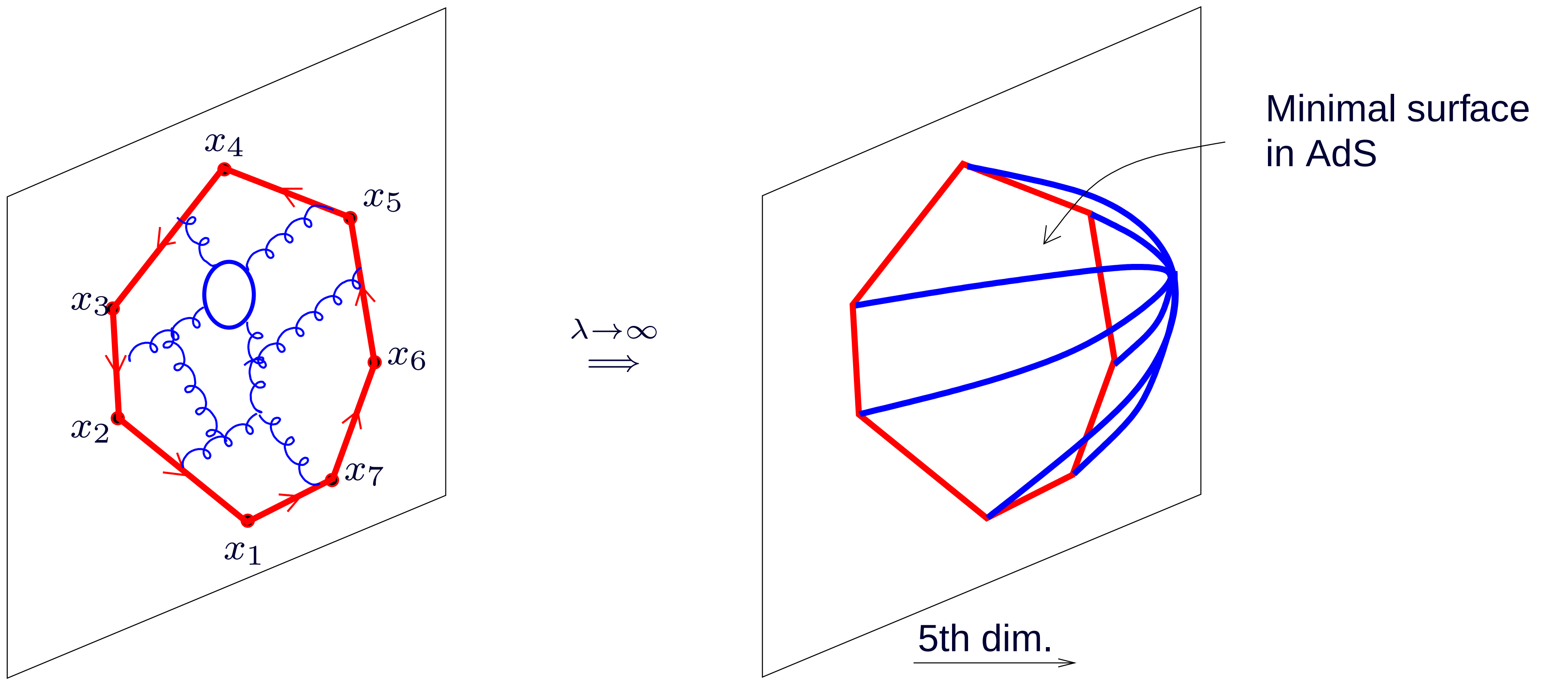
$$ds^2 = R^2 \left[e^{2\tau} \underbrace{\left(-dt^2 + dx^2 + dy^2 + dz^2\right)}_{\text{Minkowski}} + \underbrace{d\tau^2}_{\text{5th dim}} \right]$$



Hyperbolic space depicted by M.C.Escher:

- ✔ Each fish has the same coordinate-invariant "proper" size
- ✔ Appear to get smaller near the boundary because of their projection onto the flat surface of this page.
- ✔ We "live" at the circular boundary, infinitely far from the center of the disk ($\tau \to \infty$).

From Wilson loops to minimal surfaces

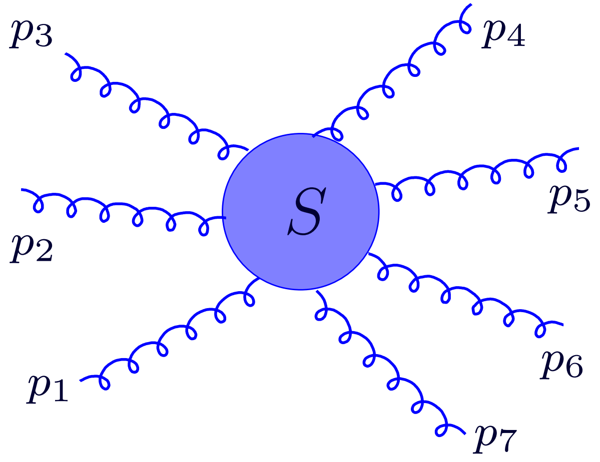


The Wilson loop at strong coupling from gauge/string duality

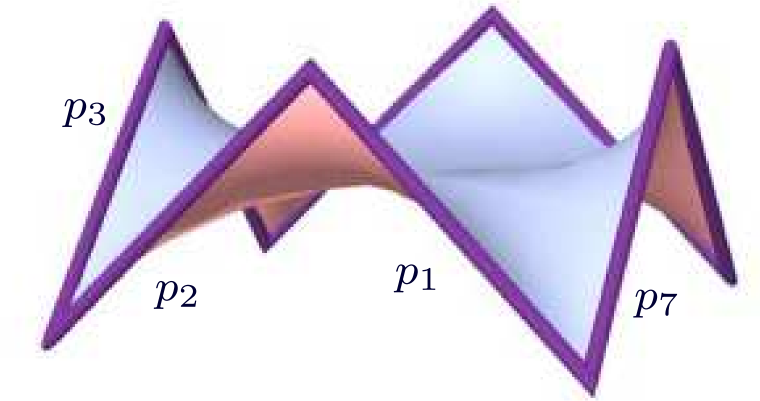
$$\text{Amplitude} = \text{Wilson loop} \sim \exp\left(-\sqrt{g^2 N_c} \times \text{Area}\right)$$

- ✔ Defined by the area of minimal surface in anti-de Sitter space
- ✔ The surface ends at the AdS boundary on a polygon given by a sequence of gluon momenta
- ✔ Solution to the Plateau's problem – soap bubbles in hyperbolic space

Computing scattering amplitudes using soap films



$\Longrightarrow$



- ✔ Explicit expressions for 4- and 5-particle scattering amplitudes for *arbitrary coupling !*
- ✔ First nontrivial results for $n-$particle scattering amplitudes at strong coupling

Maximally supersymmetric Yang-Mills theory is the simplest gauge theory – "harmonic oscillator of 21st century":

- ✔ An excellent testing ground for computing QCD scattering amplitudes needed for precise theoretical predictions at hadron colliders
- ✔ Unexpected interconnections and hidden symmetries

We are close to finding an exact solution to Maximally supersymmetric Yang-Mills theory

Stay tuned!